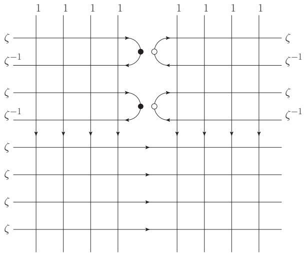
The Corner Transfer Matrix/Vertex Operator Approach to Entanglement and Fidelity



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London, June 2014

Plan

1. What You Can Compute
2. The CTM/Vertex Operator Approach
3. Example: The XXZ Model
   - Renyi Entropies
   - Representation Theory
   - Fidelity

Fidelity Refs:

- RW: 1110.2032, 1203.2326

- Builds on formalism of 'Algebraic Analysis ...' by Jimbo & Miwa (95), and boundary papers by Jimbo, Kedem, Konno, Kojima/RW, Miwa: hep-th:9411112/9502060

What you can compute

- Let $\rho =$ be reduced density matrix for infinite bipartite system. Can compute exact expressions for
  - $S_n = \frac{1}{1-n} \ln \mathrm{Tr}(\rho^n)$ - Renyi entropies
  - $S = S_1 = -\mathrm{Tr}(\rho \ln \rho)$ - entanglement entropy
  - $f = |\langle \mathrm{vac}|\mathrm{vac}\rangle'|^2$ - bipartite fidelity ($|\mathrm{vac}\rangle'$ is for cut chain)

  for a range of massive solvable lattice models.

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- More generally, the vertex operator approach gives exact expressions for arbitrary correlation functions and form factors for the geometries I will discuss.

Scaling limit

- Find
  - $S_n \sim \frac{c}{12}(1+\frac{1}{n})\ln(\xi)$
  - $S \sim \frac{c}{6}\ln(\xi)$
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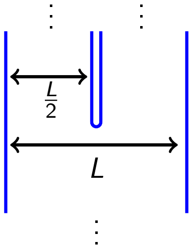
- Above are valid in regime $0 \ll \xi \ll L$.
- CFT predictions for S_n come from free energy on Reimann surface that is n-fold cover of $\mathbb{C}$ [Holzhey et al 94, Calabrese & Cardy 04].
- S_n also consistent with general QFT predictions for finite ξ [Calabrese & Cardy 04].

Scaling limit cont.

- CFT predictions for $-\ln f$ by mapping half-plane to [Dubail & Stéphan 11]



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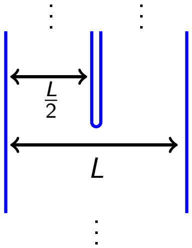
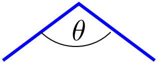


- or just by taking twice free energy

$$F = \frac{c\theta}{24\pi}(1-(\pi/\theta)^2)\ln L$$

for corner angle 

[Cardy & Peschel 88]

The CTM/Vertex Operator Approach

- Consider a solvable vertex model with bulk and boundary matrices:

$$R(\zeta_1/\zeta_2) : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n, \quad K_\bullet(\zeta), K_\circ(\zeta) : \mathbb{C}^n \to \mathbb{C}^n,$$

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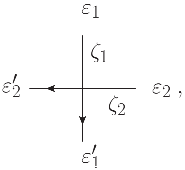
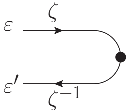
$R^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1',\varepsilon_2'}(\zeta_1/\zeta_2) =$



, $K_{\bullet}{}^{\varepsilon}_{\varepsilon'}(\zeta) =$



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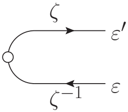

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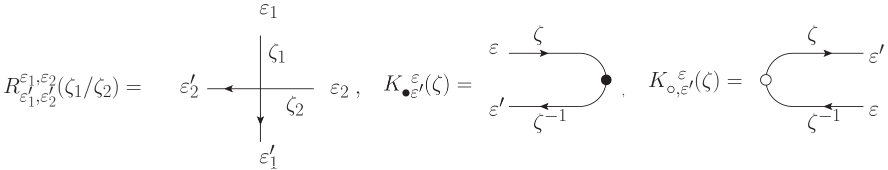
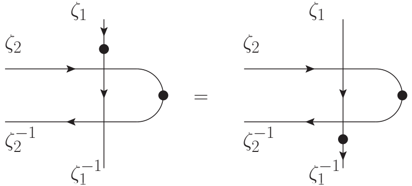
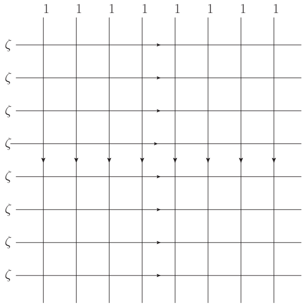


- Boundary YB is



Two Partition Functions

- We consider the partition functions of two infinite lattices with an ordered BC:



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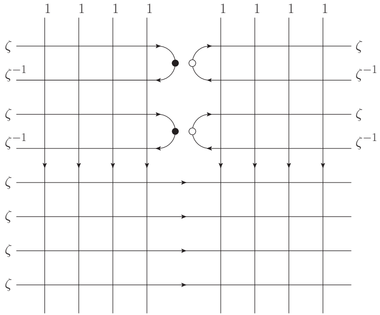

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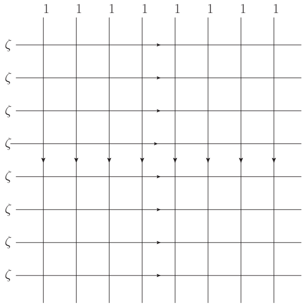

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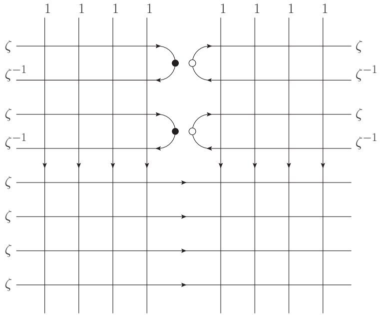
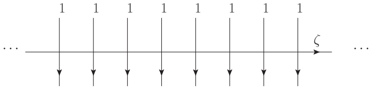


- View Hilbert space as $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R = \mathcal{H}_L \otimes \mathcal{H}_L^*$.

Transfer Matrices and Vacua

- Corresponding transfer matrices are

$T(\zeta) =$



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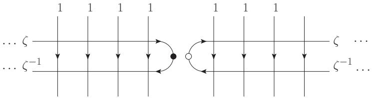

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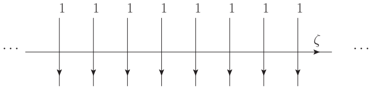

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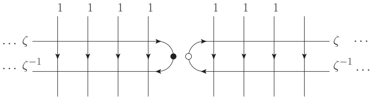


with vacua $|\text{vac}\rangle$ and $|\text{vac}\rangle' = |\text{vac}\rangle_B \otimes {}_B\langle\text{vac}|$.

- If cpts real and non-negative, and normalisation chosen s.t. eigenvalue 1, then these are PF eigenvectors

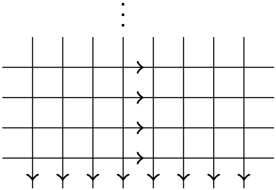
$$|\text{vac}\rangle = \lim_{N\to\infty} T^N(\zeta)|BC\rangle, \quad |\text{vac}\rangle' = \lim_{N\to\infty} {T'}^N(\zeta)|BC\rangle$$

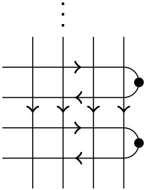
Transfer Matrices and Vacua cont.

- In terms of pictures:

$|\mathrm{vac}\rangle = \lim_{N\to\infty} T^N(\zeta)|BC\rangle,$

$|\mathrm{vac}\rangle' = |\mathrm{vac}\rangle_B \otimes {}_B\langle\mathrm{vac}| = \lim_{N\to\infty} T'^N(\zeta)|BC\rangle$ are:

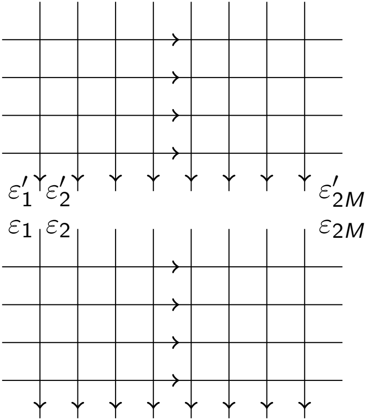
$|\mathrm{vac}\rangle \quad = \quad \cdots$  $\cdots$

$|\mathrm{vac}\rangle_B \quad = \quad \cdots$ 

The Density Matrix

- The density matrix $\tilde{\rho} = |\mathrm{vac}\rangle\langle\mathrm{vac}|$ is

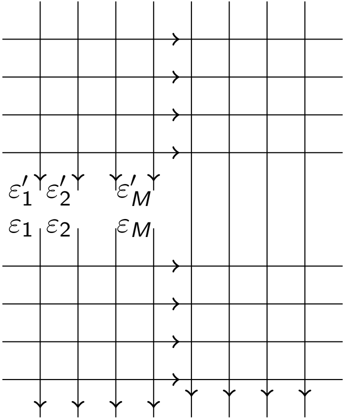
$$\tilde{\rho}^{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{2M}}_{\varepsilon'_1,\varepsilon'_2,\cdots,\varepsilon'_{2M}} =$$



The Reduced Density Matrix

- The reduced density matrix $\rho = \mathrm{Tr}_{\mathcal{H}_R}(|\mathrm{vac}\rangle\langle\mathrm{vac}|)$ is then

$$\rho^{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_M}_{\varepsilon'_1,\varepsilon'_2,\cdots,\varepsilon'_M} =$$

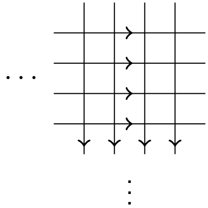


The Corner Transfer Matrix

- Key idea is to write in terms of 4 corner transfer matrices

$$\rho = A_{NW}(\zeta)A_{NE}(\zeta)A_{SE}(\zeta)A_{SW}(\zeta)$$

where $A_{SW}(\zeta) : \mathcal{H}_L \to \mathcal{H}_L$ is



Corner Transfer Matrix cont.

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$$A_{SE}(\zeta)A_{SW}(\zeta) \sim (-q)^{D}$$

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- Thus $\rho = \frac{q^{2D}}{\mathrm{Tr}(q^{2D})}$ and $\mathrm{Tr}_{\mathcal{H}_L}(\rho^n) = \frac{\chi_{\mathcal{H}_L}(q^{2n})}{\chi_{\mathcal{H}_L}(q^2)^n}$ with $\chi_{\mathcal{H}_L}(z) = \mathrm{Tr}_{\mathcal{H}_L}(z^D)$ the character

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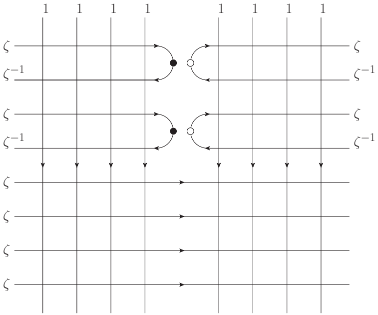
- Remember: $S_n = \frac{1}{1-n}\ln \mathrm{Tr}_{\mathcal{H}_L}(\rho^n)$

Fidelity

- Define fidelity as $f = |\langle \text{vac}|\text{vac}\rangle'|^2$
- With same reasoning as above (remembering $|\text{vac}\rangle' = |\text{vac}\rangle_B \otimes {}_B\langle \text{vac}|$):

$\langle \text{vac}|\text{vac}\rangle' =$



$$= {}_B\langle \text{vac}|A_{SE}(\zeta)A_{SW}(\zeta)|\text{vac}\rangle_B$$

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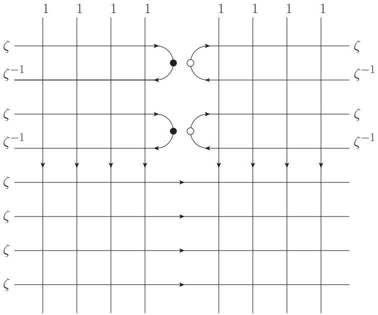

$$= \; {}_B\langle \text{vac}|A_{SE}(\zeta)A_{SW}(\zeta)|\text{vac}\rangle_B$$

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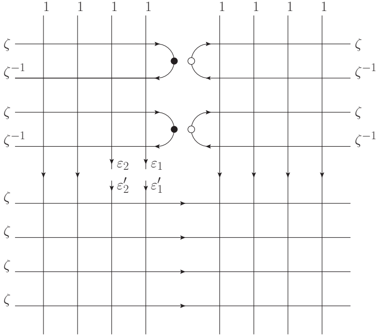
- To compute this we need to know $|\text{vac}\rangle_B \in \mathcal{H}_L$.

Vertex Operators

- More generally interested in *finite* reduced density matrix/correlation fns: $\langle \text{vac}|E^{\varepsilon_m}_{\varepsilon'_m}\cdots E^{\varepsilon_2}_{\varepsilon'_2}E^{\varepsilon_1}_{\varepsilon'_1}|\text{vac}\rangle$ or $\langle \text{vac}|E^{\varepsilon_m}_{\varepsilon'_m}\cdots E^{\varepsilon_2}_{\varepsilon'_2}E^{\varepsilon_1}_{\varepsilon'_1}|\text{vac}\rangle'$ where $E^{\varepsilon}_{\varepsilon'}(v_a)=\delta_{a,\varepsilon}v_{\varepsilon'}$.

Latter is



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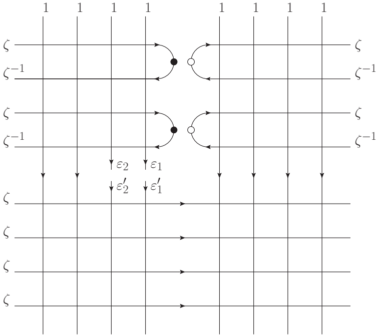
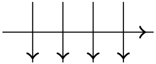
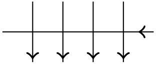


- To obtain, need to realise local operator in a nice way. We can - entirely in terms of vertex operators $\Phi_\varepsilon(\zeta),\Phi^*_\varepsilon(\zeta):\mathcal{H}_L\to\mathcal{H}_L$.

Vertex Operators cont.

- Define the Vertex Operators as lattice operators

$$\Phi_\varepsilon(\zeta) \quad = \quad \cdots$$

ε ,

$$\Phi^*_\varepsilon(\zeta) \quad = \quad \cdots$$

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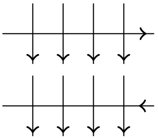

- Then, $R(1) = P$ means

$$\mathcal{O} = E^{\varepsilon_m}_{\varepsilon'_m} \cdots E^{\varepsilon_2}_{\varepsilon'_2} E^{\varepsilon_1}_{\varepsilon'_1} = \Phi^*_{\varepsilon_1}(1) \cdots \Phi^*_{\varepsilon'_m}(1)\, \Phi_{\varepsilon_m}(1) \cdots \Phi_{\varepsilon_1}(1).$$

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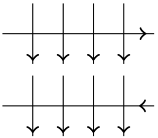

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- In general, just need to compute

$$\langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle = \mathrm{Tr}_{\mathcal{H}_L}(q^{2D}\mathcal{O}), \quad \langle \mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle' = {}_B\langle \mathrm{vac}|(-q)^D\mathcal{O}|\mathrm{vac}\rangle_B.$$

Example: The XXZ/6-vertex model

- Remember we want to compute

$$S_n = \frac{1}{1-n} \ln \frac{\mathrm{Tr}_{\mathcal{H}_L}(q^{2nD})}{\mathrm{Tr}_{\mathcal{H}_L}(q^{2D})^n}, \qquad \langle \mathrm{vac}|\mathrm{vac}\rangle' = {}_B\langle \mathrm{vac}|(-q)^D|\mathrm{vac}\rangle_B$$

- Consider the 6-vertex model with weights

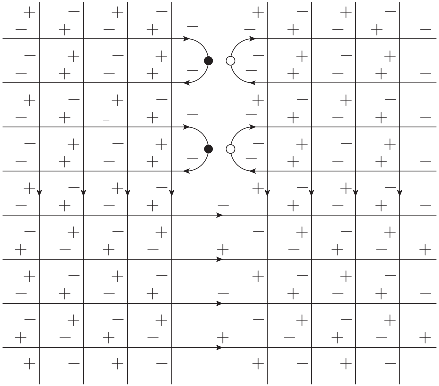
$$R(\zeta) = \frac{1}{\kappa(\zeta)} \begin{pmatrix} 1 & & & \\ & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \\ & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \\ & & & 1 \end{pmatrix} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$K_\bullet(\zeta; r) = \frac{1}{f(\zeta; r)} \begin{pmatrix} \frac{1-r\zeta^2}{\zeta^2-r} & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$$

obeying usual YB, unitarity and crossing (for bulk and boundary).

Boundary Conditions

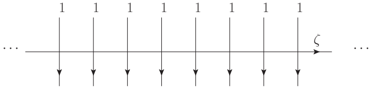
- The boundary condition is such that at finite, but arbitrarily large, distances from origin we fix BC to pattern:



Hamiltonians

- Two transfer matrices:

Bulk $T(\zeta) =$



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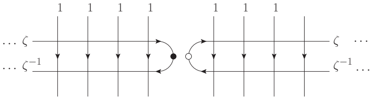

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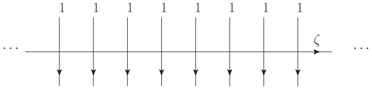

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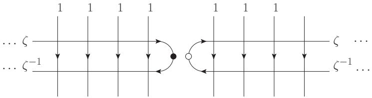

- Hamiltonians

$$H = -\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right), \qquad \Delta = \frac{q+q^{-1}}{2}$$

$$H' = H_L + H_R,$$

$$H_L = -\frac{1}{2}\sum_{n\geq 1}\left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right) + h\sigma^z_1,$$

$$H_R = -\frac{1}{2}\sum_{n\leq 0}\left(\sigma^x_i\sigma^x_{i-1} + \sigma^y_i\sigma^y_{i-1} + \Delta\sigma^z_i\sigma^z_{i-1}\right) - h\sigma^z_0, \;\; h(r) = \frac{q^2-1}{4q}\frac{1+r}{1-r}.$$

Renyi Entropies

- Remember $S_n = \dfrac{1}{1-n} \ln \dfrac{\mathrm{Tr}_{\mathcal{H}_L}(q^{2nD})}{\mathrm{Tr}_{\mathcal{H}_L}(q^{2D})^n}$

- Result known from Baxter's work in the 70s [The Book]:

$$\begin{aligned} D &= \frac{q}{1-q^2} \sum_{i=1}^{\infty} i \left(\sigma^x_{i+1}\sigma^x_i + \sigma^y_{i+1}\sigma^y_i + \Delta\sigma^z_{i+1}\sigma^z_i\right), \\ \mathrm{Tr}_{\mathcal{H}_L}(z^D) &= \prod_{m=1}^{\infty} \frac{1}{1-z^{(2m-1)}} \end{aligned}$$

- Knowing above and exact result [Johnson, Krinsky, McCoy 73]

$$\xi^{-1} = -\frac{1}{2}\ln\left(\frac{1-k'}{1-k'}\right), \qquad k' = \text{elliptic modulus}$$

gives scaling form (with $c = 1$)

$$S_n \sim \quad \frac{c}{12}\left(1+\frac{1}{n}\right)\ln(\xi), \quad \text{and} \quad S = S_1 \sim \frac{c}{6}\ln(\xi)$$

[This form and corrections to it discussed in Calabrese & Cardy 04; RW 06; Ercolessi Evangelisti Franchini, Ravanini 2011; Cardy, Castro Alvaredo,Doyon 2007, 2008; $\cdots$]

Representation Theory

- To proceed further, we need to understand $\mathcal{H}_L$ and D and $|\mathrm{vac}\rangle_B$ in terms of representation theory (and $\Phi_\varepsilon(\zeta)$ and $\Phi^*_\varepsilon(\zeta)$ for correlation functions).

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- Starting point: Note $\mathrm{Tr}_{\mathcal{H}_L}(q^{2D}) = \mathrm{Tr}_{V(\Lambda_i)}(q^{2D})$
  - Here $V(\Lambda_{0,1})$ is a character of a infinite dim. hw representation of affine algebra $\widehat{\mathfrak{sl}}_2$ (or $U_q(\widehat{\mathfrak{sl}}_2)$).
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  - D on rhs is a derivation (grading) in this algebra.
- In fact, $U_q(\widehat{\mathfrak{sl}}_2)$ has an action on $\mathbb{C}^2$ and also on $\mathcal{H} = \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots$. We have

$$[T(\zeta), U_q(\widehat{\mathfrak{sl}}_2)] = 0.$$

[Davies et al. 92, Jimbo & Miwa 94]

Representation Theory cont.

- However

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Representation Theory cont.

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- $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry lead to the identification of

  $\mathcal{H}_L \simeq V(\Lambda_i)$, and $D, \Phi(\zeta), \Phi^*(z) : V(\Lambda_i) \to V(\Lambda_{1-i})$

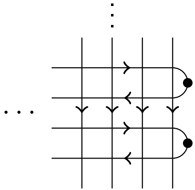
  in terms of $U_q(\widehat{\mathfrak{sl}}_2)$ representation theory.

  [Davies et al. 92, Jimbo & Miwa 94]

Representation Theory cont.

- Then

$$|\text{vac}\rangle_B \quad = $$



is given by $T_B(\zeta)|\text{vac}\rangle_B = |\text{vac}\rangle_B$ where

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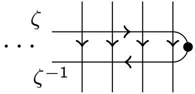
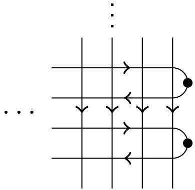


$$= \Phi^*_{\varepsilon'}(\zeta^{-1}) K_\bullet(\zeta;r)^{\varepsilon}_{\varepsilon'} \Phi_\varepsilon(\zeta)$$

Representation Theory cont.

- Then

$|\mathrm{vac}\rangle_B \quad = $ 

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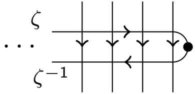
$T_B(\zeta) \quad = $ 

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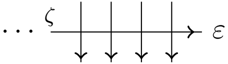
- Identification comes from



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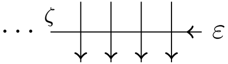

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Exact Results

- We can diagonalise $T_B(\zeta)$ in $V(\Lambda_i)$ by making use of a free-field (Fock space) realisation of both. We find [RW 12]:

$$\langle \text{vac}|\text{vac}\rangle'$$
$$= (q^2;q^4)_\infty^{\frac{1}{2}} \frac{(r^2q^{10};q^8,q^8)_\infty^2}{(r^2q^4;q^8,q^8)_\infty (r^2q^{12};q^8,q^8)_\infty} \frac{(r^2q^2;q^4,q^8)_\infty}{(r^2q^4;q^4,q^8)_\infty} \frac{(q^6;q^8,q^8)_\infty}{(q^{10};q^8,q^8)_\infty}$$

where $(a;b)_\infty = \prod_{n=0}^{\infty}(1-ab^n), \quad (a;b,c)_\infty = \prod_{n=0}^{\infty}\prod_{m=0}^{\infty}(1-ab^nc^m)$

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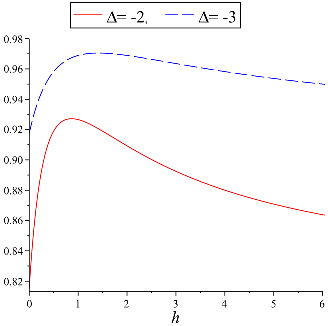
- Also get exact expression for correlation functions, including

$$\langle \mathrm{vac}|\sigma_1^z|\mathrm{vac}\rangle' = 1 + 2(1-r)\sum_{n=1}^{\infty}\frac{(-q^2)^n}{1-rq^{4n}}$$

$$= \frac{(q^2;q^2)_\infty^2}{(-q^2;q^2)_\infty^2} \quad \text{when} \quad r=-1 \quad (h=0).$$

Fidelity

- Plotting fidelity $f = |\langle \mathrm{vac}|\mathrm{vac}\rangle'|^2$ vs h:



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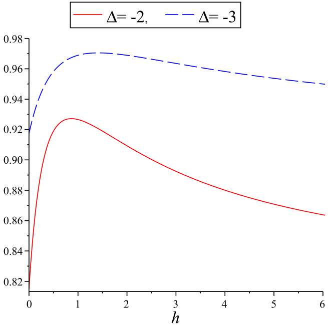

max at $h = \frac{q-q^{-1}}{4}$

- When we have zero boundary magnetic field $(r = -1)$

$$f = (q^2; q^4)_\infty \frac{(-q^4; q^4, q^4)_\infty}{(-q^2; q^4, q^4)_\infty}$$

Scaling limit

- Letting $-q = e^{-\varepsilon}$, the scaling limit is $\varepsilon \to 0$ corresponding to XXX point $\Delta = -1$.

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- Then

$$\ln(\xi) \underset{\varepsilon \to 0}{=} \frac{\pi^2}{2\varepsilon} - \ln(4) + O(\varepsilon).$$

giving

$$-\ln(f) \underset{\varepsilon \to 0}{=} \frac{\pi^2}{16\varepsilon} - \frac{1}{4}\ln(2) + O(\varepsilon),$$

and

$$-\ln(f) \underset{\varepsilon \to 0}{=} \frac{1}{8}\ln(\xi) + O(\varepsilon).$$

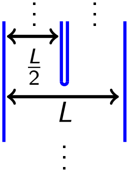
Scaling limit cont.

- $-\ln(f) \underset{\varepsilon\to 0}{=} \frac{1}{8}\ln(\xi) + O(\varepsilon)$

  is consistent with general conjecture [RW12]:

$$-\ln(f) \underset{\xi\to\infty}{\sim} \frac{c}{8}\ln(\xi).$$

- and with CFT result for size L 1D critical bipartite system



$$-\ln(f) \underset{\xi\to\infty}{\sim} \frac{c}{8}\ln(L).$$

[Cardy & Peschel 88, Dubail & Stéphan 11]

Conclusions

- $-\ln f$ put forward as measure of entanglement - reasonably easy to compute with these techniques.
- Interesting to study sub-leading corrections (c.f. predicted $L^{-1}\ln L$ term [Stéphan & Dubail 13])
- Direct QFT argument for $\frac{c}{8}$ for finite ξ?
- Can find correlation functions for these geometries as well - giving mutual entropies for example.
- Approach works for other models too: higher spin [RW06], 8-vertex [Ercolessi et al. 11], RSOS, $\mathfrak{sl}_n$, Z_N, ...