An Algebraic Setting for Defects in the XXZ and Sine-Gordon Models

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Abstract

We construct defects in the XXZ and sine-Gordon models by making use of the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$. The representations involved are generalisations of the infinite-dimensional, q-oscillator representations used in the construction of Q-operators. We present new results for intertwiners of these representations, and use them to consider both quantum spin-chain Hamiltonians with defects and quantum defects in the sine-Gordon model. We connect a specialisation of our results with the work of Bowcock, Corrigan and Zambon and present sine-Gordon soliton/defect and candidate defect/defect scattering matrices.

1 Introduction

Integrable quantum field theory and solvable lattice models started out as separate fields, the one dealing with high-energy and the other with condensed matter physics. The two fields merged as they were subsumed within the larger theory of quantum inverse scattering. The level of understanding of such models has deepened as the number of examples studied and applications considered has grown enormously. Heightened interest in this area in the last few years can be explained by a few factors: the number of new, exact results that have been found [1, 2], the appearance of quantum spin-chains in string theory [3], and the use of exact results in the description of experimental, quasi-one-dimensional systems [4].

Two aspects of these systems that have received attention in the recent renaissance are Q-operators and defects. In this paper, we demonstrate that there is a connection between these two types of object. The reason for this connection, expressed in the technical language of the quantum inverse scattering method, is that they are both constructed in terms of the monodromy matrix with an infinite-dimensional auxiliary space. The goals of this paper are to explain this statement to non-experts and to exploit the technology developed in the study of Q-operators to produce a catalogue of results applicable to defects.

Baxter introduced Q-operators as a technical tool in his solution of the 8-vertex model [5]; they allowed him to derive Bethe equations for Hamiltonian eigenvalues in the absence of a Bethe ansatz for the eigenvectors. This use of Q-operators became redundant when Baxter introduced another clever trick [6–8] that enabled him to obtain eigenvectors[1]. The Q-operator then fell into obscurity before resurfacing into the mainstream with the work of Bazhanov, Lukyanov and Zamolodchikov (BLZ). BLZ made use of a Q-operator in their construction of the transfer matrix in quantum field theory [9–12]. Their insight was that the Q-operator was a more fundamental object that the transfer matrix. In fact, the transfer matrix could be constructed in terms of their Q. BLZ constructed the Q-operator by using a monodromy matrix for $U_q\big(\widehat{\mathfrak{sl}}_2\big)$ that involved an auxiliary space that was an infinite-dimensional, q-oscillator representation of a Borel subalgebra. This approach was developed and generalised by various authors [13–16].

The interest in both conformal field theories and massive integrable models has followed a similar route. Bulk models were studied first [17], followed by models with boundaries [18], and finally models with defects [19, 20]. Models with boundaries and defects are of interest for various reasons: physical systems have them; they affect the field content; they increase the richness and complexity of the models; and they are an essential aspect of the branes/string theory paradigm. However, while there is a large body of literature on integrable models with boundaries, much less work has been done on integrable defects.

One motivation for the current work was to develop an algebraic understanding of the work of Bowcock, Corrigan and Zambon (BCZ) on defects [21–24]. BCZ start from a classical Lagrangian density for the sine-Gordon model with an integrable defect inserted at the spatial origin. This defect produces interesting field equations for the sine-Gordon field on the left and right of the defect. Away from the defect these two fields obey independent conventional sine-Gordan equations. At the defect, the two fields are related by a classical Backlund transformation. The authors consider the classical scattering of solitons with this defect. As solitons pass through the defect their topological charge can stay the same, or they can flip from soliton to anti-soliton and deposit topological charge on the defect. The defect thus carries an odd or even integer topological charge. It also carries a rapidity-like parameter. Making use of this classical scattering data, the authors go on to conjecture a corresponding quantum soliton/defect scattering matrix. They also take steps towards producing a defect/defect scattering matrix [23]. That such an objects exist might seem odd, but the defects considered in [23] can move independently and can thus scatter.

The approach of BCZ to finding the quantum soliton/defect scattering matrix is to solve a Yang-Baxter equation of the form $STT = TTS$, where S is the soliton S matrix and T is the desired soliton/defect scattering matrix. In this paper, we compute T in a more general setting by solving a simpler, linear equation. We do this in the time-honoured, quantum inverse scattering way by converting the problem into one of representation theory. In this approach, S becomes the R-matrix which is the intertwiner for the spin-1/2 representation V_ζ of the quantum affine algebra $U_q\big(\widehat{\mathfrak{sl}}_2\big)$:

$$R(\zeta_1/\zeta_2) : V_{\zeta_1} \otimes V_{\zeta_2} \to V_{\zeta_1} \otimes V_{\zeta_2}.$$

The T matrix becomes a special case of a more general intertwiner $\mathcal{L}$ of an infinite-dimensional

[1] Baxter invented and used a mapping of the 8-vertex model to the SOS model.

representation $W^{(\underline{r})}_{\zeta_1}$ and the above V_{ζ_2}:

$$\mathcal{L}(\zeta_1/\zeta_2) : W^{(\underline{r})}_{\zeta_1} \otimes V_{\zeta_2} \to W^{(\underline{r})}_{\zeta_1} \otimes V_{\zeta_2}.$$

This new Borel subalgebra representation $W^{(\underline{r})}_{\zeta}$ is parametrised by a rapidity-like parameter ζ and by a complex vector $\underline{r} = (r_0, r_1, r_2)$, and is a generalisation of the q-oscillator representations that have been used in the construction of the Q-matrix. It is a slightly modified version of the representation introduced in [13]. The Q-matrix would be given as a regularised trace over a product of such $\mathcal{L}$ operators. In the application to defects, it is $\mathcal{L}$ itself that is of interest. Finding $\mathcal{L}$ amounts to solving the linear intertwining condition $\mathcal{L}\Delta(x) = \Delta'(x)\mathcal{L}$, where $\Delta(x)$ is the Borel subalgebra coproduct and $\Delta'(x)$ is defined in Section 2.

In Section 2 of this paper, we consider the required representation theory of $U_q\big(\widehat{\mathfrak{sl}}_2\big)$ and construct the various intertwiners that we will associate with defects. We go on to consider the application to defects in the 6-vertex and XXZ models in Section 3. This connection is of course natural since $R(\zeta)$ specifies the Boltzmann weights of the 6-vertex model. In Section 4, we make the connection with the work of BCZ. A specialisation of $\underline{r}$ for our defect reproduces the scattering results of BCZ. We also consider defect/defect scattering. Finally, we summarise and suggest future work in Section 5.

2 $U_q\big(\widehat{\mathfrak{sl}}_2\big)$ Representation Theory and Defects

In this section, we develop the representation theory that we shall use to describe defects and their interactions. The starting point is the quantum affine algebra $U'_q(\widehat{sl}_2)$, in terms of which both the XXZ R-matrix and sine-Gordon S-matrix may be written. We will not reproduce too many details of this algebra, which can be found in many other places [25]. However, it simplifies the subsequent discussion of the Borel subalgebra representations if we give the relations: the algebra $U'_q(\widehat{sl}_2)$ is an associative algebra over the complex numbers generated by the elements $e_i, f_i, t_i^{\pm 1}$ with relations[2]

$$[e_i, f_j] = \delta_{i,j}\frac{t_i - t_i^{-1}}{q - q^{-1}}, \tag{2.1}$$
$$t_i e_i t_i^{-1} = q^2 e_i, \quad t_i e_j t_i^{-1} = q^{-2} e_j \quad (i \neq j), \tag{2.2}$$
$$t_i f_i t_i^{-1} = q^{-2} f_i, \quad t_i f_j t_i^{-1} = q^2 f_j \quad (i \neq j), \tag{2.3}$$
$$e_i e_j^3 - [3] e_j e_i e_j^2 + [3] e_j^2 e_i e_j - e_j^3 e_i = 0 \quad (i \neq j), \tag{2.4}$$
$$f_i f_j^3 - [3] f_j f_i f_j^2 + [3] f_j^2 f_i f_j - f_j^3 f_i = 0 \quad (i \neq j). \tag{2.5}$$

We use the coproduct $\Delta : U'_q(\widehat{sl}_2) \to U'_q(\widehat{sl}_2) \otimes U'_q(\widehat{sl}_2)$ given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i. \tag{2.6}$$

The Borel subalgebra $U_q(b_+)$ that we consider is the one generated by the elements $e_i, t_i^{\pm 1}$. The only relevant relations from the above are therefore (2.2) and the Serre relation (2.4). In the rest of this section we present representations and the associated intertwiners for both these algebras.

[2]The prime on $U'_q(\widehat{sl}_2)$ indicates that we are not including a derivation in the definition.

2.1 The Generalised Oscillator Algebra

We define a generalised oscillator algebra that we shall use in order to construct $U_q(b_+)$ representations. Let r_1, r_2 be complex numbers. Then, we define the generalised oscillator algebra, $\mathcal{O}^{(r_1,r_2)}$, to be the associative algebra generated by a, a^*, $q^{\pm D}$ with relations

$$q^D a^* q^{-D} = q a^*, \;\; q^D a q^{-D} = q^{-1} a, \;\; a a^* = (r_1 + q^{-2D})(r_2 + q^{2D}), \;\; a^* a = (r_1 + q^{2-2D})(r_2 + q^{2D-2}).$$

Note that we recover the more conventional q-oscillator algebras [26] when either $r_1 = 0$ or $r_2 = 0$, in which cases we have the q-oscillator relations

$$a^* a - q^2 a a^* = (1 - q^2), \;\; \text{or} \quad a a^* - q^2 a^* a = (1 - q^2) \quad \text{respectively.}$$

We consider the $\mathcal{O}^{(r_1,r_2)}$ module $W^{(r_1,r_2)} = \oplus_{j\in\mathbb{Z}} \mathbb{C}|j\rangle$ defined by

$$a|j\rangle = |j-1\rangle, \quad a^*|j\rangle = (r_1 + q^{-2j})(r_2 + q^{2j})|j+1\rangle, \quad q^{\pm D}|j\rangle = q^{\pm j}|j\rangle.$$

2.2 Borel Subalgebra Representations

Let $\underline{r}$ denote the vector $(r_0, r_1, r_2) \in \mathbb{C}^3$. Let $W_\zeta^{(\underline{r})}$ be the infinite-dimensional $U_q(b_+)$ module, spanned by $|j\rangle \otimes \zeta^n \in W^{(r_1,r_2)} \otimes \mathbb{C}[[\zeta, \zeta^{-1}]]$, with $U_q(b_+)$ action

$$e_0(|j\rangle \otimes \zeta^n) = \frac{1}{q - q^{-1}} a^*|j\rangle \otimes \zeta^{n+1}, \quad e_1(|j\rangle \otimes \zeta) = \frac{1}{q - q^{-1}} a|j\rangle \otimes \zeta^{n+1},$$
$$t_1(|j\rangle \otimes \zeta^n) = r_0\, q^{-2D}|j\rangle \otimes \zeta^n, \quad t_0(|j\rangle \otimes \zeta^n) = r_0^{-1} q^{2D}|j\rangle \otimes \zeta^n.$$

The module $W_\zeta^{(\underline{r})}$ is a convenient reparametrisation of the $U_q(b_+)$ module first introduced, and shown to be the most general solution of the relations (2.2) and (2.4), in the paper [13].

2.2.1 Special cases

In the special case when either $r_1 = -q^{-2n}$ or $r_2 = -q^{2n}$ $(n \in \mathbb{Z})$, we have $a^*|n\rangle = 0$, and the $U_q(b_+)$ module is modified to $(\oplus_{j\le n} \mathbb{C}|j\rangle) \otimes \mathbb{C}[[\zeta, \zeta^{-1}]]$. Also, in the root of unity case $q^{2N} = 1$ the module may be truncated to $(\oplus_{j=0}^{N-1} \mathbb{C}|j\rangle) \otimes \mathbb{C}[[\zeta, \zeta^{-1}]]$.

2.3 Evaluation Representations

We will make use of the $U_q'(\widehat{sl}_2)$ principal evaluation module $V_\zeta = (\mathbb{C}v_+ \oplus \mathbb{C}v_-) \otimes \mathbb{C}[[\zeta, \zeta^{-1}]]$ defined by

$$e_0(v_+ \otimes \zeta^n) = (v_- \otimes \zeta^{n+1}), \quad e_1(v_- \otimes \zeta^n) = (v_+ \otimes \zeta^{n+1}), \quad e_0(v_- \otimes \zeta^n) = 0, \quad e_1(v_+ \otimes \zeta^n) = 0$$
$$f_0(v_- \otimes \zeta^n) = (v_+ \otimes \zeta^{n-1}), \quad f_1(v_+ \otimes \zeta^n) = (v_- \otimes \zeta^{n-1}), \quad f_0(v_+ \otimes \zeta^n) = 0, \quad f_1(v_- \otimes \zeta^n) = 0,$$
$$t_0(v_\pm \otimes \zeta^n) = q^{\mp 1}(v_+ \otimes \zeta^n), \quad t_1(v_\pm \otimes \zeta^n) = q^{\pm 1}(v_+ \otimes \zeta^n).$$

The R-matrix $R(\zeta_1/\zeta_2) : V_{\zeta_1} \otimes V_{\zeta_2} \to V_{\zeta_1} \otimes V_{\zeta_2}$ that obeys $R(\zeta) \circ \Delta(x) = \Delta'(x) \circ R(\zeta)$ for all $x \in U_q'(\widehat{sl}_2)$ is given by[3]

$$R(\zeta) = \frac{1}{\kappa(\zeta)}\bar{R}(\zeta), \quad \bar{R}(\zeta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & 0 \\ 0 & \frac{(1-q^2)\zeta}{1-q^2\zeta^2} & \frac{(1-\zeta^2)q}{1-q^2\zeta^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

We do not give the explicit expression for $\kappa(\zeta)$ in this paper, but this, and the crossing and unitarity properties of $R(\zeta)$ that $\kappa(\zeta)$ ensures, can be found in Appendix A of [1].

In order to understand the properties of $R(\zeta)$ and other intertwiners it is very useful to work with pictures; we represent $R(\zeta)$ as shown in Figure 1. The arrows indicate that the operator acts

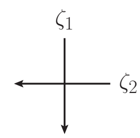

Figure 1: The Operator $R(\zeta_1/\zeta_2)$

from the North-East to the South-West.

2.4 The L-operator

We now define a new object: a $U_q(b_+)$ intertwiner $\mathcal{L}^{(\underline{r})}(\zeta_1/\zeta_2) : W^{(\underline{r})}_{\zeta_1} \otimes V_{\zeta_2} \to W^{(\underline{r})}_{\zeta_1} \otimes V_{\zeta_2}$ that obeys

$$\mathcal{L}(\zeta) \circ \Delta(x) = \Delta'(x) \circ \mathcal{L}(\zeta) \tag{2.8}$$

for all $x \in U_q(b_+)$. Solving this linear equation, we find (with a particular choice of normalisation) the expression

$$\mathcal{L}^{(\underline{r})}(\zeta, q) = \begin{pmatrix} (1 + r_2\zeta^2 q^{2-2D}) & -\zeta a^* \\ -\zeta a & (1 + r_1\zeta^2 q^{2D}) \end{pmatrix} \begin{pmatrix} q^D & 0 \\ 0 & r_0\, q^{-D} \end{pmatrix}. \tag{2.9}$$

In writing (2.9) in the form shown, and in finding the intertwiner $\mathcal{R}$ in Section 2.5, we have been led by the approach of the appendices of the papers [27, 28] that deal with the q-oscillator cases. Except when we specifically require it, we will suppress both the $\underline{r}$ superscript and q argument of $\mathcal{L}^{(\underline{r})}(\zeta, q)$, and write it as $\mathcal{L}(\zeta)$. We represent the infinite-dimensional module $W^{(\underline{r})}(\zeta_1/\zeta_2)$ by a dashed line, and depict $\mathcal{L}(\zeta_1/\zeta_2)$ by Figure 2.

2.4.1 Properties of $\mathcal{L}(\zeta)$

We have, by construction, the Yang-Baxter relation

$$R_{2,3}(\zeta_2/\zeta_3)\mathcal{L}_{1,3}(\zeta_1/\zeta_3)\mathcal{L}_{1,2}(\zeta_1/\zeta_2) = \mathcal{L}_{1,2}(\zeta_1/\zeta_2)\mathcal{L}_{1,3}(\zeta_1/\zeta_3)R_{2,3}(\zeta_2/\zeta_3), \tag{2.10}$$

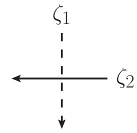

Figure 2: The Operator $\mathcal{L}(\zeta_1/\zeta_2)$

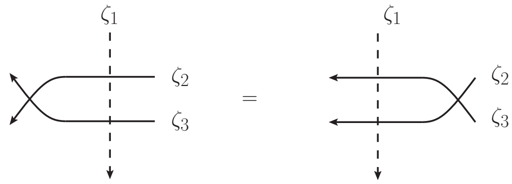

Figure 3: The Yang-Baxter relation on $W^{(\underline{r})}(\zeta_1) \otimes V_{\zeta_2} \otimes V_{\zeta_3}$

where both sides act on the space $W^{(\underline{r})}(\zeta_1) \otimes V_{\zeta_2} \otimes V_{\zeta_3}$. The relation is indicated by Figure 3.

We find that the inverse operator is given by

$$\mathcal{L}^{-1}(\zeta) = \frac{1}{(1-\zeta^2)(1-\zeta^2 r_1 r_2)} \begin{pmatrix} q^{-D} & 0 \\ 0 & r_0^{-1} q^D \end{pmatrix} \begin{pmatrix} (1 + r_1 \zeta^2 q^{2D-2}) & \zeta a^* \\ \zeta a & (1 + r_2 \zeta^2 q^{-2D}) \end{pmatrix}.$$

The operator $\mathcal{L}(\zeta)$ also obeys the crossing relation

$$\mathcal{L}^{-1}(-\zeta q) = \frac{1}{r_0(1-\zeta^2 q^2)(1-\zeta^2 q^2 r_1 r_2)} (\sigma^x \mathcal{L}(\zeta) \sigma^x)^{t_2}$$

where σ^x is the Pauli matrix, and t_2 indicates transpose with respect to the two dimensional space.

In the introduction, we mentioned the connection of our work with Q-operators. The substance of this connection is that we can identify a Q operator as a suitably regularised trace of $\mathcal{L}(\zeta)$ over the space $W_\zeta^{(\underline{r})}$ [13, 14]. The resulting operator obeys the $TQ = Q + Q$ Baxter relation as a result of the fusion properties that of $W^{(\underline{r})}(\zeta)$ and V_ζ that we will now discuss. It can be shown, following the method of [13], that we have the short exact sequence

$$0 \longrightarrow W_{\zeta q}^{(q r_0, q^{-2} r_1, r_2)} \stackrel{\iota}{\longrightarrow} W_\zeta^{(r_0, r_1, r_2)} \otimes V_\zeta \stackrel{\pi}{\longrightarrow} W_{\zeta q^{-1}}^{(q^{-1} r_0, q^2 r_1, r_2)} \longrightarrow 0.$$

This constitutes a concise statement of the fact that $W_{\zeta q}^{(q r_0, q^{-2} r_1, r_2)}$ is a submodule of $W_\zeta^{(r_0, r_1, r_2)} \otimes V_\zeta$ and that there is an isomorphism $W_\zeta^{(r_0, r_1, r_2)} \otimes V_\zeta / W_{\zeta q}^{(q r_0, q^{-2} r_1, r_2)} \simeq W_{\zeta q^{-1}}^{(q^{-1} r_0, q^2 r_1, r_2)}$. The embedding ι and projection π are given explicitly by

$$\begin{aligned} \iota : W_{\zeta q}^{(q r_0, q^{-2} r_1, r_2)} &\rightarrow W_\zeta^{(r_0, r_1, r_2)} \otimes V_\zeta \\ |j\rangle &\mapsto A_j := r_0 (q^{j-1} r_1 + q^{1-j}) |j\rangle \otimes v_+ + q^j |j-1\rangle \otimes v_-, \quad \text{and,} \end{aligned}$$

$$\begin{aligned} \pi : W_\zeta^{(r_0, r_1, r_2)} \otimes V_\zeta &\rightarrow W_{\zeta q^{-1}}^{(q^{-1} r_0, q^2 r_1, r_2)} \\ |j\rangle \otimes v_+ &\mapsto q^j |j-1\rangle, \\ A_j &\mapsto 0. \end{aligned}$$

[3] If $\Delta(x) = \sum_i a_i \otimes b_i$, then $\Delta'(x) = \sum_i b_i \otimes a_i$.

A consequence of this short exact sequence is that the $\mathcal{L}$ operator for $W_{\zeta q^{\pm 1}}^{(q^{\pm 1} r_0, q^{\mp 2} r_1, r_2)}$ is related to that of $W_\zeta^{(r_0,r_1,r_2)}$ via a fusion relation. With our choice of normalisation, we find

$$\begin{aligned}(\iota \otimes 1)\,\mathcal{L}^{(qr_0,q^{-2}r_1,r_2)}(\zeta q) &= \frac{(1-q^2\zeta^2)}{(1-\zeta^2)}\mathcal{L}_{1,3}^{(r_0,r_1,r_2)}(\zeta)\bar{R}_{2,3}(\zeta)(\iota\otimes 1),\\ \mathcal{L}^{(q^{-1}r_0,q^2 r_1,r_2)}(\zeta q^{-1})(\pi\otimes 1) &= q^{-1}(\pi\otimes 1)\mathcal{L}_{1,3}^{(r_0,r_1,r_2)}(\zeta)\bar{R}_{2,3}(\zeta),\end{aligned}$$

where the first relation acts on $W_{\zeta_1 q}^{(qr_0,q^{-2}r_1,r_2)} \otimes V_{\zeta_2}$, the second acts on $W_{\zeta_1}^{(r_0,r_1,r_2)} \otimes V_{\zeta_1} \otimes V_{\zeta_2}$, and $\zeta = \zeta_1/\zeta_2$. We include these relations because of the potential application in the study of soliton/defect fusion.

2.5 Intertwiners of Generalised Oscillator Representation

In this subsection, we consider intertwiners of the generalised oscillator representations of $U_q(b_+)$. That is, we look for an intertwiner $\mathcal{R}^{(\underline{r})}(\zeta_1/\zeta_2) : W_{\zeta_1}^{(\underline{r})} \otimes W_{\zeta_2}^{(\underline{r})} \to W_{\zeta_1}^{(\underline{r})} \otimes W_{\zeta_2}^{(\underline{r})}$ that obeys the condition

$$\mathcal{R}_{1,2}^{(\underline{r})}(\zeta_1/\zeta_2)\mathcal{L}_1^{(\underline{r})}(\zeta_1)\mathcal{L}_2^{(\underline{r})}(\zeta_2) = \mathcal{L}_2^{(\underline{r})}(\zeta_2)\mathcal{L}_1^{(\underline{r})}(\zeta_1)\mathcal{R}_{1,2}^{(\underline{r})}(\zeta_1/\zeta_2), \tag{2.11}$$

where 1 and 2 denote the first and second components, and $\mathcal{L}_1^{(\underline{r})}(\zeta_1)\mathcal{L}_2^{(\underline{r})}(\zeta_2)$ indicates 2×2 matrix multiplication of the $\mathcal{L}$-matrices. We have for example that

$$\left(\mathcal{L}_1^{(\underline{r})}(\zeta_1)\mathcal{L}_2^{(\underline{r})}(\zeta_2)\right)_{+,+} = (q^{D_1} + r_2\zeta_1^2 q^{2-D_1})(q^{D_2} + r_2\zeta_2^2 q^{2-D_2}) + \zeta_1\zeta_2 r_0 a_1^* q^{-D_1} a_2 q^{D_2}.$$

Pictorially, we have $\mathcal{R}(\zeta_1/\zeta_2)$ and the relation (2.11) given by Figures 4 and 5.

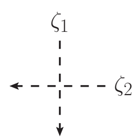

Figure 4: The Intertwiner $\mathcal{R}(\zeta_1/\zeta_2)$

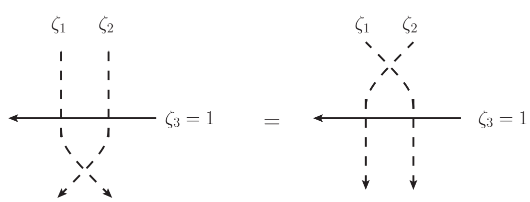

Figure 5: The Representation of Equation (2.11)

We do not have a solution of (2.11) for $\mathcal{R}^{(\underline{r})}(\zeta)$ for generic $\underline{r}$, but we do have formal solutions when either $r_1 = 0$ or $r_2 = 0$. We obtain these by extending the method of [27], which deals with this problem in the $r_1 = -1$, $r_2 = 0$ case. We make use of the the exchange matrix $P(|j\rangle \otimes |k\rangle) =$

$(|k\rangle \otimes |j\rangle)$. Following the approach of Appendix A of [27], we write the intertwining matrices in the form

$$\mathcal{R}^{(r_0,r_1,0)}(\zeta) = P\, h(\zeta, r_0\, u)\, \zeta^{D_1+D_2}, \quad R^{(r_0,0,r_2)}(\zeta) = P\, h(\zeta, r_0^{-1} u')\, \zeta^{-(D_1+D_2)} \tag{2.12}$$

where $u = a_1^* q^{-2D_1} a_2$, $u' = a_1 q^{2D_1} a_2^*$, and $h(\zeta, v)$ is a formal series in v. In both the $r_1 = 0, r_2 \neq 0$ and $r_1 \neq 0, r_2 = 0$ cases, Equation (2.11) then reduces to the following single condition for the formal series:

$$h(\zeta, v)(1 + \zeta v) \quad = \quad h(\zeta, q^2 v)(1 + \zeta^{-1} v). \tag{2.13}$$

If we assume that $h(\zeta, 0) = 1$, then Equation (2.13) has the unique solution

$$h(\zeta, v) \quad = \quad \sum_{n \geq 0} (-q^{-1} v)^n \prod_{m=1}^{n} \frac{(\zeta^{-1} q^{m-1} - \zeta q^{1-m})}{q^m - q^{-m}}.$$

Using the q-binomial theorem, this can be rewritten for $|v\zeta| < 1$, and $|q| < 1$ as[4]

$$h(\zeta, v) = \frac{(-v\zeta^{-1}; q^2)_\infty}{(-v\zeta; q^2)_\infty}.$$

We have solved (2.11) to obtain $\mathcal{R}^{(\underline{r})}(\zeta)$ for the most general $\underline{r}$ cases that we can. The expense of this generality is that the solutions (2.12) are formal series in u or u'. However, in the special cases $r_1 = -q^{-2n}$ or $r_2 = -q^{2n}$ discussed in Section 2.2.1, the operators u and u' are nilpotent and the series are well defined. In the more restrictive case $r_1 = 0$, $r_2 = 0$, to be considered in connection with the work of BCZ in our Section 4, we have two well-defined, independent solutions given by (2.12).

3 Defects in the 6-Vertex and XXZ Models

The components of the R-matrix (2.7) specify the vertex weights of the 6-vertex model. We can use the $\mathcal{L}(\zeta)$ operator to define a new Boltzmann weight, that is we can define components of $\mathcal{L}(\zeta)$ by

$$\mathcal{L}(\zeta)(|j\rangle \otimes v_\varepsilon) = \sum_{j',\varepsilon'} \mathcal{L}(\zeta)_{j',\varepsilon'}^{j,\varepsilon} (|j'\rangle \otimes v_{\varepsilon'}),$$

where $\varepsilon, \varepsilon' \in \{+1, -1\}$, and $j \in \mathbb{Z}$ (at least in the generic $\underline{r}$ case). Note that the only non-zero contribution to the above sum comes from $j' = j + (\varepsilon' - \varepsilon)/2$. These components may then be associated with the Boltzmann weight for the edge configuration of Figure 6. We can use this weight to introduce a defect line into the 6-vertex model as shown in Figure 7. The simplest scenario is to assume an $N \times N$ finite lattice with periodic boundary conditions.

We can consider an anisotropic 6-vertex model with a defect with different ζ 'spectral' parameters on different vertical and horizontal lines. In fact, in order to produce a simple quantum

[4]The infinite product is defined as $(a; b)_\infty = \prod_{n=0}^{\infty} (1 - a\, b^n)$.

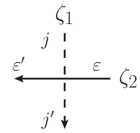

Figure 6: The Boltzmann Weight $\mathcal{L}(\zeta_1/\zeta_2)_{j',\varepsilon'}^{j,\varepsilon}$

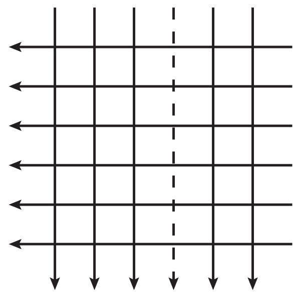

Figure 7: The 6-vertex model with a defect line

spin-chain Hamiltonian, it is convenient to consider the case when all horizontal lines have spectral parameter 1, all vertical lines with the exception of the defect line have spectral parameter ζ, and the defect line itself has a spectral parameter $\zeta - 1$. The horizontal transfer matrix associated with this choice is

$$T(\zeta) = \mathrm{Tr}_{\mathbb{C}^2}\big(R_{1,0}(\zeta)R_{2,0}(\zeta)\cdots R_{j-1,0}(\zeta)\mathcal{L}^{(r)}_{j,0}(\zeta-1)R_{j+1,0}(\zeta)\cdots R_{N-1,0}(\zeta)R_{N,0}(\zeta)\big).$$

This operator acts on the space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes W^{(r_1,r_2)} \otimes \mathbb{C}^2 \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and the trace is over the two dimensional horizontal auxiliary space labelled by '0'. This transfer matrix is represented in Figure 8. The solvability/integrability property $[T(\zeta), T(\zeta')] = 0$ follows from the Yang-Baxter equation for R and from Equation (2.10).

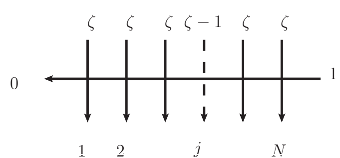

Figure 8: The transfer matrix $T(\zeta)$ for the XXZ model with a defect

The XXZ Hamiltonian with a defect is given in terms of the logarithmic derivative of $T(\zeta)$ at $\zeta = 1$. One of the goals of this paper is to communicate the mysteries of the quantum inverse scattering method to a non-expert readership, and so rather than just writing down the Hamiltonian, we will give a derivation, emphasising the simplifying role of pictures. The starting point is to note that $R(1) = P$, the permutation operator[5]. This is represented graphically by Figure 9. We also note that $\mathcal{L}^{-1}(\zeta_1/\zeta_2)$ can be represented by Figure 10 with the relation $\mathcal{L}^{-1}(\zeta_1/\zeta_2)\mathcal{L}(\zeta_1/\zeta_2) = 1$ represented by Figure 11.

Hence, the relation $T(1)^{-1}T(1) = 1$ can be represented by Figure 12. The Hamiltonian however

[5]The normalisation function $\kappa(\zeta)$ has the property $\kappa(1) = 1$.

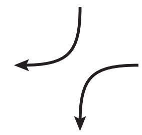

Figure 9: Representation of $R(1) = P$

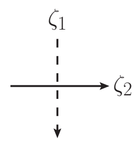

Figure 10: Representation of $\mathcal{L}^{-1}(\zeta_1/\zeta_2)$

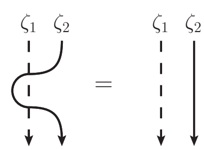

Figure 11: Representation of $\mathcal{L}^{-1}(\zeta_1/\zeta_2)\mathcal{L}(\zeta_1/\zeta_2) = 1$

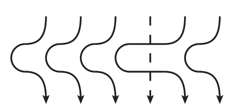

Figure 12: Representation of $T(\zeta^{-1})T(\zeta) = 1$

is given in terms of $\frac{d\ln(T(\zeta))}{d\zeta}\big|_{\zeta=1} = T^{-1}(\zeta)T'(\zeta)\big|_{\zeta=1}$. Let us represent $R'(\zeta = 1)$ and $\mathcal{L}'(\zeta = 0)$ by the vertices with black bullets as shown in Figure 13. Then finally we arrived at the pictorial

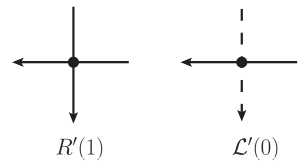

Figure 13: Representation of $R'(\zeta = 1)$ and $\mathcal{L}'(\zeta = 0)$

representation of $\frac{d\ln(T(\zeta))}{d\zeta}\big|_{\zeta=1}$ shown in Figure 14.

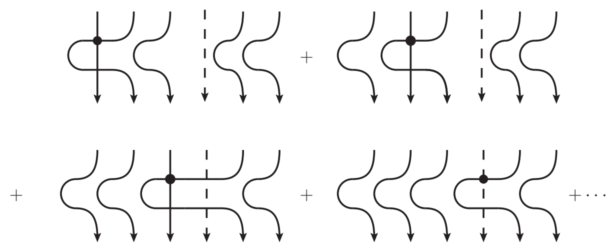

Figure 14: Representation of $\frac{d\ln(T(\zeta))}{d\zeta}\big|_{\zeta=1}$

We can then essentially read off the Hamiltonian from Figure 14 after noting the explicit form of $R'(1)$ and $\mathcal{L}'(0)$. We have

$$\frac{d\ln(R(\zeta))}{d\zeta}\Big|_{\zeta=1} = -\kappa'(1) + \frac{q\Delta}{1-q^2} + \frac{2q}{1-q^2}h,$$

$$\text{where} \qquad h := -\frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta\, \sigma^z \otimes \sigma^z) = \begin{pmatrix} -\frac{\Delta}{2} & & & \\ & \frac{\Delta}{2} & -1 & \\ & -1 & \frac{\Delta}{2} & \\ & & & -\frac{\Delta}{2} \end{pmatrix},$$

$$\text{and} \qquad \frac{d\ln(\mathcal{L}^{(\underline{r})}(\zeta))}{d\zeta}\Big|_{\zeta=0} = -\begin{pmatrix} 0 & r_0 a^*\, q^{-1-2D} \\ r_o^{-1} a\, q^{2D-1} & 0 \end{pmatrix}.$$

If we define $H = \frac{1-q^2}{2q}\frac{d\ln(T(\zeta)}{d\zeta}\big|_{\zeta=1}$, we have the integrable Hamiltonian for the XXZ model with a defect:

$$H = const + \sum_{i\neq j-1,j} h_{i,i+1} + \begin{pmatrix} q^{-D} & 0 \\ 0 & r_0^{-1} q^D \end{pmatrix}_{j,j+1} h_{j-1,j+1} \begin{pmatrix} q^D & 0 \\ 0 & r_0 q^{-D} \end{pmatrix}_{j,j+1} + \frac{q-q^{-1}}{2q} \begin{pmatrix} 0 & r_0 a^*\, q^{-2D} \\ r_0^{-1} a\, q^{2D} & 0 \end{pmatrix}_{j,j+1}.$$

Note, that the Hamiltonian for the XXZ model without a defect is simply $\sum_i h_{i,i+1}$.

4 Defects in the Sine-Gordon Model

Bowcock, Corrigan, and Zambon have considered defects in the sine-Gordan and other affine-Toda models is a series of papers [21–24]. Their starting point is a classical sine-Gordon Lagrangian with a delta function term localised at the spatial origin. They match solitonic solutions on either side of the defect, and interpret the results as classical, reflectionless, soliton/defect scattering. They go on to consider quantum scattering, and follow the S-matrix programme of solving quantum Yang-Baxter-like equations and finding a solution that is consistent with the classical data. Schematically, if S represents the known soliton/soliton scattering matrix and T the soliton/defect transmission matrix, then BCZ solve a Yang-Baxter equation of the form $STT = TTS$ to find T. This is a difficult quadratic relation to solve, even for the relatively simple scattering process considered in [23]. In Section 3 of this paper, we have presented an algebraic scheme. In this section, we will identify S with our R and the soliton/defect scattering matrix with our $\mathcal{L}$. Rather than solving (2.10) directly, we have had the luxury of solving the much simpler relation (2.8). The resulting $\mathcal{L}$ is a more complicated object which carries extra parameters, but, as we shall show, it coincides with the T of BCZ under a certain specialisation.

Let us first summarise the results of BCZ concerning the sine-Gordon model[6]. Their soliton S-matrix is [23]

$$S(\theta,q)=\rho(\theta)\begin{pmatrix} qx^{-1}-xq^{-1} & & & \\ & q-q^{-1} & x-x^{-1} & \\ & x-x^{-1} & q-q^{-1} & \\ & & & qx^{-1}-xq^{-1}\end{pmatrix},\quad x=e^{\gamma\theta},\quad q=e^{i\pi\gamma},$$

where θ is the rapidity and γ is related to the conventional sine-Gordon coupling constant by $\gamma=\frac{8\pi-\beta^2}{\beta^2}$. Defects are characterised by a continuous parameter η and by an integer (denoted by lower case Greek letters α or β). The conjectured soliton/defect transmission matrix for a soliton or antisoliton with rapidity θ colliding with a defect labelled by (α,η) and going to a soliton or antisoliton with the same rapidity and a defect labelled by (β,η) is [23]

$$T(\theta,\eta,q)_\alpha^\beta=f(q,x)\begin{pmatrix}\nu^{-\frac{1}{2}}Q^\alpha\delta_\alpha^\beta & q^{-\frac{1}{2}}e^{\gamma(\theta-\eta)}\delta_\alpha^{\beta-2}\\ q^{-\frac{1}{2}}e^{\gamma(\theta-\eta)}\delta_\alpha^{\beta+2} & \nu^{\frac{1}{2}}Q^{-\alpha}\delta_\alpha^\beta\end{pmatrix},\quad Q^2=-q.$$

The identification of the results of BCZ and the objects of Section 3 of this paper is relatively straightforward. First of all, we identify

$$\frac{1}{(qx^{-1}-q^{-1}x)\rho(\theta)}S(\theta,q)=P\bar{R}(x^{-1},-q). \tag{4.14}$$

Then, letting

$$U(r_0,q)=\begin{pmatrix}1 & \\ & r_0^{\frac{1}{2}}q^{-D-\frac{1}{2}}\end{pmatrix},$$

we define a gauge-transformed $\mathcal{L}$ operator

$$\widetilde{\mathcal{L}}^{(\underline{r})}(\zeta,q)=r_0^{-\frac{1}{2}}U(r_0,q)\mathcal{L}^{(\underline{r})}(\zeta,q)U^{-1}(r_0,q)=\begin{pmatrix}r_0^{-\frac{1}{2}}(q^D+r_2\zeta^2q^{2-D}) & -\zeta q^{\frac{1}{2}}a^* \\ -\zeta q^{\frac{1}{2}}a & r_0^{\frac{1}{2}}(q^{-D}+r_1\zeta^2q^D)\end{pmatrix}.$$

[6] Corrigan and Zambon have recently considered a more complicated defect [29]; we consider only the simple case of [23].

This gauge transformation does not effect the $R\mathcal{L}\mathcal{L} = \mathcal{L}\mathcal{L}R$ relations of Equation (2.10), i.e., we have the same relations with the same R but $\mathcal{L} \to \widetilde{\mathcal{L}}$. We can then identify

$$\frac{1}{f(q,x)} T(\theta,\eta,q) = \widetilde{\mathcal{L}}^{(r_0=\nu, r_1=0, r_2=0)}(ie^{\gamma(\theta-\eta-i\pi)}, -q). \tag{4.15}$$

Note that we also identify $\alpha = 2j$. Hence, the $STT = TTS$ soliton-defect transmission-matrix relations of [23] follow immediately from our Equation (2.10).

The defect/defect scattering matrix U is defined in [23] via an equation of the for $UTT = TTU$. In our picture U corresponds to the intertwiner $\mathcal{R}$ satisfying (2.11). It may be read off from (2.12) in the special case $r_1 = r_2 = 0$ corresponding to the identification (4.15). As mentioned in Section 2, (4.15) gives have two independent solutions in this special case. Further physical requirements will be needed in order to pin down the defect/defect scattering matrix, but our method clearly provides a set of candidates from which such a scattering matrix may be selected.

5 Discussion

In this paper, we have developed the representation theory of generalised oscillator algebras in the $U_q(\widehat{\mathfrak{sl}}_2)$ case. We have produced $\mathcal{L}$ and $\mathcal{R}$ operators, and demonstrated their use in the theory of lattice and sine-Gordon defects. We have made the connection between our $\mathcal{L}$ operator under the $r_1 = r_2 = 0$ specialisation and the work of BCZ on quantum defects in the sine-Gordan model. It would be interesting to attempt to find a classical Lagrangian description of a sine-Gordan defect related to our more general $\mathcal{L}$. Corrigan and Zambon have indeed recently considered a classical sine-Gordon defect with an extra parameter [29]. It is possible that the as yet unknown quantum soliton/defect scattering matrix in this case may be of our more general class. It is also possible that it may correspond to a tensor product of generalised oscillator algebra representations. This possibility is suggested by the observation in [29] that the classical soliton/defect scattering in the extra-parameter case is related to that of a soliton with two simple defects. Tensor products of q-oscillator representations have already been considered in the Q-matrix context in [10,14] and [27], where they lead to relations of the symbolic form $T = QQ + QQ$. We encourage experts in field theory to make use of such representation theory constructions in order to simplify the development of new results in the theory of defects.

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