Bethe Ansatz Results for the Partially Asymmetric Exclusion Process with Open Boundaries

Fabian Essler (Oxford)

Jan de Gier (Melbourne)

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Outline

A. Introduction

Stochastic processes/Definition of the PASEP/Mapping to non-Hermitian Quantum Spin Chains/Stationary State Phase Diagram

B. Relaxation Rates at Late Times

Definition/Bethe Ansatz/"Dynamical" Phase Diagram

C. Current Fluctuations

Setup/Bethe Ansatz/Results for "gapped" phases

Master Equation for classical non-equilibrium systems

C = configurations of a system of classical particles

P(C,t) = prob. for system to be in configuration C at time t

Time evolution described by Master Equation:

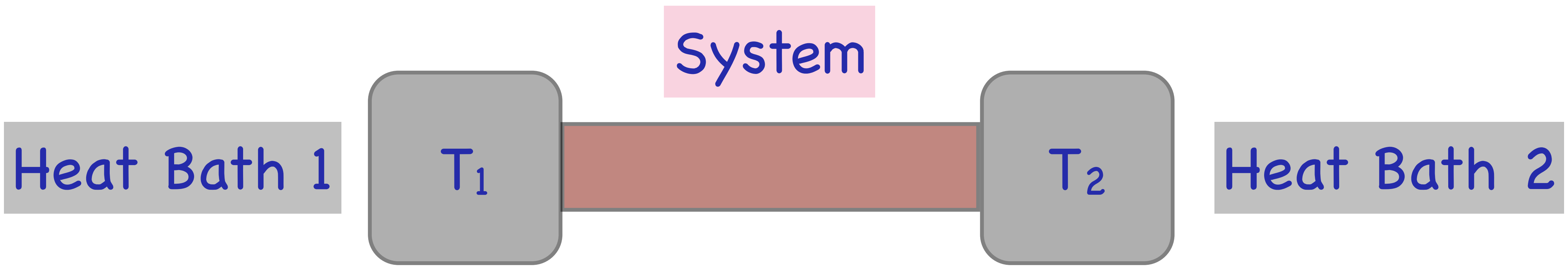
$$P(\mathbf{C}, t+dt) = P(\mathbf{C}, t) + \left[\sum_{\mathbf{C}' \neq \mathbf{C}} M_{\mathbf{C}' \to \mathbf{C}} \, P(\mathbf{C}', t) - M_{\mathbf{C} \to \mathbf{C}'} \, P(\mathbf{C}, t) \right] dt.$$

$M_{\mathbf{C} \to \mathbf{C}'}$ "Transition rates" from C to C'

Given P(C,t) one can calculate average of observable O by

$$\langle \mathcal{O} \rangle = \sum_{\mathbf{C}} P(\mathbf{C}, t) \mathcal{O}(\mathbf{C}).$$

Non-Equilibrium Steady States



Equilibrium ($T_1=T_2$): $P_{eq}(C) = Z^{-1}\, e^{-\beta E(C)}$

Non-equilibrium ($T_1 \neq T_2$):

$$\lim_{t\to\infty} P(\mathbf{C},t) = P_{\mathrm{stat}}(\mathbf{C}) \ , \qquad \frac{dP_{\mathrm{stat}}(\mathbf{C})}{dt} = 0.$$

$P_{stat}(C)$ = ??

Some Questions to ask:

- What is $P_{stat}(C)$ for a given system?
- Calculate stationary averages $\langle \mathcal{O} \rangle_{\mathrm{stat}} = \sum_{\mathbf{C}} P_{\mathrm{stat}}(\mathbf{C}) \mathcal{O}(\mathbf{C})$.
- Late time behaviour: how does $P(C,t)$ approach $P_{stat}(C)$?
- Probability distributions of observables at late times ?

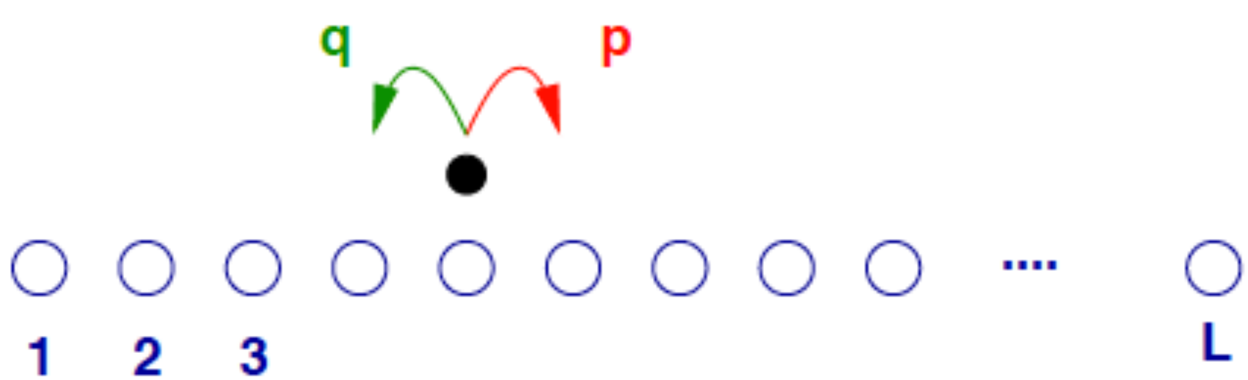
Definition of the PASEP

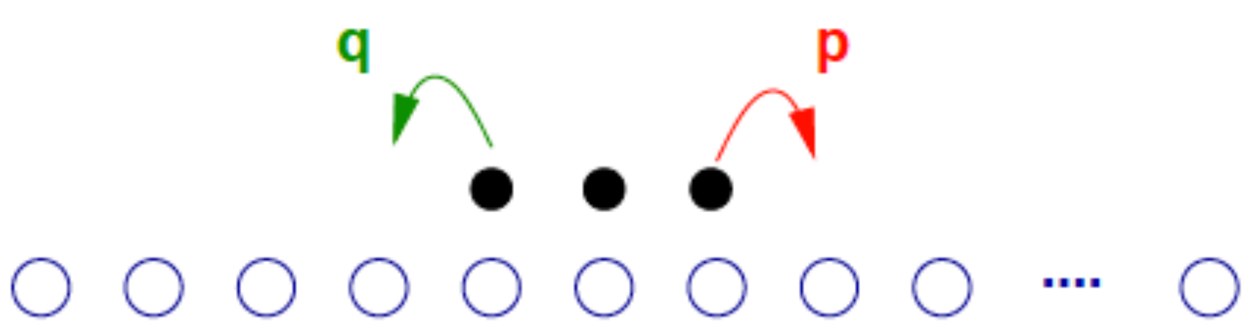
Hard core particles hopping along 1D lattice of L sites.

site j $\begin{cases} \text{empty} \\ \text{occupied} \end{cases}$ $\Rightarrow$ $\tau_j = \begin{cases} 0 \\ 1 \end{cases}$

2^L configurations $C=(\tau_{1,\ldots,}\ \tau_L)$; Prob. distr. $P(\tau_{1,\ldots,}\ \tau_L,t)$

Transition Rates:

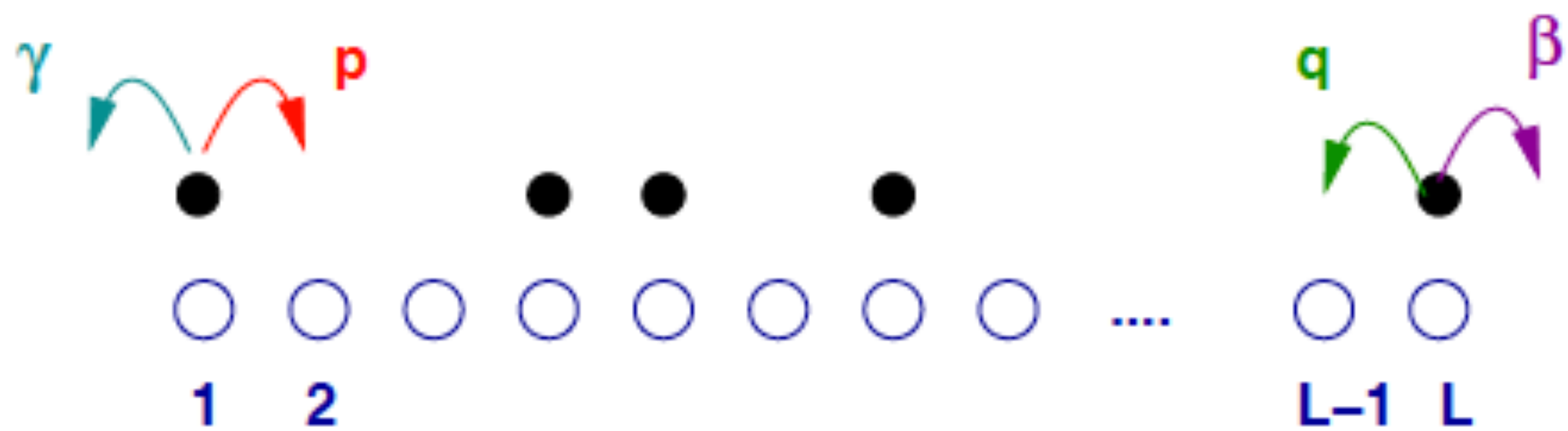




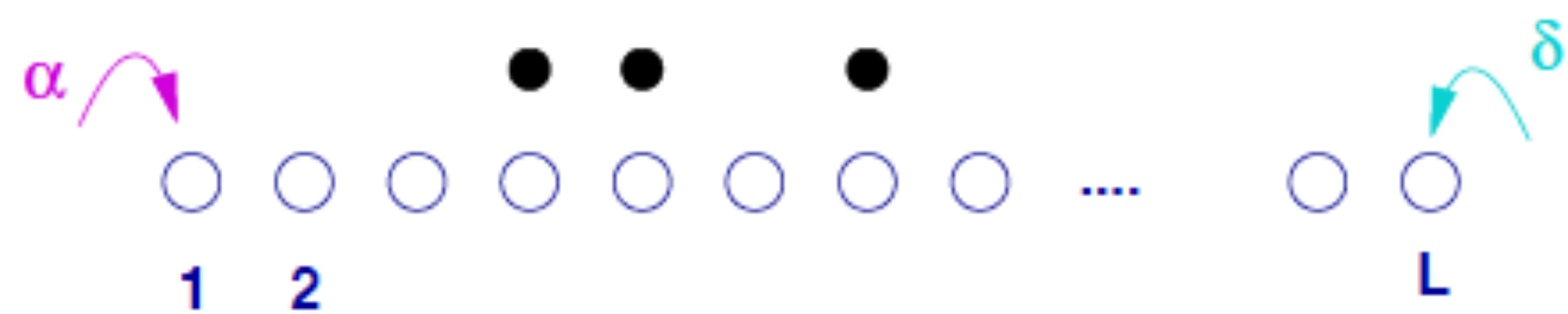
Other moves blocked: "Exclusion"

PASEP with open boundaries

Particle Extraction:



Particle Injection:



Particle number not conserved!

Special case ("TASEP"): $\gamma=\delta=q=0$.

Mapping to non-Hermitian Quantum Spin Chain

(Alcaraz, Droz, Henkel & Rittenberg '94)

- 2^L dim Hilbert space with basis states $|\tau_{1,\ldots,}\tau_L\rangle$
- Probability distr. → State $|P(t)\rangle = \sum_{\tau_j} P(\tau_1,\ldots,\tau_L,t)|\tau_1,\ldots,\tau_L\rangle.$
- Master eqn. → imaginary time Schrödinger eqn

$$\frac{\partial |P(t)\rangle}{\partial t} = M|P(t)\rangle$$

M = non-Hermitian "Hamiltonian"

- left eigenstates different from right eigenstates
- stationary state: eigenstate with eigenvalue $E_0=0$:

left: $\langle 0| = \sum_{\tau_j} \langle \tau_1,\ldots,\tau_L|.$

- Averages:

$$\langle \mathcal{O}(t)\rangle = \sum_{\{\tau_j\}} \mathcal{O}(\tau_1,\ldots,\tau_L)P(\tau_1,\ldots,\tau_L) = \langle 0|\widehat{\mathcal{O}}|P(t)\rangle$$

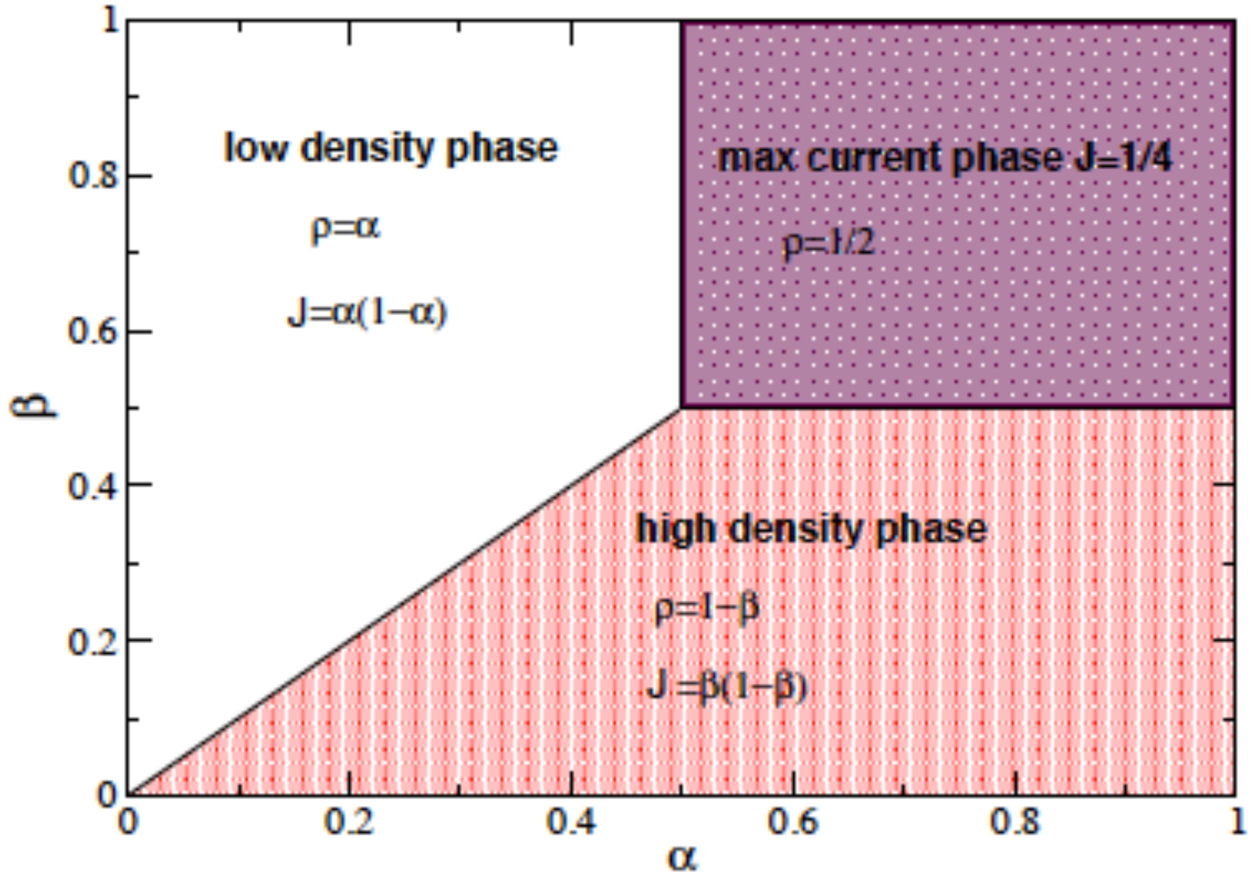
≠QM!

Stationary State Properties of the TASEP

(Derrida, Evans, Hakim & Pasquier '93; Schütz/Domany '93)

- Can be worked out by matrix product state techniques
- Several Phases depending on boundary rates α,β

$\Rightarrow$ "boundary-induced phase transitions"



PASEP: Sandov '95, Essler/Rittenberg '96, Sasamoto '99 Blythe et al '00

Approach to Stationarity

Master Eqn: $$\frac{\partial |P(t)\rangle}{\partial t} = M|P(t)\rangle$$

M diagonalizable $\Rightarrow$ **M** $|P_n\rangle = -E_n |P_n\rangle$, $\mathrm{Re}(E_n) \geq 0$.

Eigenvalues with smallest real parts → approach to stationarity

$$\langle \widehat{\mathcal{O}}(t) \rangle = \langle 0 | \widehat{\mathcal{O}} | P(t) \rangle = \sum_n e^{-E_n t} \langle 0 | \widehat{\mathcal{O}} | P_n \rangle \langle P_n | P(0) \rangle$$

→ **Calculate "leading" E_n**

Relation to spin-1/2 Heisenberg XXZ "Ferromagnet"

Sandow '95, Essler/Rittenberg '96

Similarity Transformation: $\widehat{M} = \sqrt{pq}\, U_\lambda \widehat{H} U_\lambda^{-1}$

$$\widehat{H} = -\frac{1}{2}\sum_{j=1}^{L-1}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \Delta\sigma_j^z\sigma_{j+1}^z + h(\sigma_{j+1}^z - \sigma_j^z) + \Delta\right] + B_1 + B_L.$$

$$\Delta = -\frac{1}{2}(Q+Q^{-1}), \quad h = \frac{1}{2}(Q-Q^{-1}), \quad Q = \sqrt{\frac{q}{p}},$$

$$B_L = \frac{\beta+\delta-(\beta-\delta)\sigma_L^z - \frac{2\beta}{\lambda Q^{L-1}}\sigma_L^+ - 2\delta\lambda Q^{L-1}\sigma_L^-}{2\sqrt{pq}},$$

$$B_1 = \frac{\alpha+\gamma+(\alpha-\gamma)\sigma_1^z - 2\alpha\lambda\sigma_1^- - \frac{2\gamma}{\lambda}\sigma_1^+}{2\sqrt{pq}}.$$

B_1 and B_L break particle number conservation.

5 free parameters: $\alpha,\beta,\gamma,\delta,\lambda$ (spectrum indep. of λ)

Integrability of the open spin-1/2 XXZ Chain

H is integrable: ∃ infinite number of local integrals of motion $[H,I_n]=0$, $[I_m,I_n]=0$.

Sklyanin '88
deVega&Gonzales-Ruiz '93

6 free parameters (general boundary fields)

Bethe Ansatz ???

(Nepomechie '03, '04; Cao et al'04)

Algebraic Bethe Ansatz/Functional Eqns

if the boundary fields satisfy a constraint!

Since then lots of work on general case

Murgan&Nepomechie, Yang et al, Galleas, Baseilhac, Simon, Frahm et al, Crampe et al,...

Bethe Ansatz Eqns for the PASEP

deGier/Essler '05

Eigenvalues of H parametrized by L-1 complex numbers $\{z_1, \ldots, z_{L-1}\}$

$$E = \alpha + \beta + \gamma + \delta + \sum_{j=1}^{L-1} \frac{\left(Q^2 - 1\right)^2 z_j}{(Q - z_j)(Qz_j - 1)},$$

Bethe Ansatz Equations:

$$\left[\frac{z_j Q - 1}{Q - z_j}\right]^{2L} K(z_j, \alpha, \gamma) K(z_j, \beta, \delta) = \prod_{l \neq j}^{L-1} \frac{z_j Q^2 - z_l}{z_j - z_l Q^2} \frac{z_j z_l Q^2 - 1}{z_j z_l - Q^2}, \; j = 1, \ldots, L-1.$$

$$K(z, \alpha, \gamma) = \frac{-\alpha z^2 + Qz(Q^2 - 1 + \alpha - \gamma) + \gamma Q^2}{\gamma Q^2 z^2 + Qz(Q^2 - 1 + \alpha - \gamma) - \alpha}.$$

checked for small systems that these are **complete!**

Solution of the BAE for large L

Take log of BAE ⇒ $Y_L(z_j) = \frac{2\pi}{L} I_j, \ (I_j \text{ integers}) \quad j = 1, \ldots, L-1,$

$Y_L(z)$ = **"counting function"**

$$iY_L(z) = g(z) + \frac{1}{L} g_b(z) + \frac{1}{L} \sum_{l=1}^{L-1} K(z_l, z),$$

$$g(z) = \ln\left[\frac{z(1-Qz)^2}{(z-1)^2}\right], \quad K(w,z) = -\ln\left[\frac{w-Qz}{1-Qw/z}\frac{1-Q^2zw}{1-wz}\right],$$

$$g_b(z) = \ln\left[\frac{z(1-Q^2z^2)}{1-z^2}\right] + \ln\left[\frac{z+a}{1+Qaz}\frac{1+c/z}{1+Qcz}\right] + \ln\left[\frac{z+b}{1+Qbz}\frac{1+d/z}{1+Qdz}\right].$$

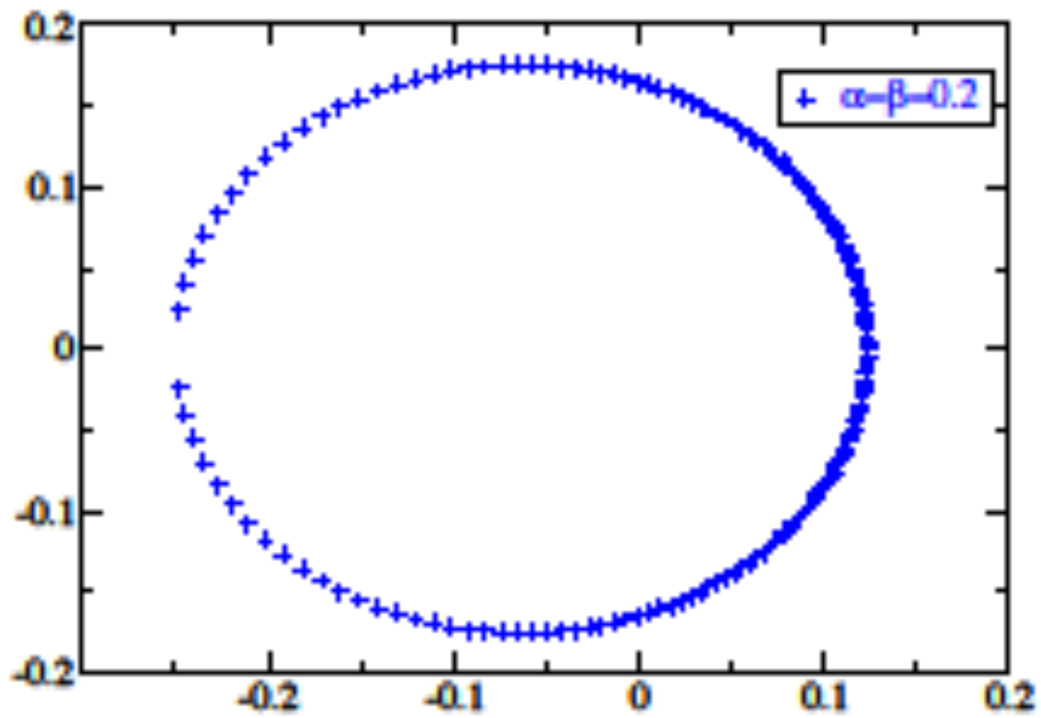
Programme:

set of int. $\{I_j\}$ ⟶ corresp. $Y_L(z)$ ⟶ Eigenvalue E

Solution of the BAE for large L

A. Numerics for L≤14: distributions $\{I_j\}$ for "low lying states"

I_j consecutive integers!



L=160

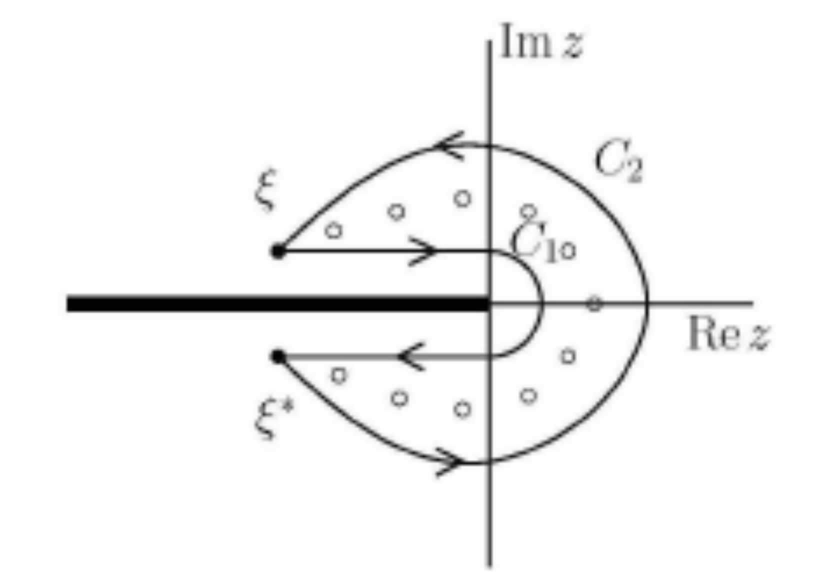
$p = \gamma = \delta = 0, \; \alpha = \beta = 0.2$

Solution of the BAE for large L

B. Consider large-L limit of these distributions $\{I_j\}$

BAE → Integro-differential eqns for $Y_L(z)$

$$iY_L(z) = g(z) + \frac{1}{L} g_b(z) + \int_{\xi^*}^{\xi} \frac{dw}{2\pi} K(w,z) Y_L'(w)$$
$$+ \int_{C_1} \frac{dw}{2\pi} \frac{K(w,z)Y_L'(w)}{1-e^{-iLY_L(w)}} + \int_{C_2} \frac{dw}{2\pi} \frac{K(w,z)Y_L'(w)}{e^{iLY_L(w)}-1}$$

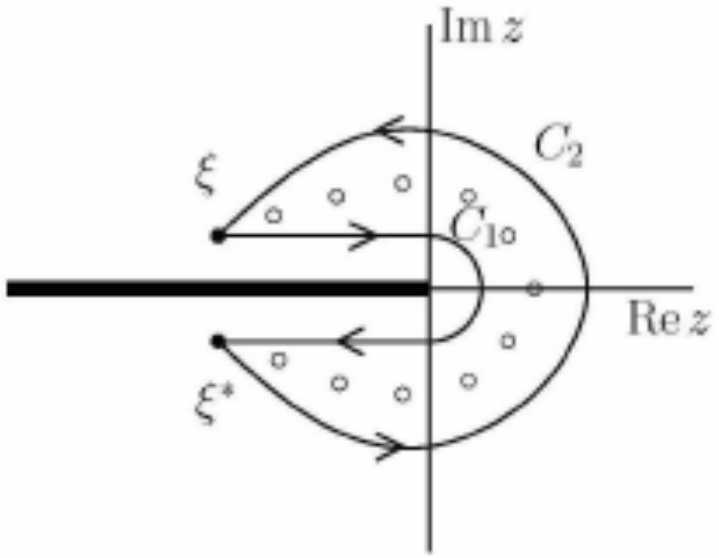


Endpoints fixed, e.g. $Y_L(\xi^*) = -\pi + \frac{\pi}{L}$, $\quad Y_L(\xi) = \pi - \frac{\pi}{L}$.

Solution of the BAE for large L

B. Consider large-L limit of these distributions $\{I_j\}$

$$iY_L(z) = g(z) + \frac{1}{L} g_b(z) + \int_{\xi^*}^{\xi} \frac{dw}{2\pi} K(w,z) Y_L'(w)$$
$$+ \int_{C_1} \frac{dw}{2\pi} \frac{K(w,z)Y_L'(w)}{1 - e^{-iLY_L(w)}} + \int_{C_2} \frac{dw}{2\pi} \frac{K(w,z)Y_L'(w)}{e^{iLY_L(w)} - 1}$$



Expand in powers of L^{-1}: $Y_L(z) = \sum_{n=0} L^{-n} Y_n(z)$, $\xi = z_c + \sum_{n=1} \delta_n L^{-n}$

⇒ System of **linear** integro-differential eqns

Key to solution: $Y_L(z)$ **analytic** close to contour

Explicit results for the counting functions:

$$Y_L(z) = \sum_{n=0} L^{-n} Y_n(z)$$

$$\begin{aligned}
Y_0(z) &= -i \ln\left[-\frac{z}{z_c}\left(\frac{1-z_c}{1-z}\right)^2\right], \\
Y_1(z) &= -i \ln\left[-\frac{z}{z_c}\frac{1-z_c^2}{1-z^2}\right] + \omega_1 + \lambda_1 \ln\left[\frac{(Qz_c/z;Q)_\infty (Qzz_c;Q)_\infty^2}{(Qz/z_c;Q)_\infty (Qz_c^2;Q)_\infty^2}\frac{z-z_c^{-1}}{z_c-z_c^{-1}}\right] \\
&\quad - i \ln\left[\frac{(-c/z;Q)_\infty(-cz;Q)_\infty(-z/a;Q)_\infty(-Qaz_c;Q)_\infty}{(-c/z_c;Q)_\infty(-cz_c;Q)_\infty(-z_c/a;Q)_\infty(-Qaz;Q)_\infty}\right] \\
&\quad - i \ln\left[\frac{(-d/z;Q)_\infty(-dz;Q)_\infty(-z/b;Q)_\infty(-Qbz_c;Q)_\infty}{(-d/z_c;Q)_\infty(-dz_c;Q)_\infty(-z_c/b;Q)_\infty(-Qbz;Q)_\infty}\right] \\
Y_2(z) &= \ldots
\end{aligned}$$

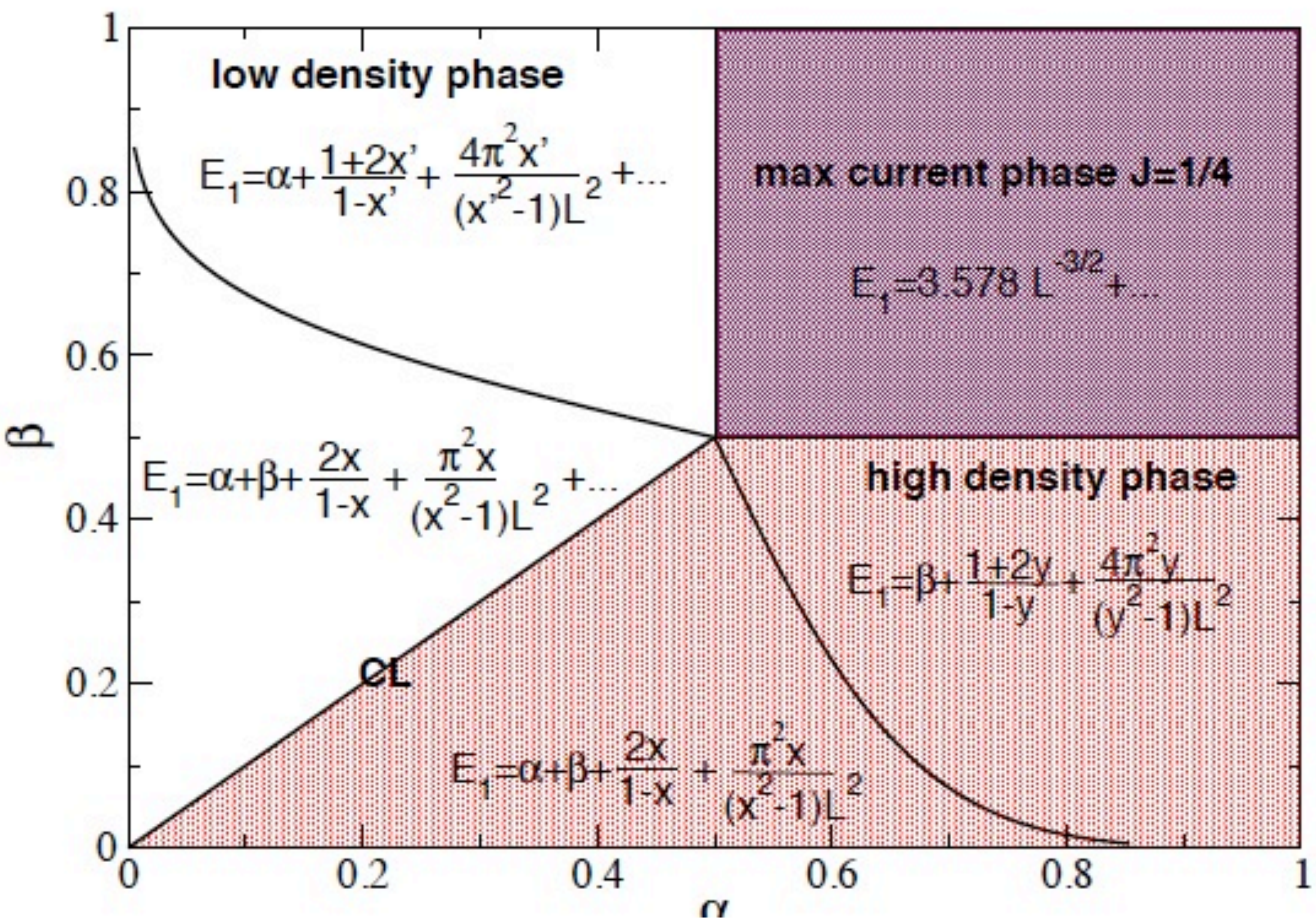
$(a;Q)_\infty = \prod_{k=0}^{\infty}(1-aQ^k)$ q-Pochhammer symbol

$a,b,c,d,z_c,\lambda_1,\omega_1$ known elementary fns of $\alpha,\beta,\gamma,\delta$.

Given $Y_L(z)$ can calculate eigenvalues E.

“Dynamical” Phase Diagram

TASEP:



$$x = -\sqrt{\frac{\alpha\beta}{(1-\alpha)(1-\beta)}}$$

$$x' = -\left[\frac{\alpha}{1-\alpha}\right]^{1/3}$$

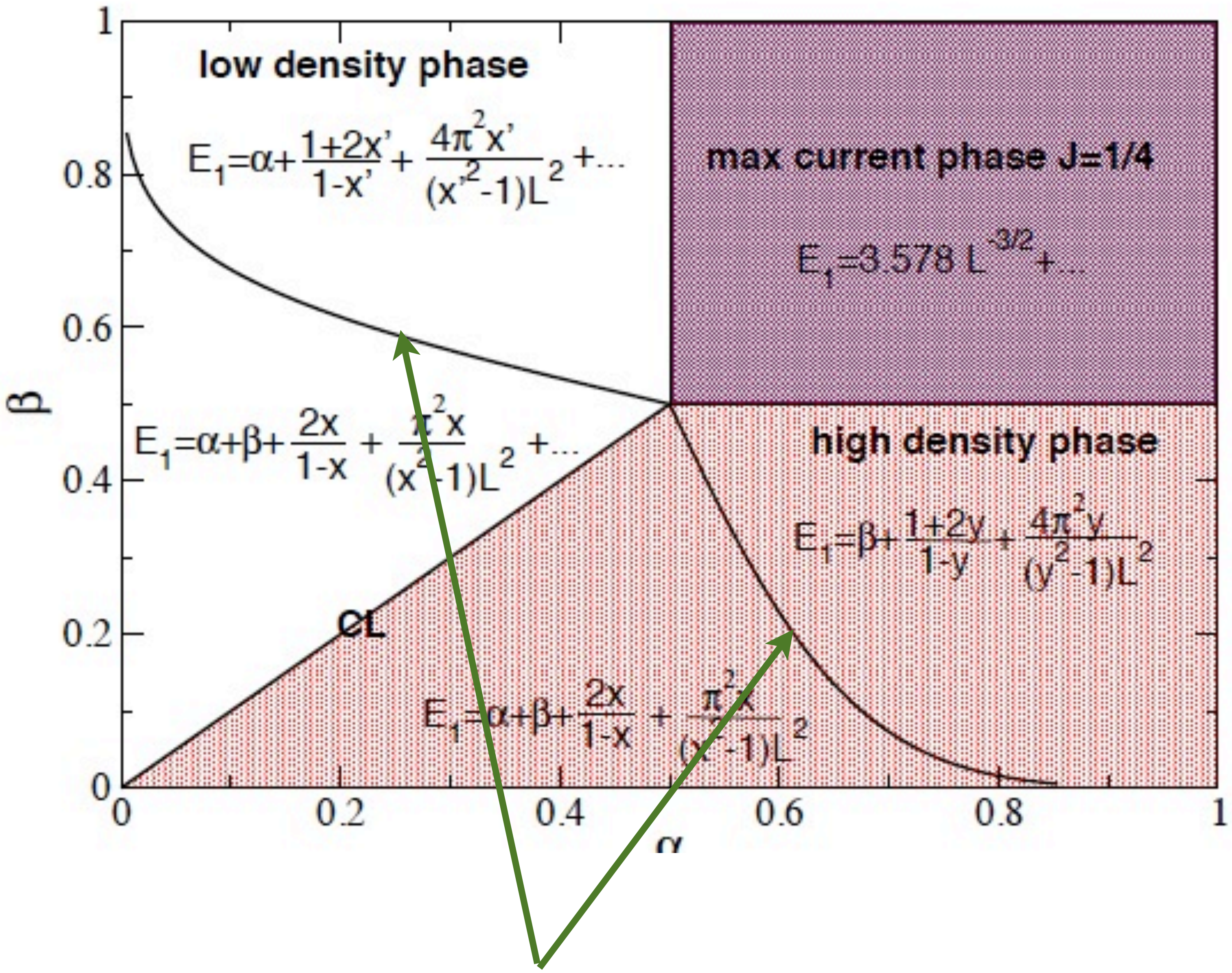
$$y = -\left[\frac{\beta}{1-\beta}\right]^{1/3}$$

$$\text{CL: } E_1 = \frac{\pi^2\alpha(1-\alpha)}{(1-2\alpha)L^2}$$

Explicit answers except in max current phase (numerics)

"Dynamical" Phase Diagram

TASEP:



$$x=-\sqrt{\frac{\alpha\beta}{(1-\alpha)(1-\beta)}}$$

$$x'=-\left[\frac{\alpha}{1-\alpha}\right]^{1/3}$$

$$y=-\left[\frac{\beta}{1-\beta}\right]^{1/3}$$

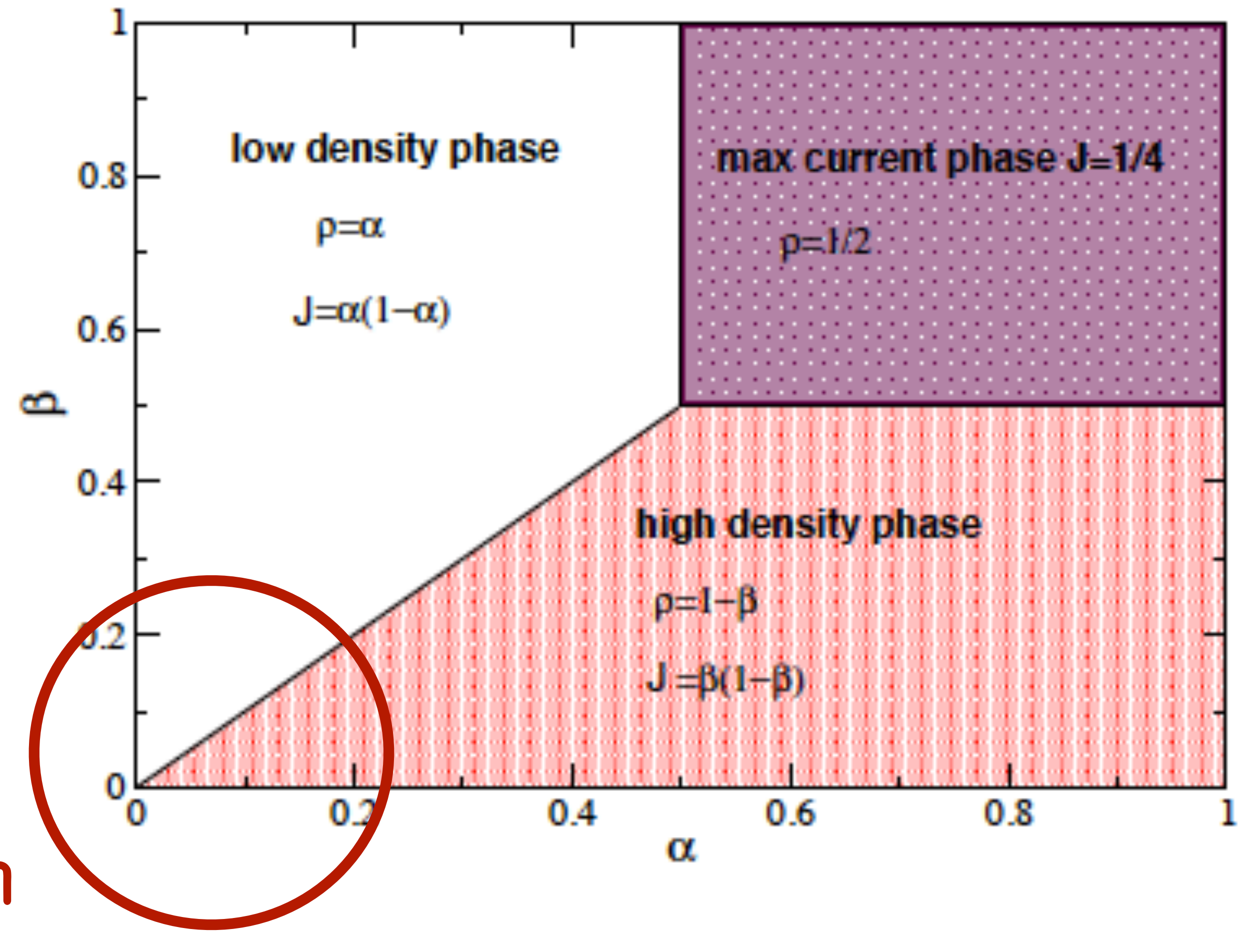
$$\text{CL: } E_1=\frac{\pi^2\alpha(1-\alpha)}{(1-2\alpha)L^2}$$

Non-analytic jump in E_1 ⇒ change in relaxational mechanism

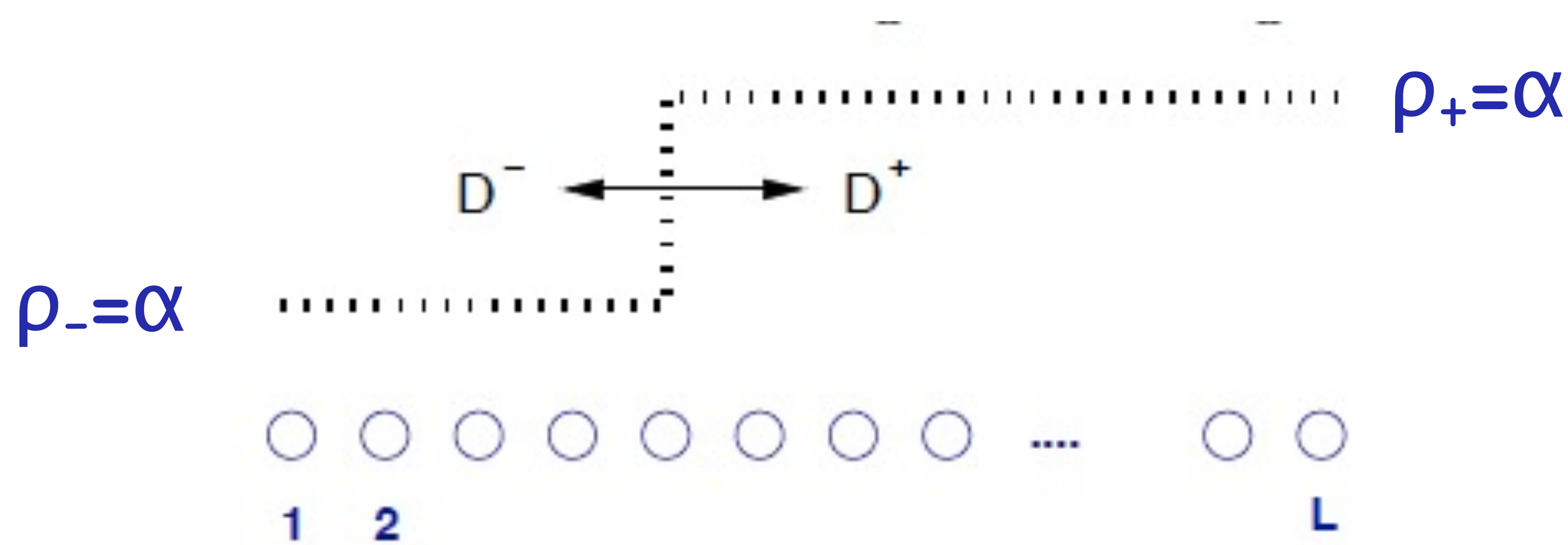
Effective Domain Wall Theory

Derrida, Evans & Mallick '95
Schütz et al '98, '00

Recall stationary state phase diagram:



<u>Assumption:</u> relaxation due to diffusion of **domain walls** between high and low-density phases

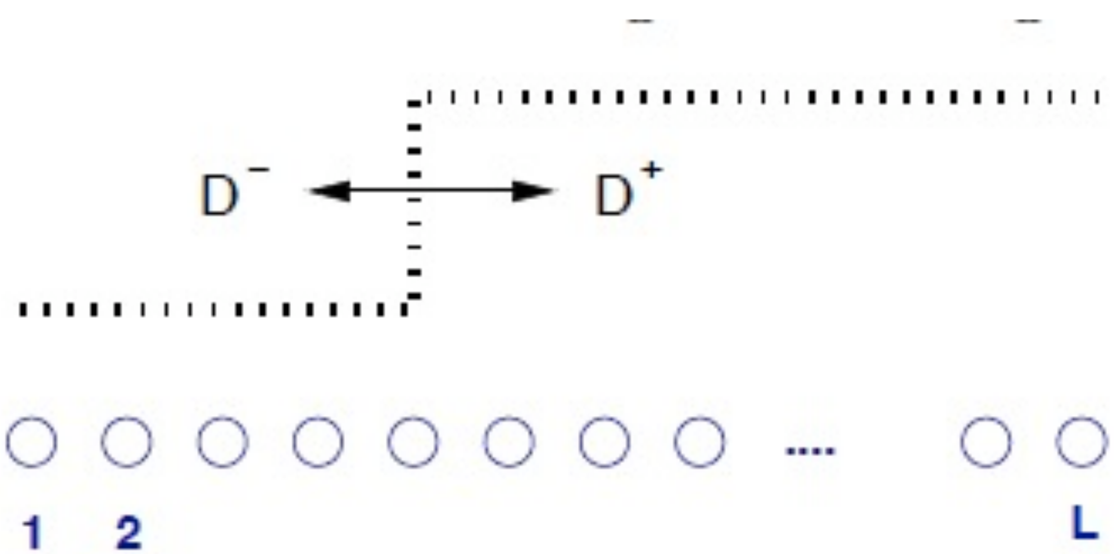


$D^{\pm}(\rho_+ - \rho_-) = j_{\pm}$
$= \rho_{\pm}(1-\rho_{\pm})$

Effective Domain Wall Theory

Derrida, Evans & Mallick '95
Schütz et al '98, '00

Consider a **single** diffusing domain wall with reflecting boundaries



TASEP:

$$D^+ = \frac{\beta(1-\beta)}{1-\alpha-\beta}$$

$$D^- = \frac{\alpha(1-\alpha)}{1-\alpha-\beta}$$

Master eqn:

$$\begin{aligned}
\frac{\partial}{\partial t}P(x,t) &= D^- P(x+1,t) + D^+ P(x-1,t) - (D^+ + D^-)P(x,t) \\
\frac{\partial}{\partial t}P(1,t) &= D^- P(2,t) + D^+ P(1,t) \\
\frac{\partial}{\partial t}P(L,t) &= D^+ P(L-1,t) - D^- P(L,t).
\end{aligned}$$

Solution:

$$P(t,n) = \sum_p A_p(0) e^{-\epsilon t}\left[e^{ipn} + B_p u^{2n} e^{-ipn}\right]$$

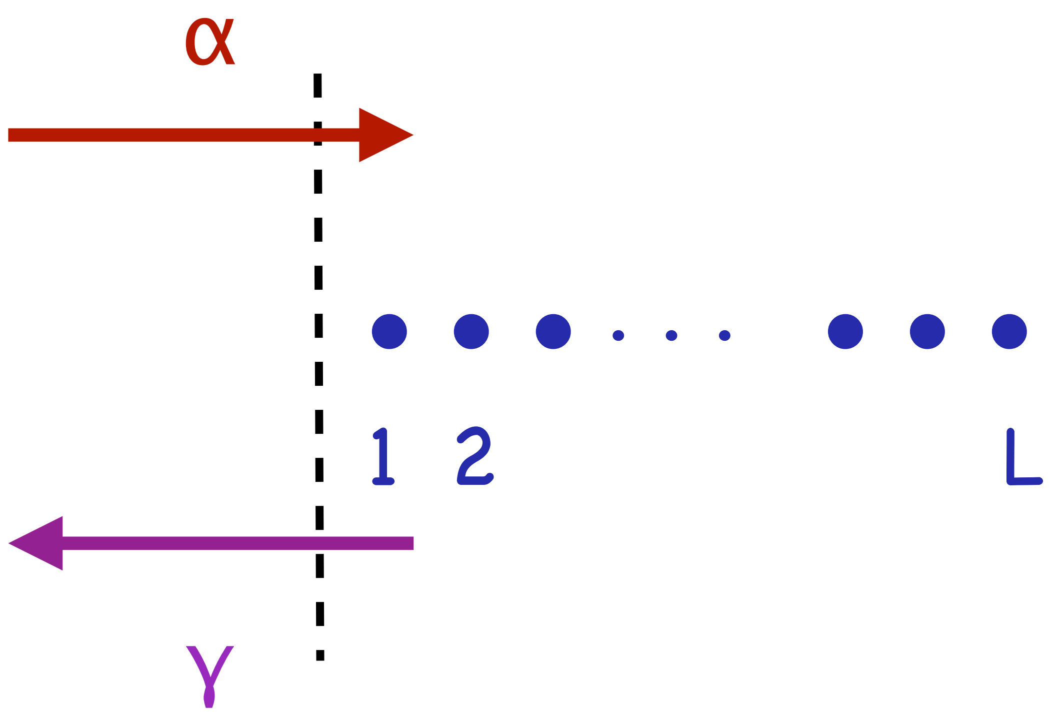
Relaxation rates ε agree with exact results for small α,β!

Some Open Questions on this Part

- Analytic calculation of E_1=3.578 $L^{3/2}$+... in the MC phase
- What is the relaxational mechanism in general?
  modified DW diffusion (boundary conditions?)? something else?
- Full correlation functions through "form factors"?

Current Fluctuations

Derrida et al '98–'11



$Q_1(t)$: net # of particles crossing the dashed line up to time t

$Q_1(t) \sim j_1 t$ for $t \to \infty$ j_1=average current on first site

Goal: calculate $\langle e^{\lambda Q_1(t)} \rangle$ ⟶ cumulants of $Q_1(t)$

Previously Known Results

Average Current:

$$\lim_{t\to\infty} \frac{\langle Q_1(t)\rangle}{t} = (p-q)\rho(1-\rho)$$

Derrida, Evans, Hakim & Pasquier '93;
Schütz/Domany '93

Diffusion Constant:

$$\lim_{t\to\infty} \frac{\langle Q_1^2\rangle - \langle Q_1\rangle^2}{t} = (p-q)\rho(1-\rho)(1-2\rho)$$

Derrida, Evans
& Mallick '95

Relation to Spectrum of M

Derrida &Lebowitz '98

Modify Boundary Term in XXZ:

$$B_1 = \frac{\alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha\sigma_1^- - 2\gamma\sigma_1^+}{2\sqrt{pq}}$$

$$B_1 = \frac{\alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha e^{\lambda}\sigma_1^- - 2\gamma e^{-\lambda}\sigma_1^+}{2\sqrt{pq}}$$

Then

$$\lim_{t\to\infty} \langle e^{\lambda Q_1(t)} \rangle = e^{E_0(\lambda)t}$$

$E_0(\lambda)$= eigenvalue of $M(\lambda)$ with smallest real part

Great! Problem reduced to calculating an energy.

Problem: model remains integrable, but can no longer fulfil constraint that allows for BA solution.

Way out: Constraint $\left[q^{L/2+k} - e^{\lambda}\right]\left[\alpha\beta e^{\lambda} - q^{L/2-k-1}\gamma\delta\right] = 0$

$$k \in \mathbb{Z} \;,\; |k| \leq \frac{L}{2}$$

Sequence I: $\lambda_n^{(1)} = n\ln(q)$ n=0,1,...,L

Sequence II: $\lambda_n^{(2)} = \ln\left(\gamma\delta q^{n-1}/\alpha\beta\right)$ n=0,1,...,L

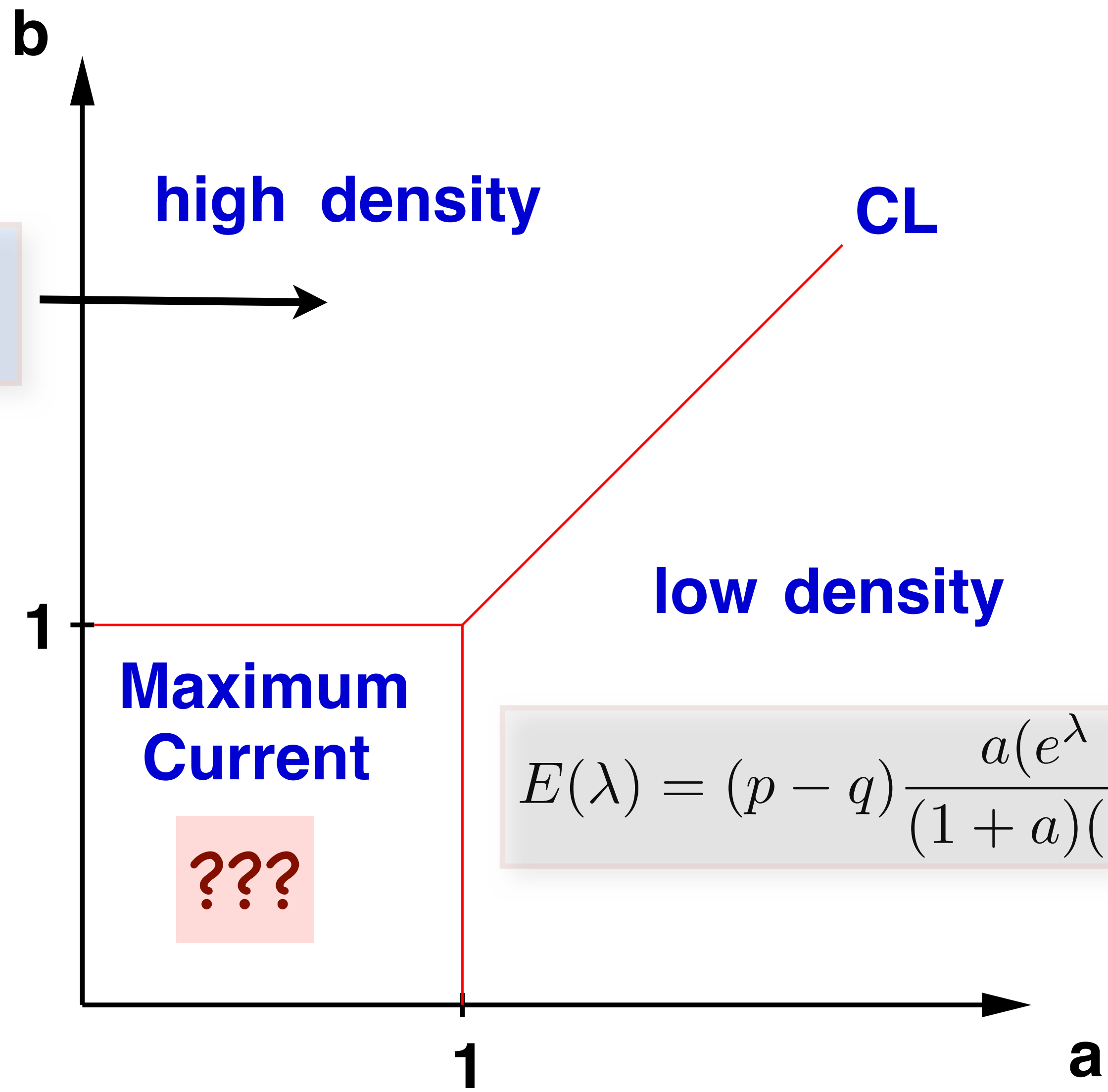
} calculate $E(\lambda_n^{(j)})$

→ Try to restore full λ dependence from these two sequences

Results for the PASEP

$$b = \frac{p - q - \beta + \delta \pm \sqrt{(p - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta}$$

$$E(\lambda) = (p-q)\frac{b(e^{\lambda}-1)}{(1+b)(e^{\lambda}+b)}$$



$$E(\lambda) = (p-q)\frac{a(e^{\lambda}-1)}{(1+a)(e^{\lambda}+a)}$$

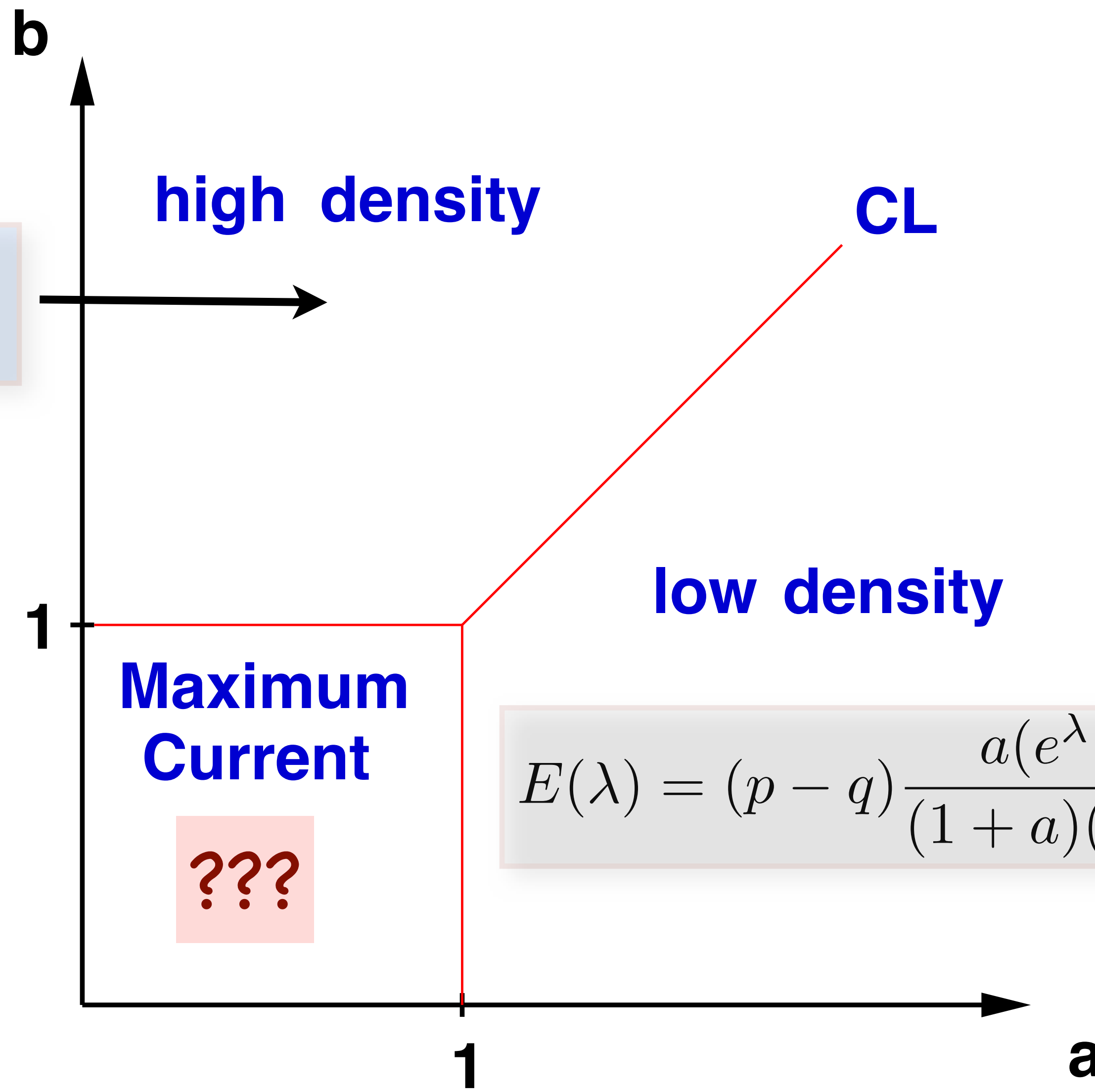
$$a = \frac{p - q - \alpha + \gamma \pm \sqrt{(p - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

These are the leading terms for L→∞

Results for the PASEP

$$b = \frac{p - q - \beta + \delta \pm \sqrt{(p - q - \beta + \delta)^2 + 4\beta\delta}}{2\beta}$$

$$E(\lambda) = (p - q)\,\frac{b(e^\lambda - 1)}{(1 + b)(e^\lambda + b)}$$



$$E(\lambda) = (p - q)\,\frac{a(e^\lambda - 1)}{(1 + a)(e^\lambda + a)}$$

Confirmed by completely different approach (giving more results):

Lazarescu & Mallick '11

$$a = \frac{p - q - \alpha + \gamma \pm \sqrt{(p - q - \alpha + \gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

Some Open Questions on this Part

- Direct calculation from Bethe Ansatz: need solution for arbitrary λ
- Finite-size corrections from Bethe Ansatz (have some results).
- Maximum Current Phase using Bethe Ansatz?

$$Q_1(t) \sim j_1 t + \mathcal{O}\left(t^{1/3}\right)$$

What can we say about the $t^{1/3}$ contribution?
→ Tracy-Widom

Summary

- Obtained Bethe Ansatz Equations for PASEP with most general open boundary conditions.
- Derived analytic expressions for eigenvalues of "Hamiltonian" with smallest real part → describes relaxation to stationary state.
- Obtained "Dynamical Phase Diagram"
- Calculated eigenvalues of "excited states".
- Used Bethe Ansatz to compute current fluctuations.
- Full correlation functions through "Lehmann rep."??