The Asymmetric Exclusion Process: an exactly solvable nonequilibrium system

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Plan: Asymmetric Exclusion Process

Plan

I Definition of model

II Solution by matrix approach

III q-deformed generalisations

IV multi species generalisations

Review:

R. A. Blythe and M.R.Evans, Nonequilibrium steady states of matrix-product form: a solver's guide, J. Phys. A.: Math. Theor. **40** R333-R441

I Definition of Totally Asymmetric Exclusion Process

TASEP

Usually consider 1d lattice of Length N, $\mathbb{Z}_N$

- at most one particle per site (exclusion)
- particles hop forward with rate p (totally asymmetric hopping)

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TASEP

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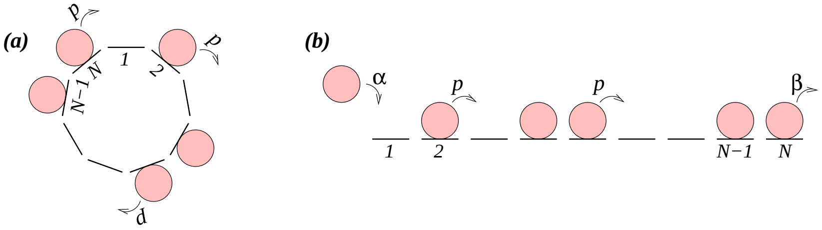
Boundary conditions:

a) on ring (periodic boundary conditions)
b) or on open lattice



Motivation

- The ASEP was first introduced in 1968 as a model for RNA translation by ribosomes
  (MacDonald, Gibbs and Pipkin, *Biopolymers* 1968)
- Now a general model for traffic (both vehicular and biophysical)
- It is a *nonequilibrium system*, since a current always flows — stationary state not known a priori
- Exhibits phase transitions in 1 dimension
  (Krug, *Phys. Rev. Lett.* 1991)
- Exactly solvable model

Correlation functions

Use indicator variable $\tau_i = 1, 0$ for particle, hole respectively
Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tau_i \rangle = p\langle \tau_{i-1}(1-\tau_i)\rangle - p\langle \tau_i(1-\tau_{i+1})\rangle$$

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$\langle\tau_i(t)\rangle$ is the density at site i at time t

$J_{i,i+1} = p\langle\tau_i(1-\tau_{i+1})\rangle$ is the current from site i to $i+1$

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Hierarchy of correlation functions e.g.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tau_i(1-\tau_{i+1})\rangle = -p\langle \tau_i(1-\tau_{i+1})\rangle + p\langle \tau_{i-1}(1-\tau_i)(1-\tau_{i+1})\rangle + p\langle \tau_i\tau_{i+1}(1-\tau_{i+2})\rangle$$

... difficult to solve generally

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N. B. take $p = 1$ from now on

II Exact Solution of Stationary State

Matrix Product Solution for Open Boundaries

● $\rightarrow D$ and _ $\rightarrow E$ where D, E are matrices

e.g. Prob [_ ● _ ●] $= \dfrac{\langle W|EDED|V\rangle}{Z_4}$

where

$$Z_N = \langle W|C^N|V\rangle \qquad C = D + E$$

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Necessary and Sufficient Conditions

$$\begin{aligned} DE &= D + E \\ \beta D|V\rangle &= |V\rangle \\ \alpha\langle W|E &= \langle W| \end{aligned}$$

(Derrida, Evans, Hakim, Pasquier 1993)

Form of matrices

Several possible representations are possible, for example :

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$\langle W| = \kappa \left(1, \quad a, \quad a^2 \quad . \quad . \right) \qquad |V\rangle = \kappa \begin{pmatrix} 1 \\ b \\ b^2 \\ . \\ . \end{pmatrix},$$

where $a = \dfrac{1-\alpha}{\alpha}$ $\qquad b = \dfrac{1-\beta}{\beta}$ and $\kappa^2 = (\alpha + \beta - 1)/\alpha\beta$

- matrices generally (semi) infinite except along $\alpha + \beta = 1$
- calculation of matrix product elements corresponds to enumeration of lattice paths

Exact Solution of Stationary State

Calculation of the Current

$$J_N = \alpha\langle(1-\tau_1)\rangle = \langle\tau_i(1-\tau_{i+1})\rangle = \beta\langle\tau_N\rangle$$

$$= \alpha\frac{\langle W|EC^{N-1}|V\rangle}{Z_N} = \frac{\langle W|C^{i-1}DEC^{N-i-1}|V\rangle}{Z_N} = \beta\frac{\langle W|C^{N-1}D|V\rangle}{Z_N}$$

$$= \frac{Z_{N-1}}{Z_N}$$

ratio of nonequilibrium partition functions

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ratio of nonequilibrium partition functions

Calculation of the Z_N

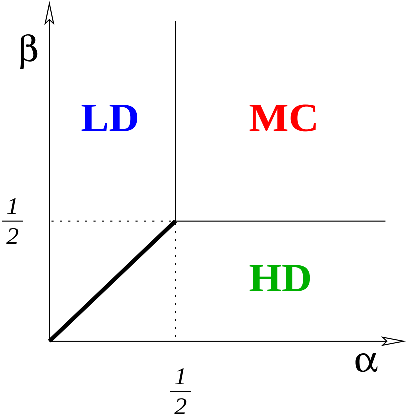
$$C^0 = 1\,, \quad C^1 = D+E\,, \quad C^2 = D^2+E^2+ED+D+E\,, \quad \ldots$$

$$C^N = \sum_{n,m} a_{n,m}E^nD^m \qquad \langle W|C^N|V\rangle = \sum_{n,m} a_{n,m}\frac{1}{\alpha^n}\frac{1}{\beta^m}$$

for large N, $\langle W|C^N|V\rangle \sim c^N N^{-\lambda}$ so

$$J \to \frac{1}{c}$$

Stationary Phase Diagram



HD - high density phase controlled by right boundary

$\rho = \beta$, $J = \beta(1-\beta)$

LD - low density phase controlled by left boundary

$\rho = \alpha$, $J = \alpha(1-\alpha)$

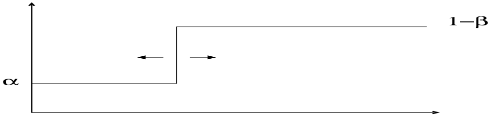
MC - maximal current phase $\rho = 1/2$, $J = 1/4$

First-order line and domain wall dynamics

$\alpha = \beta < 1/2$ is first-order transition line

Along first-order line stationary state is superposition of shocks

(domain walls)

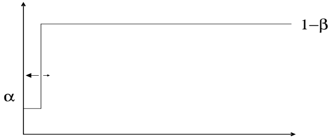


which generates an exact linear average profile.

Extending this picture to regime HD I/ LD I gives an *effective* description of dynamics in terms of domain walls moving stochastically with bias velocity $v_s = \dfrac{\beta(1-\beta) - \alpha(1-\alpha)}{1-\beta-\alpha} = \beta - \alpha$

ld

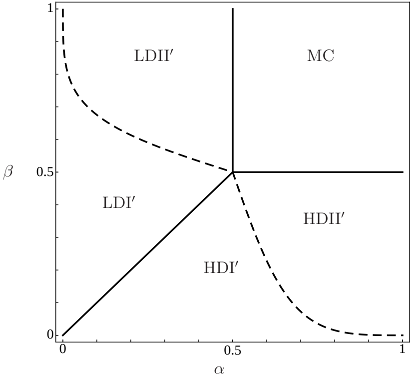
hd



(Dudzinsky, Schütz 2000)

Dynamical Phase Diagram

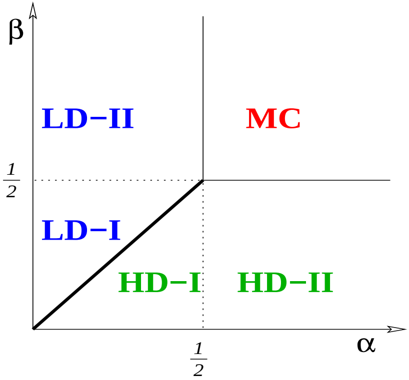
(de Gier and Essler 2005)



Explanation?

Stationary Phase Diagram

(Derrida, Evans, Hakim, Pasquier 1993), (Schütz, Domany 1993)



Dynamical Phase Diagram - de Gier-Essler Line

Dynamical Phase Diagram

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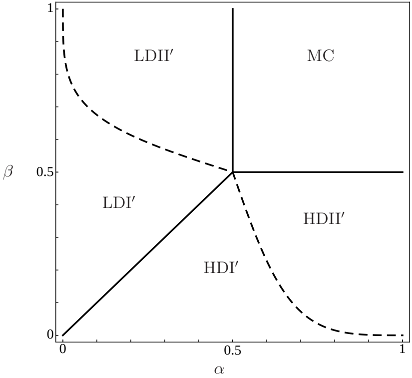

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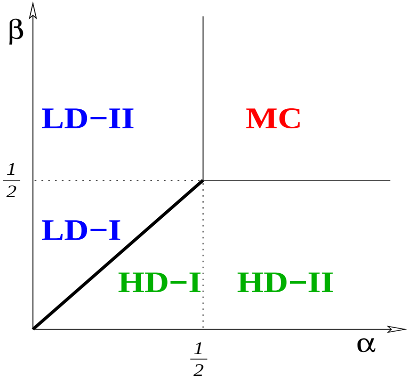


Explanation? It's a mystery!
but deGE line does exist (Proeme, Blythe, Evans 2011)

Stationary Phase Diagram: Lee Yang Theory

- Consider normalisation as a nonequilibrium partition function

$$Z_N = \sum_{p=1}^{N} \frac{p!(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{(1/\beta) - (1/\alpha)}$$

- Generalise real rates α, β to complex parameters and consider zeros of Z_N in e.g. the complex α plane

- Phase transitions occur when complex zeros of Z_N 'pinch' the real axis as $N \to \infty$

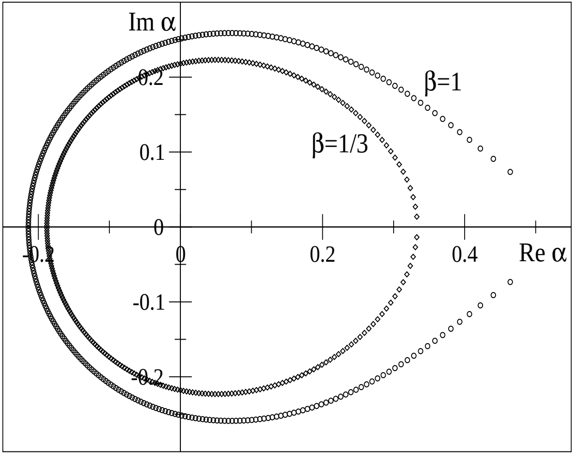
first order line: finite density of zeros pinch at angle $\pi/2$

second order line: vanishing density of zeros pinch at angle $\pi/4$

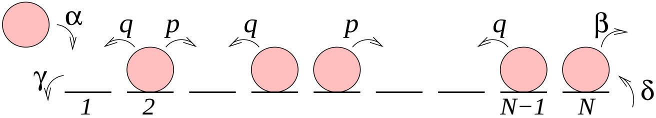
- $\lim_{N\to\infty} \frac{1}{N} \ln Z_N = \ln J$ so current plays role of free energy

Stationary Phase Diagram: Lee Yang Theory

Zeros of $Z_N(\alpha, \beta)$ in complex alpha plane



II Partial Asymmetry



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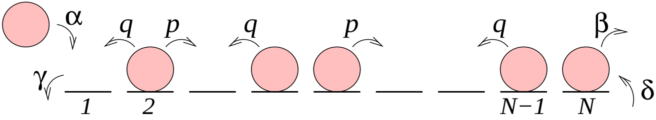
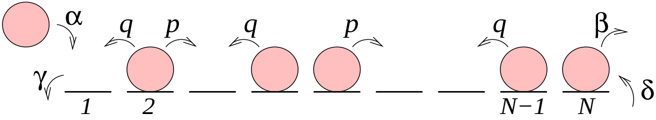


quadratic algebra

$$\begin{aligned} DE - qED &= D + E \\ (\beta D - \delta E)|V\rangle &= |V\rangle \\ \langle W|(\alpha E - \gamma D) &= \langle W| \end{aligned}$$

II Partial Asymmetry



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Let $D = \frac{\mathbf{1}}{1-q} + \frac{\hat{a}}{(1-q)^{1/2}} \quad E = \frac{\mathbf{1}}{1-q} + \frac{\hat{a}^\dagger}{(1-q)^{1/2}}$

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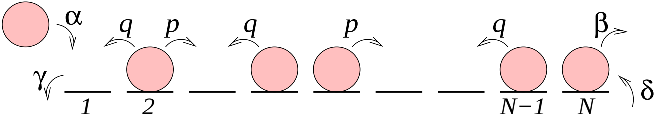

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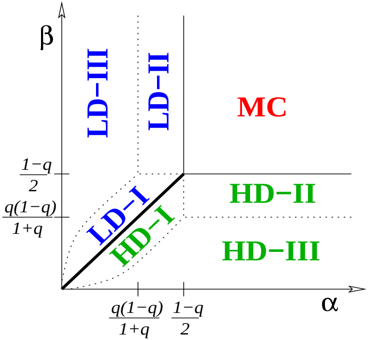
$\Rightarrow$ $\boxed{\hat{a}\hat{a}^{\dagger} - q\hat{a}^{\dagger}\hat{a} = 1}$ q-deformed harmonic oscillator

III Partial Asymmetry: Results for $\gamma = \delta = 0$

• $q < 1$ (forward bias)



Sasamoto 2000

• $q > 1$ (reverse bias)

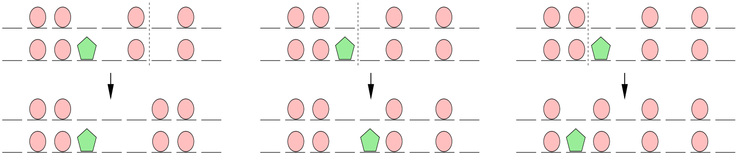
$$J \simeq \left(\frac{\alpha\beta(q-1)^2}{(q-1+\alpha)(q-1+\beta)} \right)^{1/2} q^{-N/2+1/4}$$

Blythe, Evans, Calaori, Essler 2000

• $q = 1$ (symmetric) linear profile and $J \simeq \dfrac{1}{N}$

IV Many Species: 2 species ASEP

- Motion of an **excess particle**



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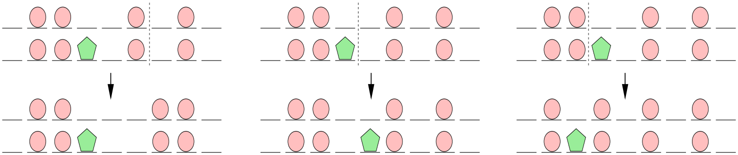

- **Second class particle dynamics**

  $1\ 0 \rightarrow 0\ 1$

  $2\ 0 \rightarrow 0\ 2$

  $1\ 2 \rightarrow 2\ 1$

- Second class particle moves forward in low density environment, moves backward in high density environment. It therefore **tracks** ‘**shocks**’ (discontinuities in density profile)

2-ASEP: Matrix Solution on Ring

$\tau_i = 0, 1, 2$, let there be P_1 first class, P_2 second class and $P_0 = N - P_1 - P_2$ holes

Stationary measure Derrida, Janowsky, Lebowtiz and Speer 1993

$$\mathrm{Prob}(\{\tau_i\}) = \frac{\mathrm{Tr}\left[\prod_{i=1}^{N} X_{\tau_i}\right]}{Z(P_0, P_1, P_2)}$$

matrices now $X_0 = E$, $X_1 = D$, $X_2 = A$

Quadratic algebra

$$\begin{aligned} DE &= D + E \\ DA &= A \\ AE &= A \end{aligned}$$

Where

$$A = |0\rangle\langle 0|$$

is a projector

Multispecies TASEP: 'n-TASEP'

n Classes of particle and vacancies

$K\ 0 \to 0\ K$ for $n \geq K \geq 1$

$K\ J \to J\ K$ for $n \geq J > K \geq 1$

e.g. **n= 3**

$1\ 0 \to 0\ 1$

$2\ 0 \to 0\ 2$

$3\ 0 \to 0\ 3$

$1\ 2 \to 2\ 1$

$1\ 3 \to 3\ 1$

$2\ 3 \to 3\ 2$

Matrix Solution for 3-TASEP

$$
\begin{aligned}
X_1 &= \mathbf{1} \otimes \mathbf{1} \otimes D + \delta \otimes \epsilon \otimes A + \delta \otimes \mathbf{1} \otimes E \\
X_2 &= A \otimes \mathbf{1} \otimes A + A \otimes \delta \otimes E \\
X_3 &= A \otimes A \otimes E \\
X_0 &= \mathbf{1} \otimes \mathbf{1} \otimes E + \mathbf{1} \otimes \epsilon \otimes A + \epsilon \otimes \mathbf{1} \otimes D
\end{aligned}
$$

where

$$\delta = D - \mathbf{1} \qquad \epsilon = E - \mathbf{1}$$

.

Example of algebraic conditions

$$
\begin{aligned}
X_1 X_2 &= X_2 \hat{X}_1 - \hat{X}_2 X_1 \\
-X_1 X_2 &= X_1 \hat{X}_2 - \hat{X}_1 X_2
\end{aligned}
$$

Example of hat matrix: $\hat{X}_1 = (\mathbf{1} - \delta) \otimes \mathbf{1} \otimes \mathbf{1}$

Hierarchical Solution for n-TASEP Evans, Ferrari, Mallick 2009

$$X_K^{(n)} = \sum_{M=0}^{n-1} a_{KM}^{(n)} \otimes X_M^{(n-1)} \text{ for } 1 \leq K \leq n\,.$$

$$X_0^{(n)} = -X_0^{(n)} + \sum_{M=0}^{n-1} a_{0M}^{(n)} \otimes X_M^{(n-1)}$$

$X_K^{(n)}$: the lower index K denotes the class of the particle; the upper index n gives the number of classes in the system.

$a_{JM}^{(n)}$ themselves tensor products of fundamental matrices $A, D, E, \mathbf{1}$

Algebraic Conditions

$$\left[X_j^{(n)}, \hat{X}_j^{(n)}\right] = 0 \qquad 0 \leq J \leq n$$

$$X_J^{(n)} X_K^{(n)} = \hat{X}_J^{(n)} X_K^{(n)} - X_J^{(n)} \hat{X}_K^{(n)} \qquad J < K$$

$$= \hat{X}_K^{(n)} X_J^{(n)} - X_K^{(n)} \hat{X}_J^{(n)} \quad \text{or } K = 0$$

Structure of ‘Matrices’

$\dim(n)$ = no. fundamental matrices $D, E, \mathbf{1}$ in tensor product = $\binom{n}{2}$

n	$\dim(n)$	
1	0	scalars
2	1	matrices
3	3	

What is the interpretation?

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What is the interpretation?

no. queue counters required for $n-1$ priority queue system =

$$\sum_{i=1}^{n-1} i = \binom{n}{2} = \dim(n)$$

Queueing Interpretation: 2- TASEP

Recall

$$D = \sum_{n=0} |n\rangle \left[\langle n| + \langle n+1|\right] \quad E = \sum_{n=0} \left[|n\rangle + |n-1\rangle\right]\langle n| \quad A = |0\rangle\langle 0|$$

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Think of $|n\rangle$ as state of a queue (no. customers waiting).
$t(i) = N - i$ is discrete time

$$\begin{aligned} \tau_i &= 0 && \text{no service} \\ \tau_i &= 1 && \text{used service} \\ \tau_i &= 2 && \text{unused service} \end{aligned}$$

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Weight of config. $\{\tau_i\}$ = no. queue trajectories consistent with $\{\tau_i\}$

$\tau_i = 0 \qquad E|n\rangle \;=\; |n\rangle + |n+1\rangle$

no arrival new arrival

$\tau_i = 1 \qquad D|n\rangle \;=\; |n-1\rangle + |n\rangle$

$\tau_i = 2 \qquad A|n\rangle \;=\; |0\rangle\delta_{n,0}$ queue must be empty

Queueing Interpretation: 3- TASEP

Now think of 2 **tandem queues**:

q1 contains first class customers which get served and go to q2
q2 contains first class customers arriving from q1 and second class customers arriving from outside

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where l is no. first class in $q2$, m is no. second class in $q2$, n is no. first class in $q1$

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\tau_i &= 1 \quad \text{first class service in} \quad q2 \\
\tau_i &= 2 \quad \text{second class service in} \quad q2 \\
\tau_i &= 3 \quad \text{unused service in} \quad q2
\end{aligned}$$

Queueing Interpretation: 3- TASEP

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e.g. $\tau_i = 3$ unused service in $q2 \Rightarrow l = m = 0$;
possible arrival at $q1 \Rightarrow n \to n$ or $n \to n+1$

$$X_3|l\rangle|m\rangle|n\rangle = \delta_{l,0}|l\rangle\delta_{m,0}|m\rangle\left[|n\rangle + |n+1\rangle\right] = A|l\rangle A|m\rangle E|n\rangle$$

Conclusions

- A fundamental class of nonequilibrium stationary states may be solved exactly as a matrix product.
- Algebraic proof of stationary measure requires **quadratic algebras and generalisations**
- Construct stationary measure hierarchically from $(n-1)$-TASEP stationary measure
- The ‘Matrices’ act on space of queue counters

What I've left out

- Algebraic proof of stationary measure
- Simple mean field theory correctly predicts phase diagram and is first recourse for many systems
- Biophysical systems imply generalisations to dynamically extending 1d lattices and coupled one lattices
- Current fluctuations and large deviations

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- Current fluctuations and large deviations
- . . . **and much, much more**