Dynamics of Ultracold Gases

in one spatial dimension

Thomas Gasenzer

Institut für Theoretische Physik
Ruprecht-Karls Universität Heidelberg

Philosophenweg 16 • 69120 Heidelberg • Germany

email: t.gasenzer@uni-heidelberg.de
www: www.thphys.uni-heidelberg.de/~gasenzer

Center for Quantum Dynamics

EMMI

Equilibration



?

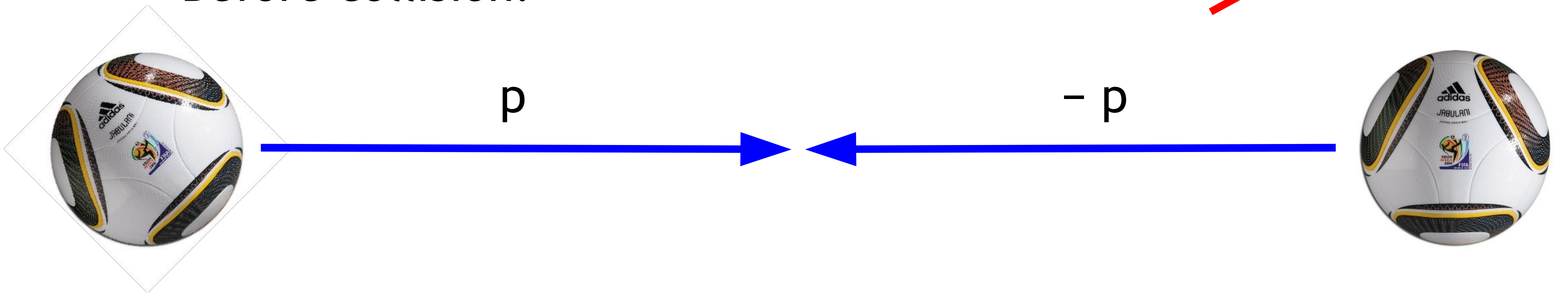
It seems 2b good to work in 1D.

Because it's just a bit...
...complicated to keep track
of dynamics in 3D.

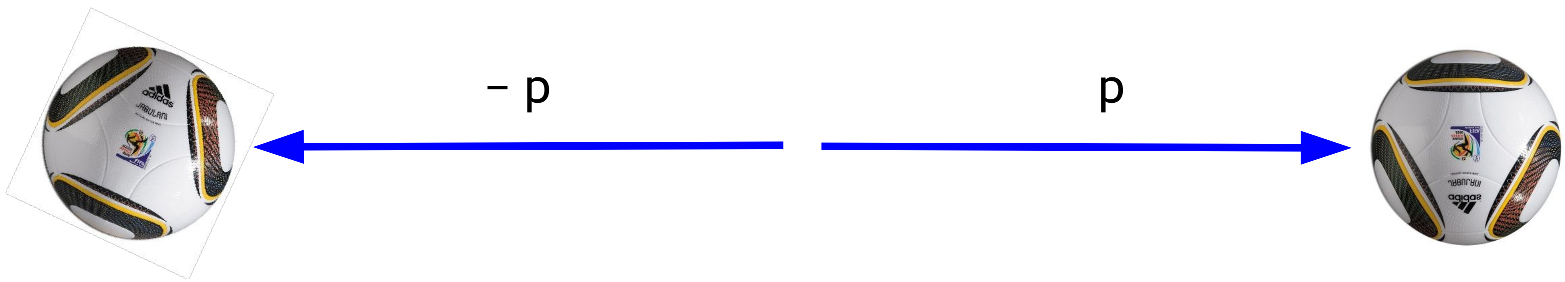
Elastic scattering of classical point-like particles in $d > 1$ dimensions

Elastic scattering of classical point-like particles in **one** dimension

Before collision:

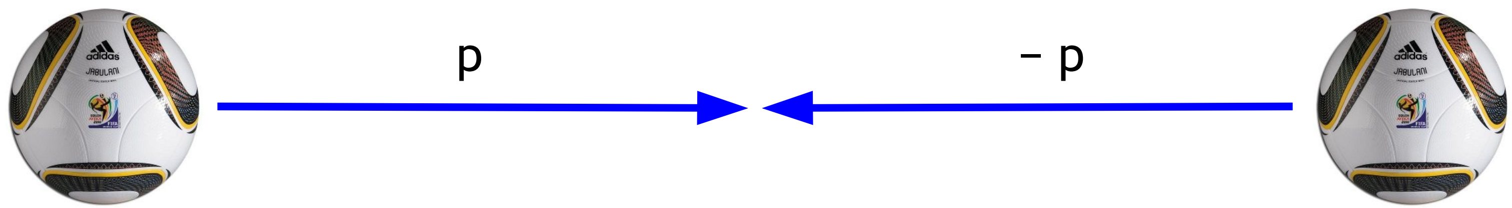


After collision:

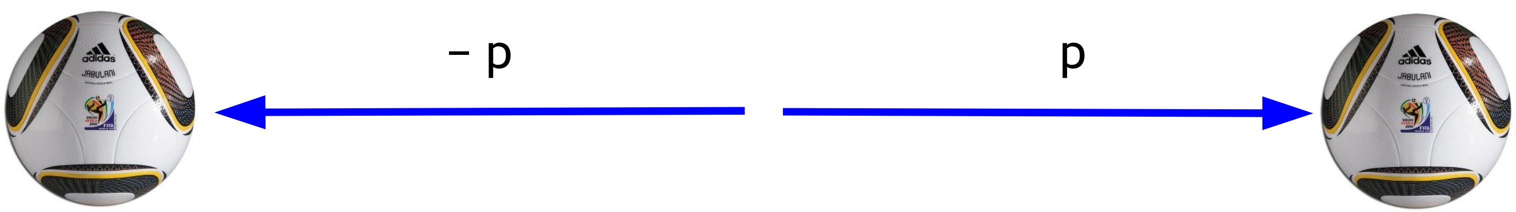


Elastic scattering of **quantum** point-like particles in one dimension

Before collision:



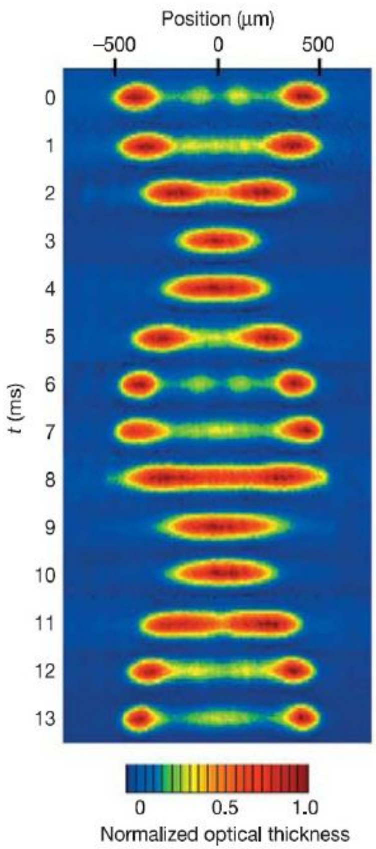
After collision:



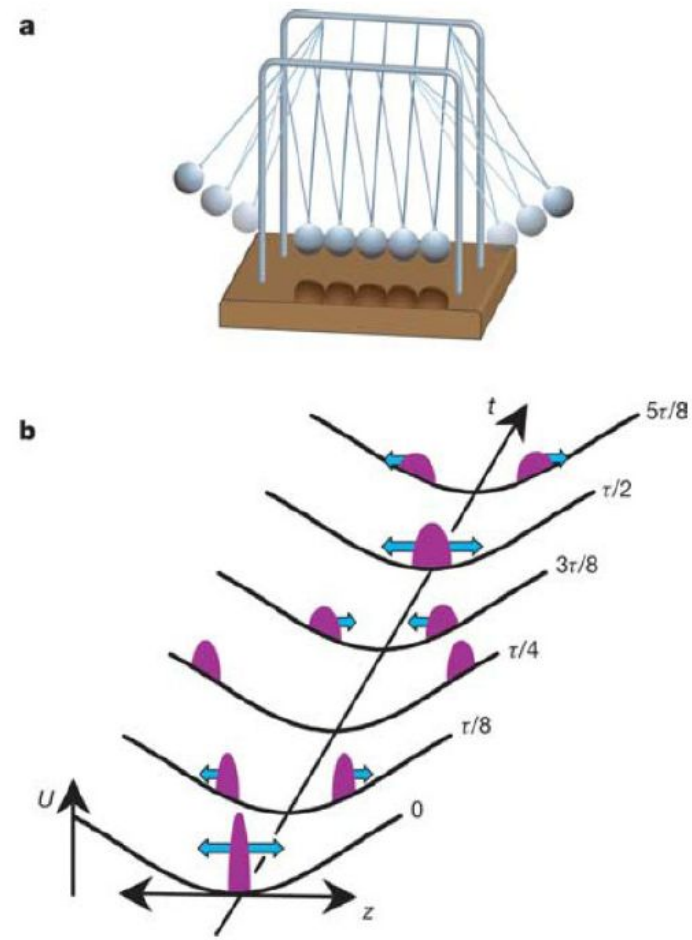
Long-time dynamics of a 1D quantum gas?

A quantum Newton's cradle.

[T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440** (06)]



Indication for strong suppression of damping



Thermalisation of a closed quantum system

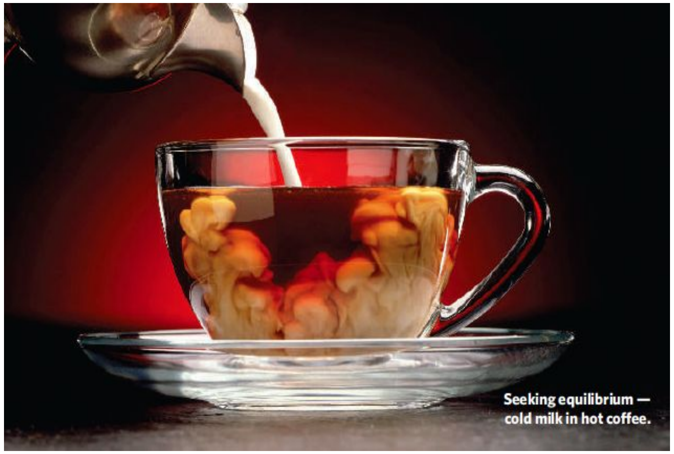
Thermalisation of a classical system

N particles in *d* dimensions: (*2Nd*)-dimensional phase space

Condition for thermalization:

→ Ergodicity

- phase-space averages = time averages
- phase-space trajectories trace out energy hypersurface uniformly



Ergodicity and Thermalisation in a quantum system?

**QM: Dephasing
Recurrence**

Thermalisation of a quantum system

v. Neumann (1929): **“Quantum Ergodicity”**

Def.: $$\hat{\rho}_{\rm mc}(E) = \sum_{\alpha\in\mathcal{H}(E)} 1/\mathcal{N}\,|\Psi_\alpha\rangle\langle\Psi_\alpha|$$

of $\mathcal{N}$ states on Hilbert subspace $\mathcal{H}(E)$ on energy shell between E and $E+\delta E$

However:

From initial state $$|\Psi_0\rangle = \sum_{\alpha\in\mathcal{H}(E)} c_\alpha|\Psi_\alpha\rangle$$

Ergodicity requires $|c_\alpha|^2 = 1/\mathcal{N}$

one obtains, at late times, the time average

$$\overline{|\Psi(t)\rangle\langle\Psi(t)|} = \sum_\alpha |c_\alpha|^2|\Psi_\alpha\rangle\langle\Psi_\alpha| = \hat{\rho}_{\rm diag}$$

(“diagonal ensemble”)

[Rigol et al., PRL 98 (07); Polkovnikov et al., RMP 83 (11)]

Thermalisation of a quantum system

v. Neumann (1929): **Quantum Ergodicity rather for observables!**

Take:

$$\overline{\langle\Psi(t)|M_\beta(t)|\Psi(t)\rangle} = \mathrm{Tr}[M_\beta\hat{\rho}_{\mathrm{diag}}] = \langle M_\beta\rangle_{\mathrm{mc}}$$

for set of macroscopic observables $\{M_\beta\}$ (coarse-grained on $\mathcal{H}(E)$ & commuting)

Eigenstate Thermalisation Hypothesis

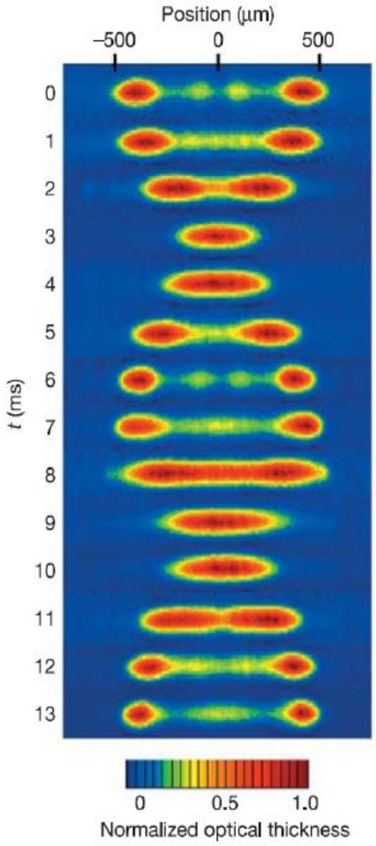
[Deutsch, PRA 43 (91); Srednicki, PRE 50 (94)]

[Polkovnikov et al., RMP 83 (11)]

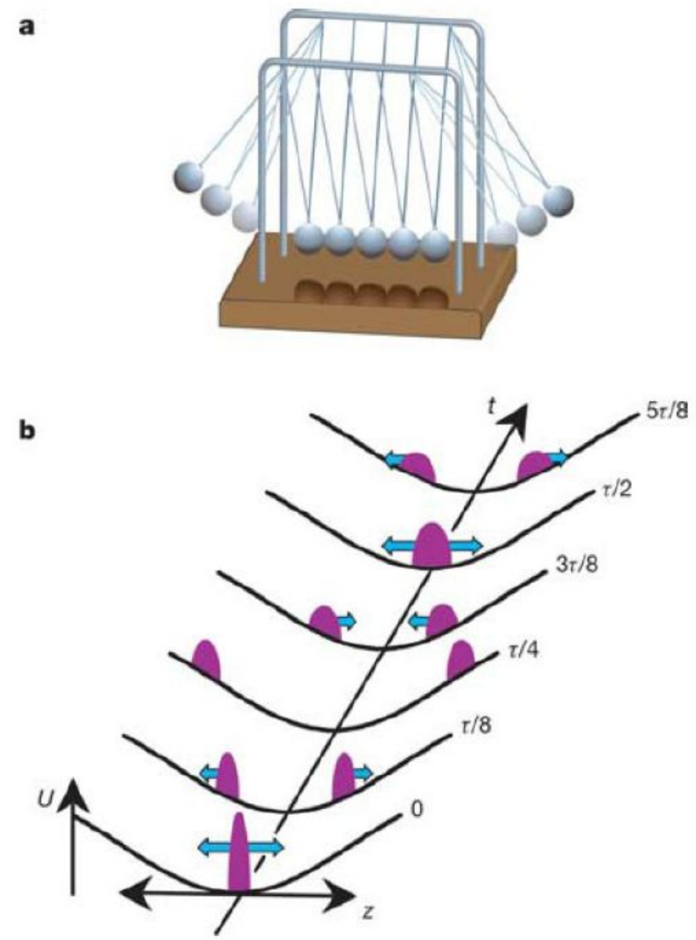
Experiment: Ergodicity is not guaranteed!

– if the closed system is (nearly) integrable.

[T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440** (06)]



Indication for strong suppression of damping



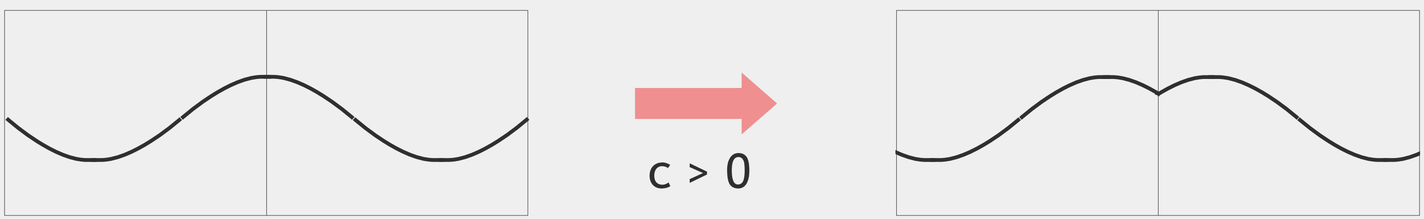
Interacting 1D Bose gas

Lieb-Liniger model: integrable

Lieb-Liniger Bose gas in 1D: [PR **130**, 1605 & 1616 (63)]

$$i\frac{\partial \psi_B}{\partial t} = -\sum_{i=1}^{N} \frac{\partial^2 \psi_B}{\partial x_i^2} + \sum_{1\le i<j\le N} 2c\,\delta(x_i - x_j)\psi_B$$

Solution via Bethe-Ansatz with cusp conditions @ particle contact hyperplanes



$x_i = x_j$ Dirichlet-von Neumann

Dynamics: H. Buljan, R. Pezer, & TG, PRL **100** (2008)

Lieb-Liniger model: integrable

Lieb-Liniger Bose gas in 1D: [PR **130**, 1605 & 1616 (63)]

$$i\frac{\partial\psi_B}{\partial t} = -\sum_{i=1}^{N}\frac{\partial^2\psi_B}{\partial x_i^2} + \sum_{1\le i<j\le N} 2c\,\delta(x_i - x_j)\psi_B$$

Quantum **integrable**:

- Solutions derived by Bethe Ansatz

***N* integrals of motion for *N* particles**

Generalised Gibbs Ensemble (GGE)

Jaynes' maximum-entropy principle [PR **106**, 620 (57)] :

implies **GGE** at $t \to \infty$:

$$\hat{\rho} = Z^{-1} \exp\Big[-\sum_m \lambda_m \hat{I}_m \Big]$$

$Z = \mathrm{Tr}[\exp(-\sum_m \lambda_m \hat{I}_m)]$

$$\mathrm{Tr}[\hat{I}_m \hat{\rho}] = \langle \hat{I}_m \rangle (t=0) \quad \text{fixes} \quad \{\lambda_m\}$$

Maximisation of entropy under constraints for macroscopic observables.

Rigol et al., PRL **98**, 050405 (07), P. Reimann, PRL **101**, 190403 (08).

1D Bose gas

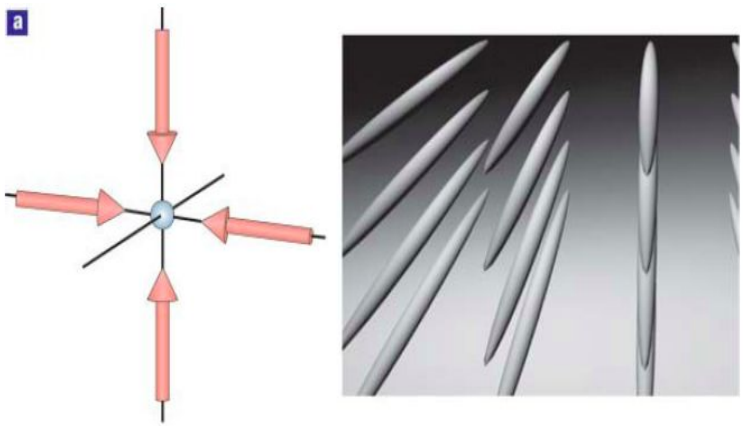
@ low energies

1D Bose gas

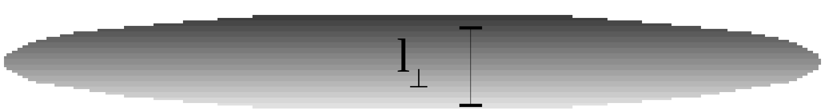
Many-body Hamiltonian:

$$[\Phi_{x,t}, \Phi^\dagger_{x',t}] = \delta(x - x')$$

$$H = \int \mathrm{d}x \left[-\Phi^\dagger_x \frac{\partial_x^2}{2m} \Phi_x + \frac{g}{2} \Phi^\dagger_x \Phi^\dagger_x \Phi_x \Phi_x \right]$$



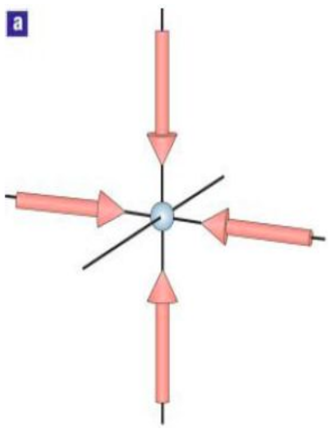
1D Coupling: $g = \dfrac{2a}{m l_\perp^2}$

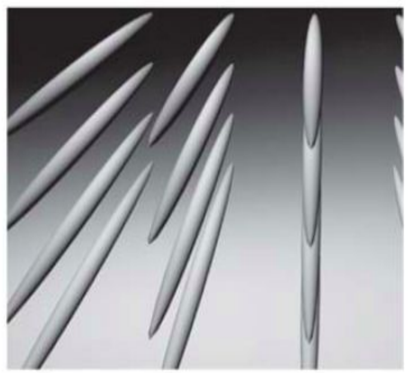


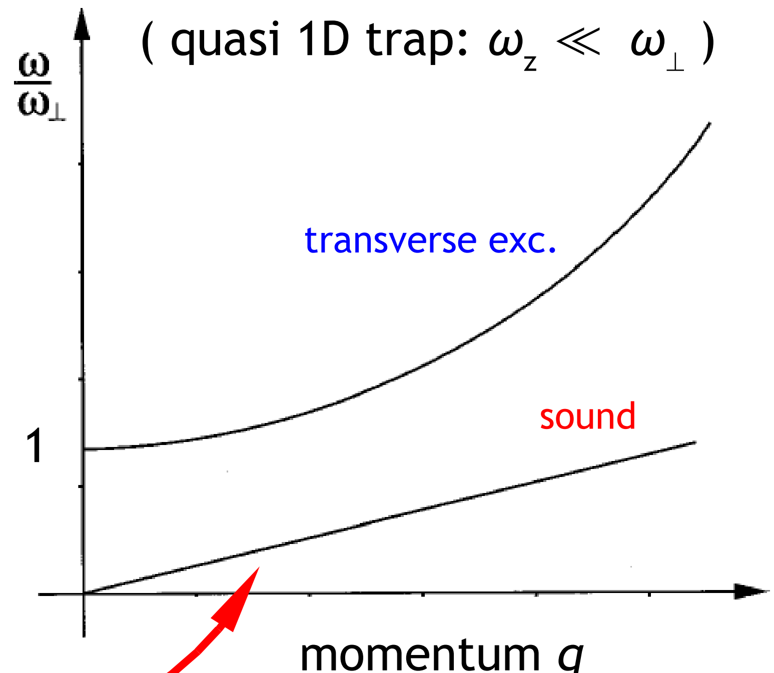
transverse oscillator length

Bogoliubov sound

Linearisation around mean field $\phi_x = \langle \Phi_x \rangle$







sound/particle dispersion:

$$\omega_q^{\,2} = c_s^{\,2} q^{\,2} + (q^2/2m)^2$$

Equilibration from
dynamical Quantum Field Theory

Dynamical Quantum Field Theory

$$\left[\Phi(t,\mathbf{x}),\Phi^\dagger(t,\mathbf{y})\right]_\mp \;=\; \delta(\mathbf{x}-\mathbf{y})$$

Calculate 2-point, time-ordered 2-time Green function:

$$G(x,y) \;=\; \langle \mathcal{T}\Phi^\dagger(x)\Phi(y)\rangle; \qquad\qquad x \;=\; (t,\, \mathbf{x})$$

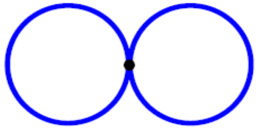
contains single-particle densities

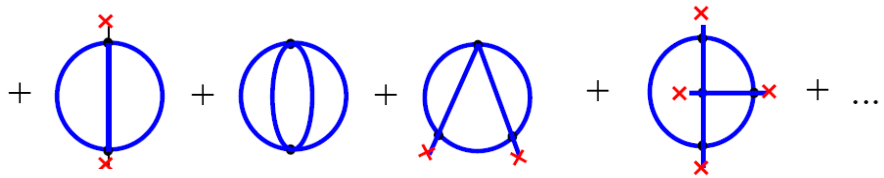
$$\mathrm{n}(t,p) = \int \mathrm{d}r\; G(t,r;\, t,0)\; e^{ipr}$$

2PI Effective Action (Φ-Functional)

[Luttinger, Ward (60); Baym (62); Cornwall, Jackiw, Tomboulis (74)]

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}(\ln G^{-1} + G_0^{-1}(\phi)G) + $$



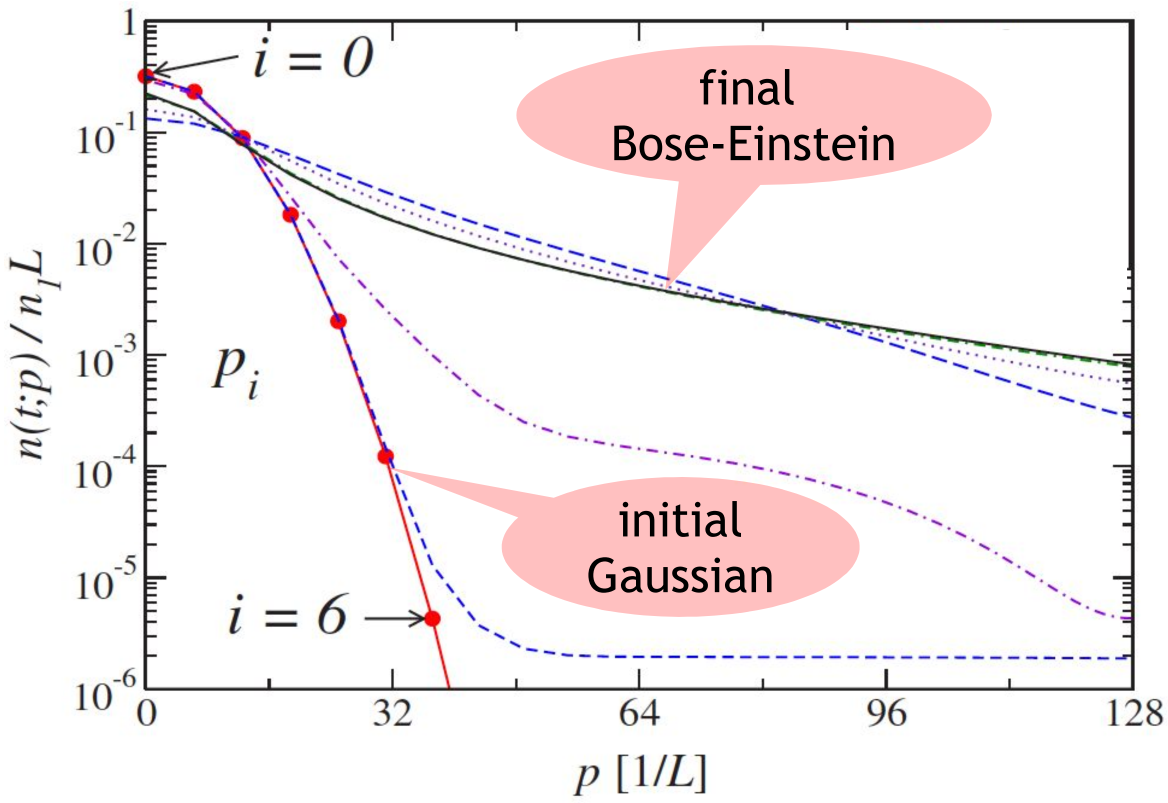


Only 2-particle irreducible (2PI) loop diagrams

Conservation laws fulfilled

Equilibration of the 1D Bose gas

Number of particles n(t,p) with momentum p:

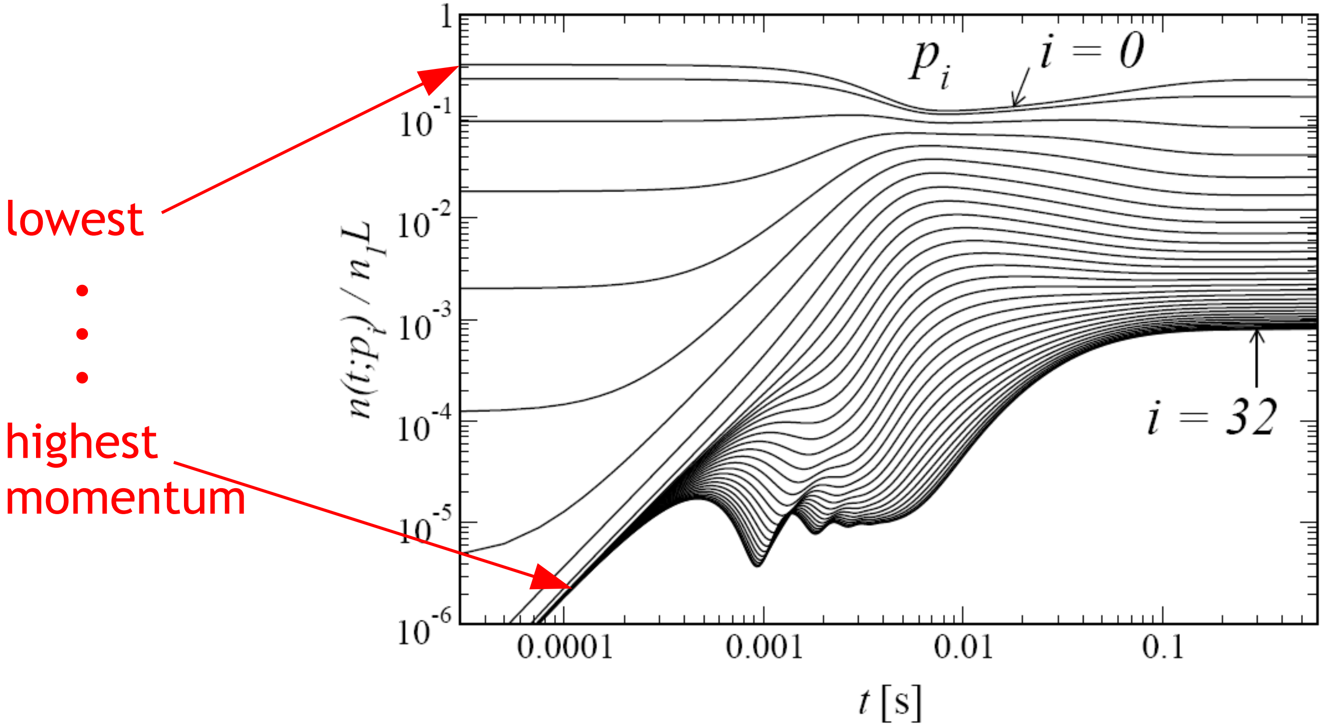


[TG, J. Berges, M. Seco & M.G.Schmidt, PRA **72** (05); J. Berges & TG, PRA **76** (07)]

initial state: ^{23}Na atoms in 1D, $n_1 = 10^7\,\mathrm{m}^{-1}$
interaction parameter $\gamma = \lambda m/(\hbar^2 n_1) = 7.5 \cdot 10^{-4}$

Equilibration of a 1D Bose gas

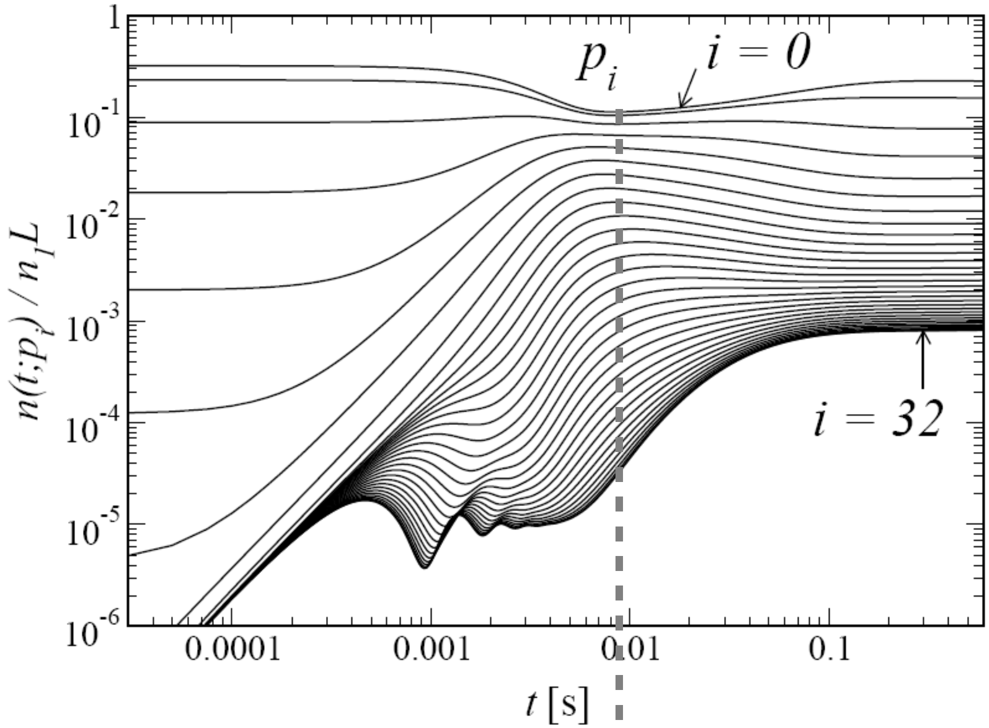
No. of particles $n(t,p_i)$ with momentum p_i:

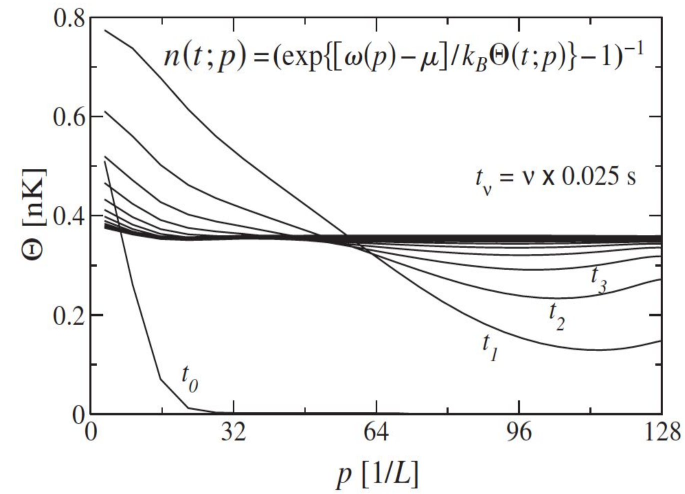


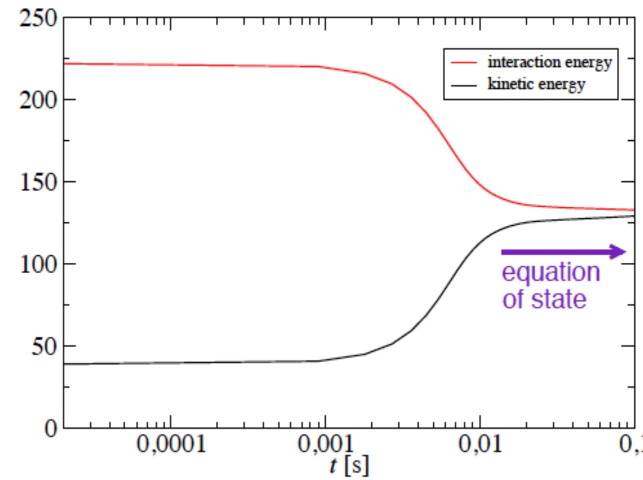
[TG, J. Berges, M.G. Schmidt, and M. Seco, PRA **72** (05); J. Berges & TG, PRA **76** (07)]

Equilibration of a 1D Bose gas

No. of particles $n(t,p_i)$ with momentum p_i:







Far from equilibrium
(no fluct.-dissip. rel.)

Near equilibrium
(fluct.-dissip. rel.)

[TG, J. Berges, M.G. Schmidt, and M. Seco, PRA **72** (05); J. Berges & TG, PRA **76** (07)]

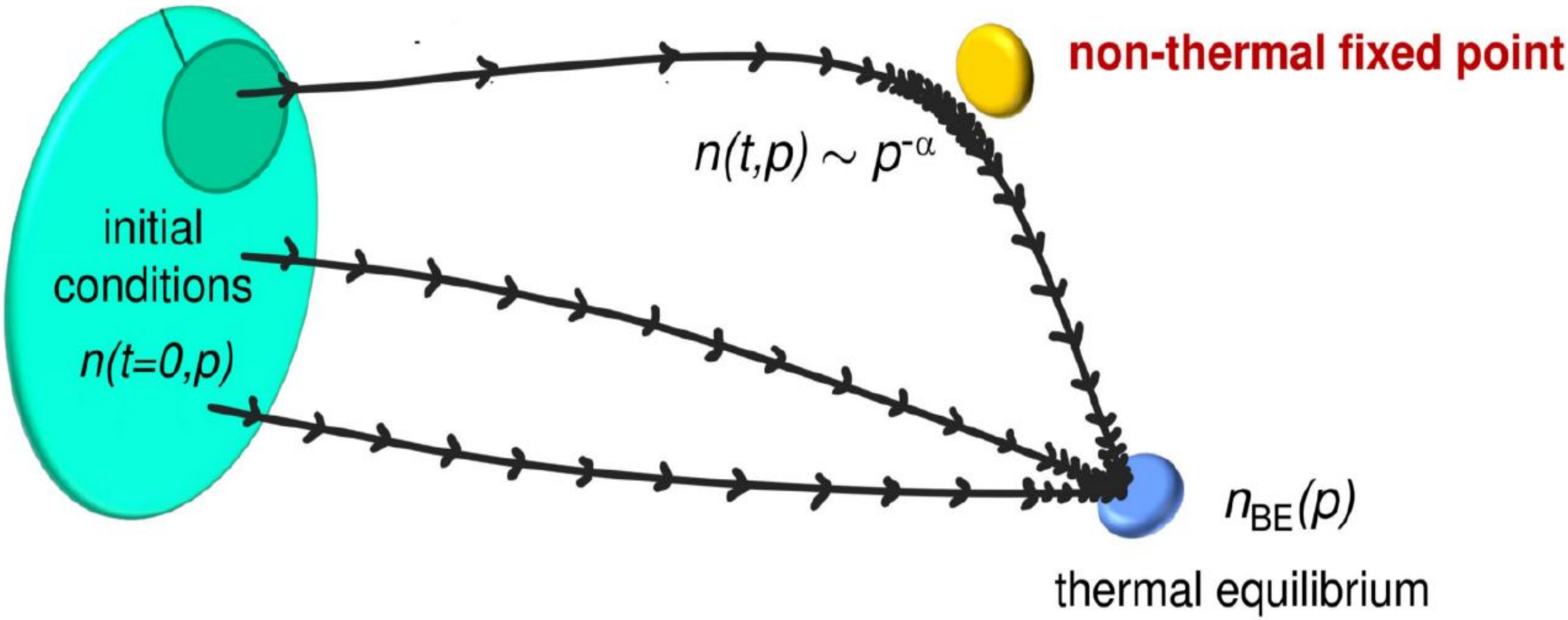
Nonthermal Fixed Points

Equilibration



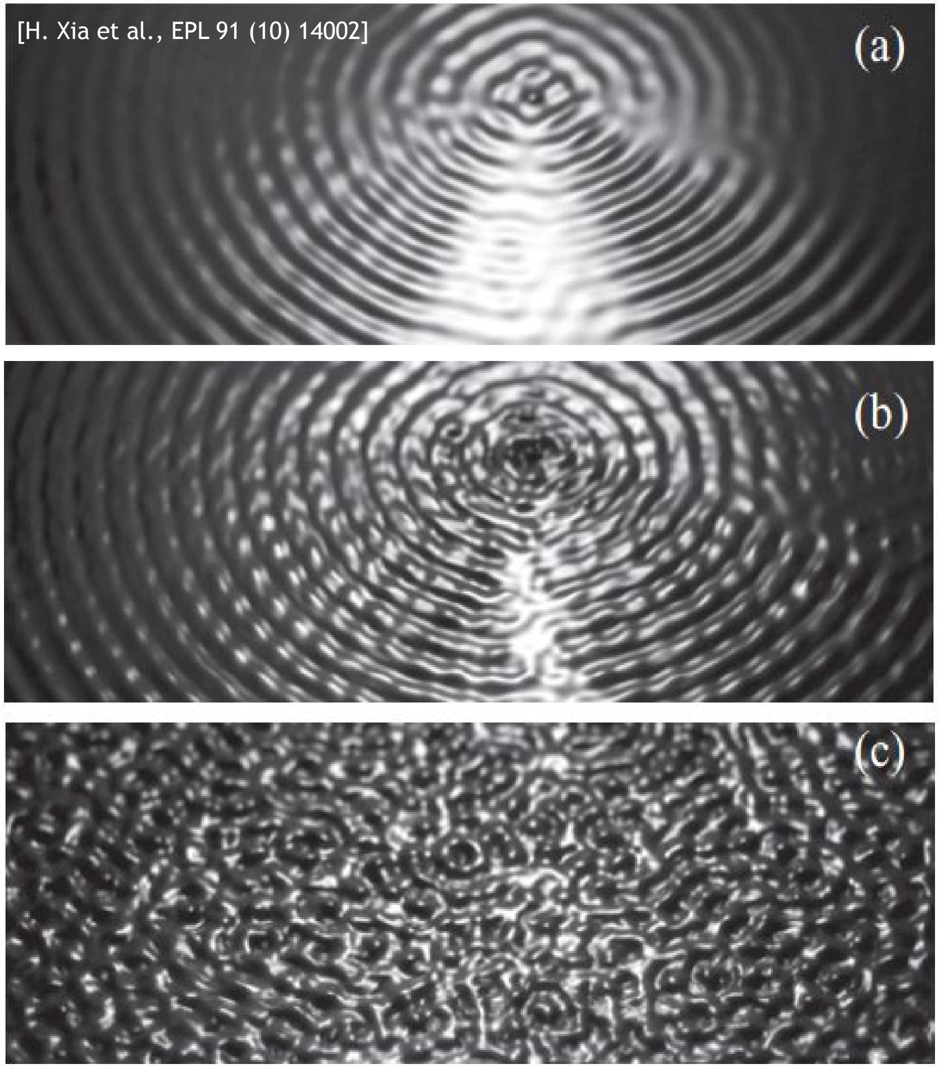
Transient, metastable state
e.g. Turbulence
Non-thermal fixed point

Non-thermal fixed point



[Fig. courtesy: J. Berges '08]

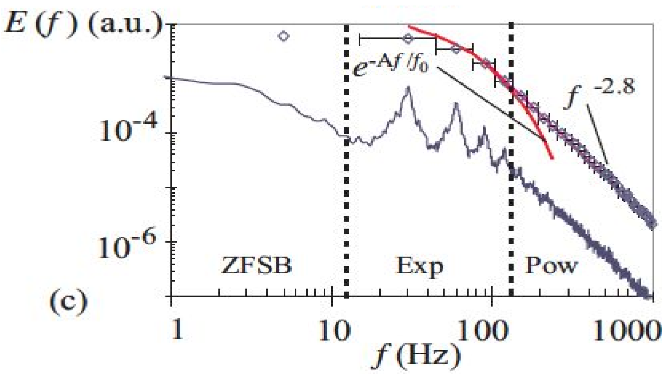
Wave Turbulence – e.g. on water



Theory prediction:

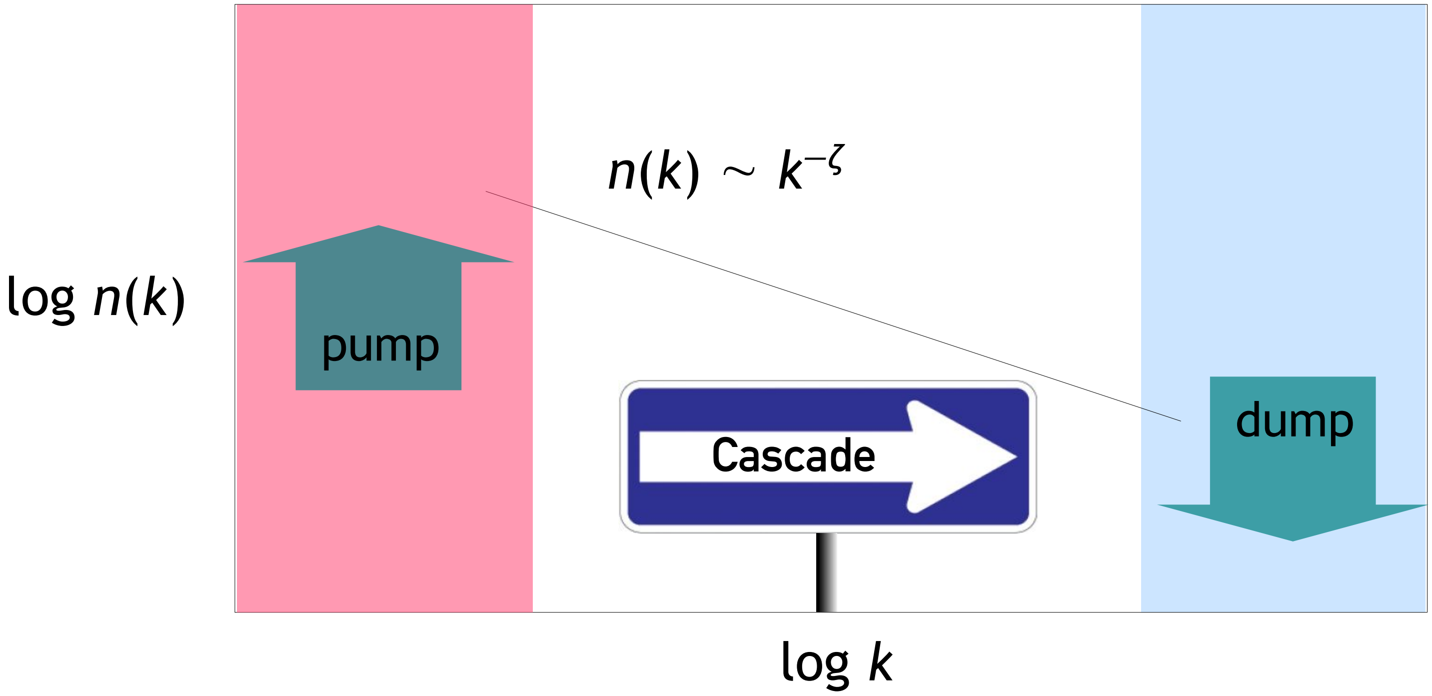
$$E_\omega \sim \omega^{-17/6}.$$

[Zakharov & Filonenko (67)]



Wave turbulence

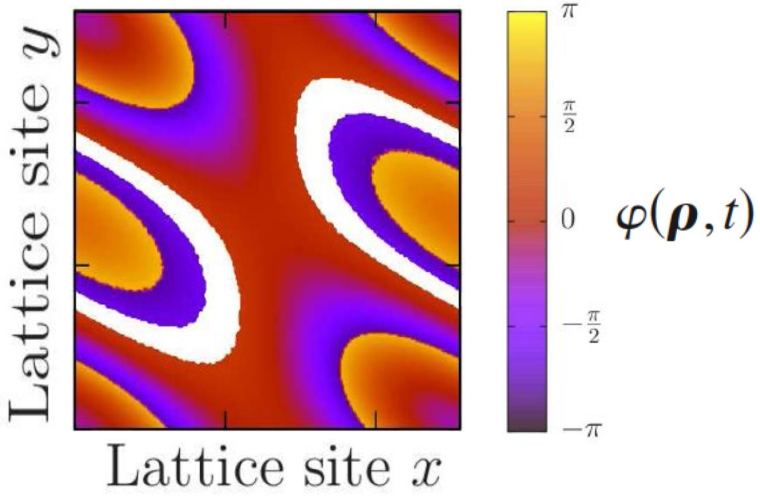
Stationary scaling $n(k)$ within inertial region:

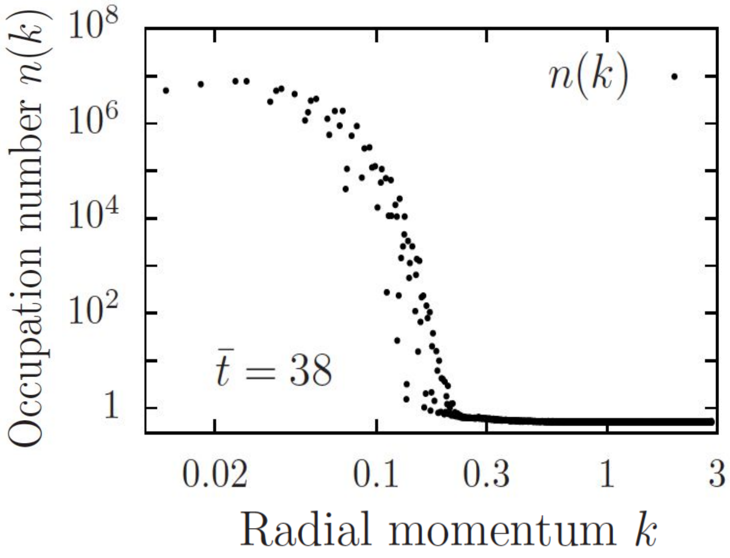


Movie 1: Phase evolution & Spectrum

$$\Psi(\boldsymbol{\rho},t)=\sqrt{n(\boldsymbol{\rho},t)}\exp[i\varphi(\boldsymbol{\rho},t)]$$

$$n(k) = \langle \Psi^*(\mathbf{k})\Psi(\mathbf{k})\rangle\big|_{\text{angle average}}$$



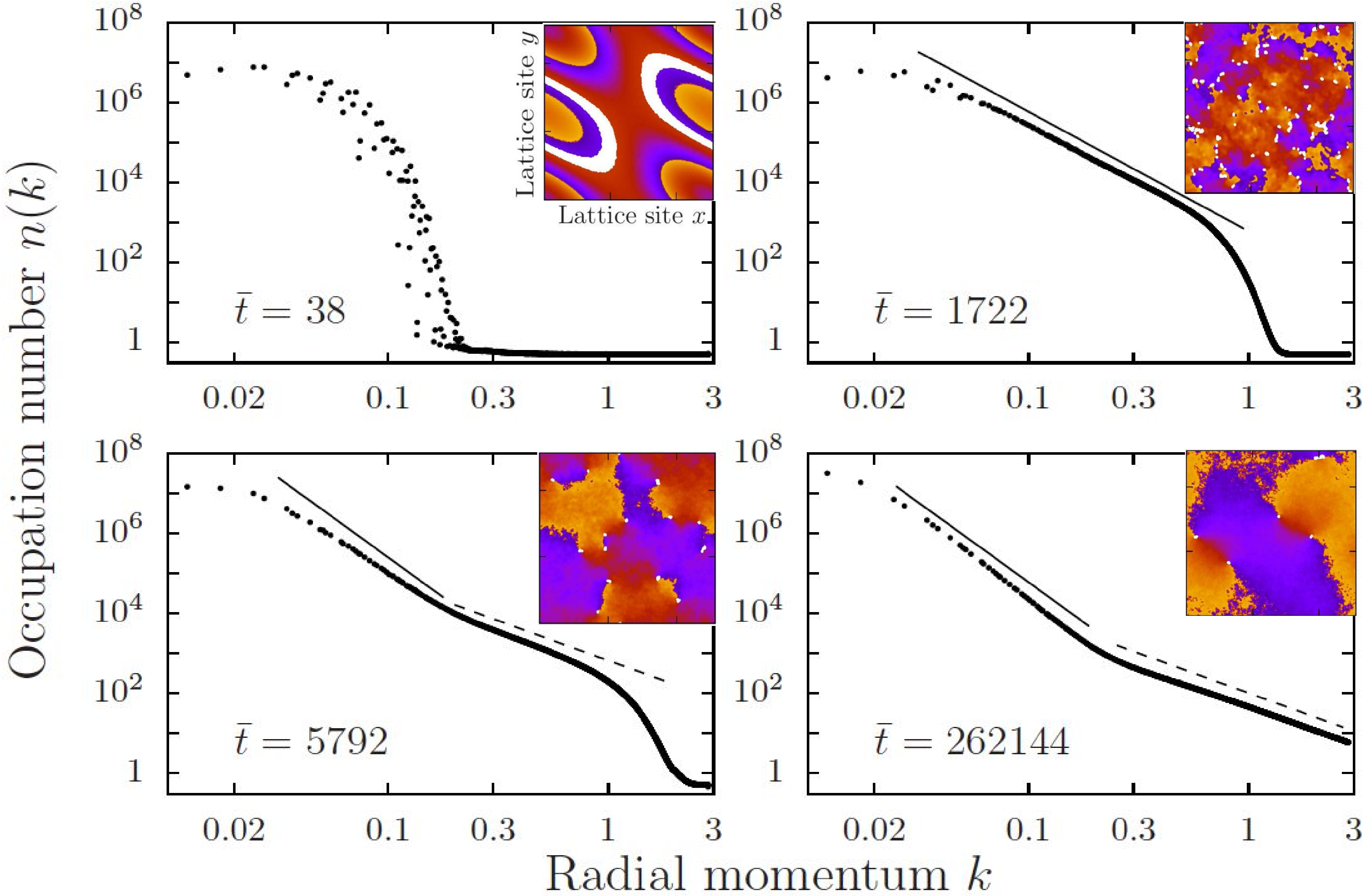


http://www.thphys.uni-heidelberg.de/~smp/gasenzer/videos/boseqt.html

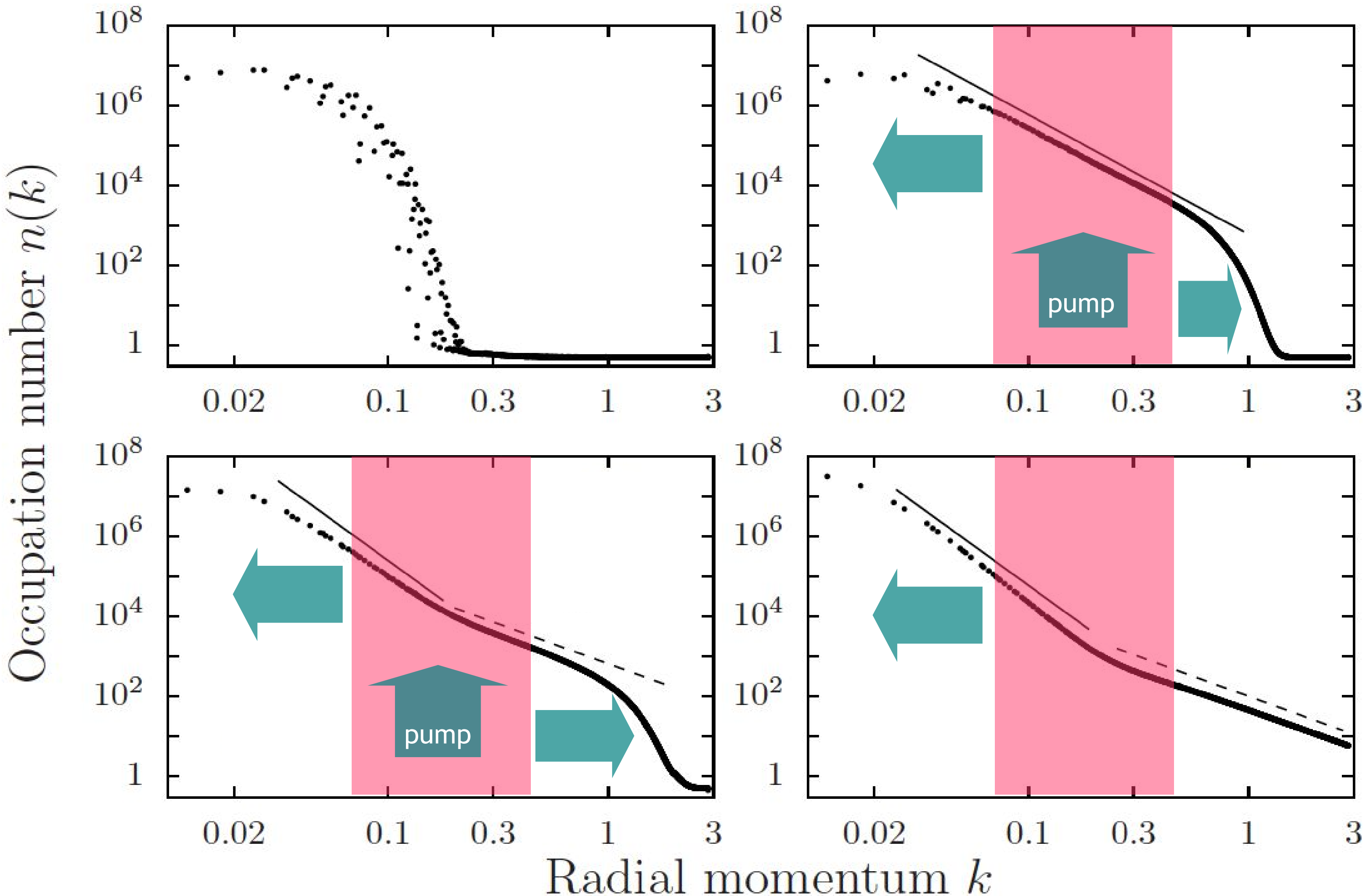
Movie by Jan Schole

B. Nowak, D. Sexty, TG, PRB **84**: 020506(R), 2011

Spectrum in 2+1 D



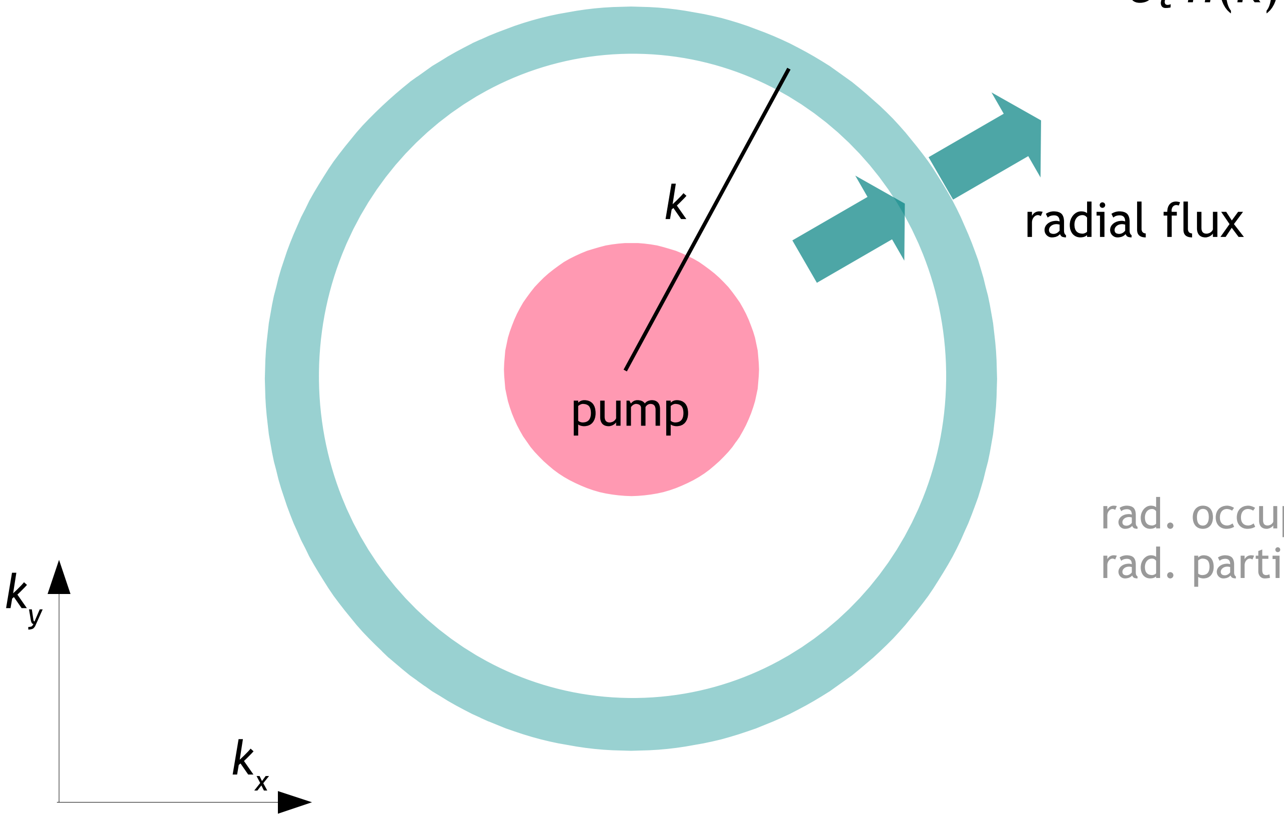
Cascades in 2+1 D



Transport in momentum space

Imagine you had a balance equation for the radial flux

$$\partial_t \, n(k) = - \partial_k Q(k)$$



rad. occupation no. n
rad. particle flux Q

Transport in momentum space

Transport equation (Quantum Boltzmann eq.):

$$\partial_t\, n(k) = -\,\partial_k Q(k) \;\sim\; k^{d-1}\, J(k)$$

$$= k^{d-1}\, \mathrm{d}\Omega_k \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})\, \delta(\omega_\mathbf{k}+\omega_\mathbf{p}-\omega_\mathbf{q}-\omega_\mathbf{r})$$

coupling *mom. conservation* *energy conservation*

$$\times \left[(n_\mathbf{k}+1)(n_\mathbf{p}+1)n_\mathbf{q}n_\mathbf{r} \;-\; n_\mathbf{k}n_\mathbf{p}(n_\mathbf{q}+1)(n_\mathbf{r}+1)\right]$$

in-scattering rate *out-scattering rate*

dilute Bose gas: $T_{\mathrm{kpqr}} \equiv g = 4\pi a_0/m = \mathit{const}.$

Transport in momentum space

Radial transport equation (Quantum Boltzmann):

$$\partial_t n(k) = -\partial_k Q(k) \sim k^{d-1} J(k)$$

$$= k^{d-1} \mathrm{d}\Omega_k \int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}^d r\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})\, \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$

coupling *mom. conservation* *energy conservation*

$$\times \left[(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)\right]$$

in-scattering rate *out-scattering rate*

Stationary distribution $n(k,t) \equiv n(k)$ if $Q(k) \equiv Q$

This requires a particular scaling of $n(k) \sim k^{-\zeta}$

Transport in momentum space

Quantum Boltzmann breaks down for large n, once $|T_{kpqr}|n_k \gg O(1)$

$$\partial_t n(k) = -\partial_k Q(k) \sim k^{d-1} J(k)$$

$$= k^{d-1}\, d\Omega_k \int d^dp\, d^dq\, d^dr\, |T_{\mathbf{kpqr}}|^2 \delta(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})\, \delta(\omega_{\mathbf{k}}+\omega_{\mathbf{p}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$

coupling *mom. conservation* *energy conservation*

$$\times\left[(n_{\mathbf{k}}+1)(n_{\mathbf{p}}+1)n_{\mathbf{q}}n_{\mathbf{r}} - n_{\mathbf{k}}n_{\mathbf{p}}(n_{\mathbf{q}}+1)(n_{\mathbf{r}}+1)\right]$$

in-scattering rate *out-scattering rate*

here: $T_{\mathrm{kpqr}} \equiv g = const.$

Cured by
effective many-body T-Matrix:

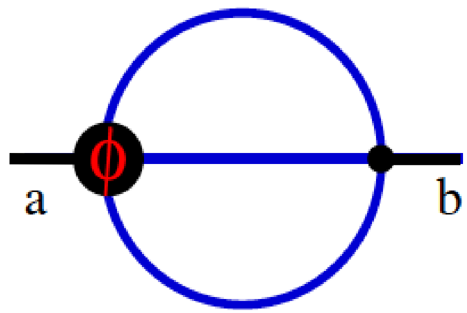
$$|T|^2 = g^2 \;\rightarrow\; |T_k^{MB}|^2 \sim \frac{g^2}{1+(g k n_k)^2}$$

Dyn. QFT: Resummed Vertex

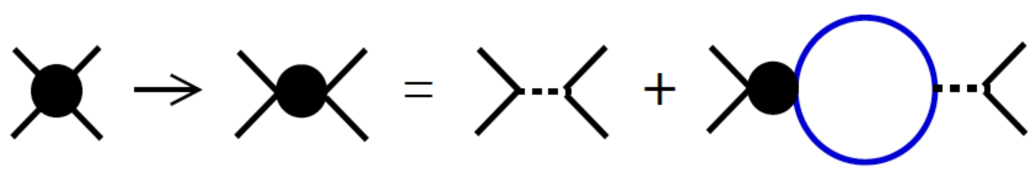
$p = (p_0, \mathbf{p})$:

$$J(p) := \Sigma^{\rho}_{ab}(p) F_{ba}(p) - \Sigma^{F}_{ab}(p)\rho_{ba}(p) \stackrel{!}{=} 0$$

$\Sigma_{ab}(x,y)$ =



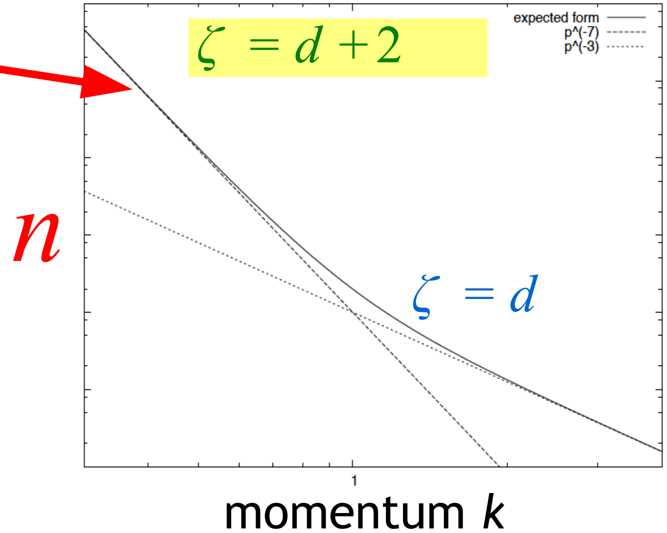
Vertex bubble resummation:
(e.g. 2PI to NLO in $1/N$)



[*Dynamics:* J. Berges, (02); G. Aarts et al., (02);
Nonthermal fixed points: J. Berges, A. Rothkopf, J. Schmidt, PRL (08)]

Bose gas in *d* spatial dimensions $n \sim k^{-\zeta}$

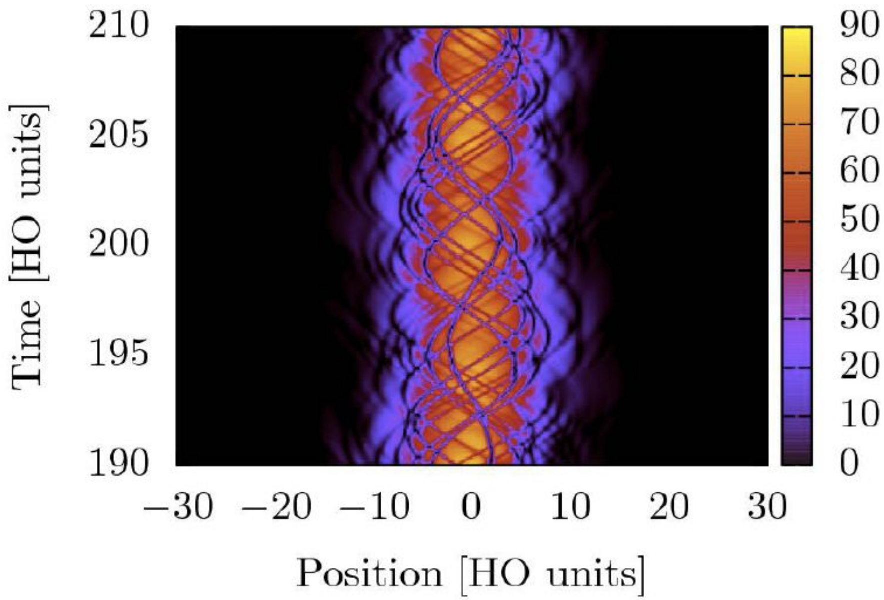
New exponent
beyond
Quantum Boltzmann!

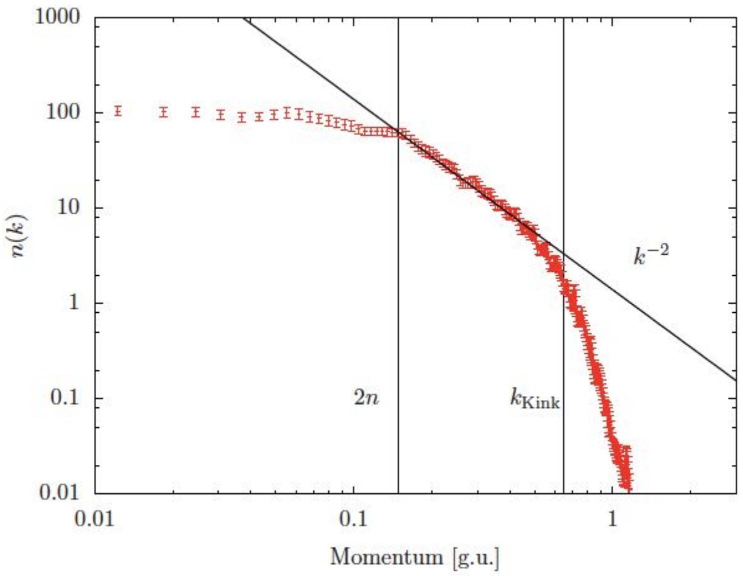


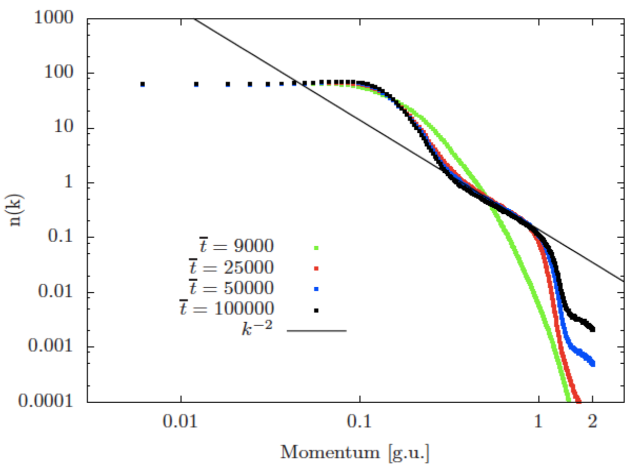


J. Berges, A. Rothkopf, J. Schmidt, PRL **101** (08) 041603; J. Berges, G. Hoffmeister, NPB 813, 383 (2009)
C. Scheppach, J. Berges, TG PRA **81** (10) 033611

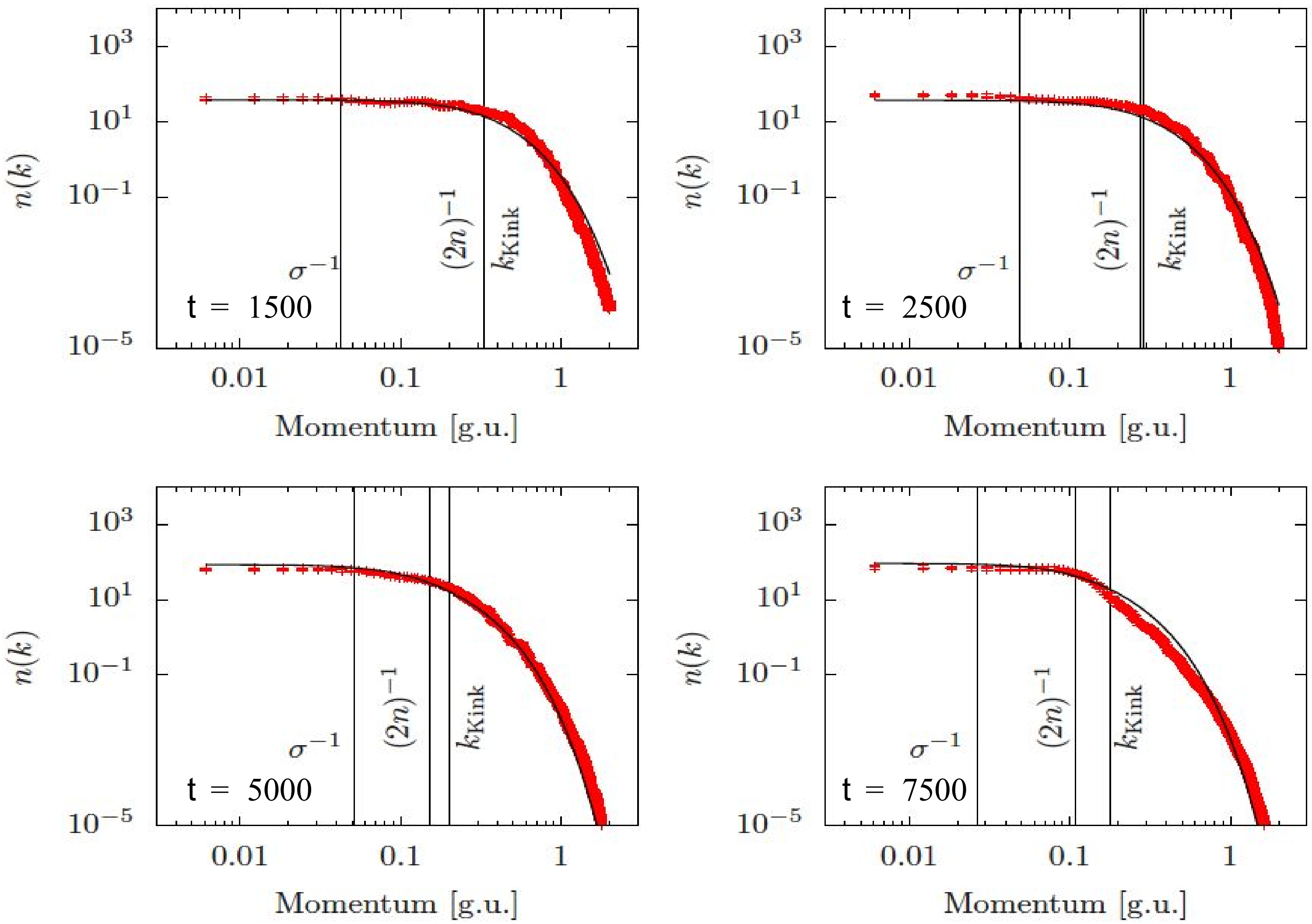
Solitons in 1 spatial dimension



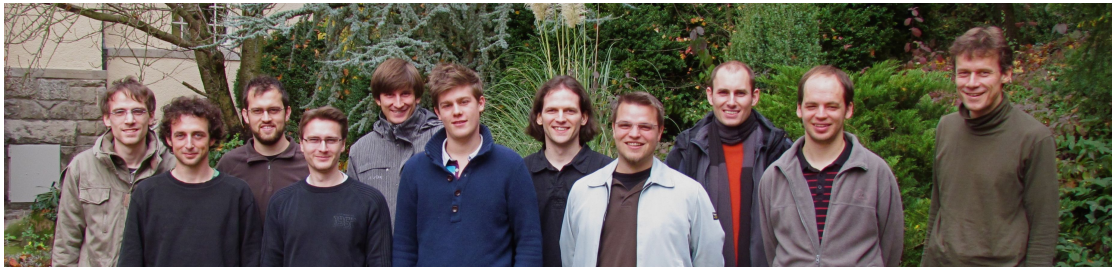




Solitons in 1 spatial dimension



Thanks & credits to...



...my work group in Heidelberg:

Sebastian Bock
Sebastian Erne
Martin Gärttner
Roman Hennig
Markus Karl
Steven Mathey
Boris Nowak
Nikolai Philipp
Maximilian Schmidt
Jan Schole
Dénes Sexty
Martin Trappe
Jan Zill

...my former students:

Cédric Bodet (→ NEC), Alexander Branschädel (→ KIT Karlsruhe), Stefan Keßler (→ U Erlangen), Matthias Kronenwett (→ R. Berger), Christian Scheppach (→ Cambridge, UK), Philipp Struck (→ Konstanz), Kristan Temme (→ Vienna)

€€€...

Alexander von Humboldt Stiftung / Foundation
Heisenberg-Programm
DFG Deutsche Forschungsgemeinschaft
RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG
LGFG BaWue
DAAD Deutscher Akademischer Austausch Dienst German Academic Exchange Service
Center for Quantum Dynamics
EMMI
HELMHOLTZ GEMEINSCHAFT