Lattice Paths and the PASEP

Des Johnston Richard Blythe
Wolfhard Janke Ralph Kenna

Heriot-Watt, Edinburgh, Leipzig, Coventry

Edinburgh, Dec 11

Plan of Talk

Definition of the model

Solution of the model (matrix product ansatz)

Relation to Lattice Path Models - ASEP

Relation to Lattice Path Models - PASEP

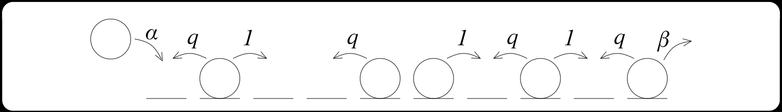
Continued fraction representation of generating function

Definition of the model

One dimensional lattice, N sites

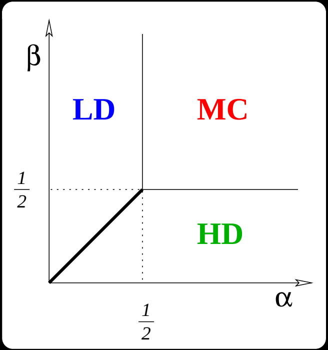
"Hard" particles

Forcing



The Phase Diagram ($q = 0$)

Give away the punchline



Roll of honour (TASEP $q = 0$): Derrida, Evans, Hakim Pasquier
Roll of honour (PASEP $q \neq 0$): Blythe, Evans, Colaiori, Essler

Setup for solution

Configuration, weights

$$\mathcal{C} \;, \quad f(\mathcal{C})$$

Normalized probability

$$P(\mathcal{C}) = f(\mathcal{C})/Z$$

Normalization

$$Z = \sum_{\mathcal{C}} f(\mathcal{C})$$

Master Equation

$$\frac{\partial P(\mathcal{C},t)}{\partial t} = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[P(\mathcal{C}',t) W(\mathcal{C}' \to \mathcal{C}) - P(\mathcal{C},t) W(\mathcal{C} \to \mathcal{C}') \right]$$

Setup for (Matrix Product) solution

Represent ball with

$$X_i = D$$

Represent space with

$$X_i = E$$

Represent $P(\mathcal{C})$ as

$$P(\mathcal{C}) = \frac{\langle W|X_1 X_2 \ldots X_N|V\rangle}{Z_N}$$

Make sure behaviour of D, E is compatible with dynamics:

$$\begin{aligned} DE - qED &= D + E \\ \alpha\langle W|E &= \langle W| \\ \beta D|V\rangle &= |V\rangle \end{aligned}$$

Now what?

Various ways to proceed

Purely algebraic - "normal order"

Get a representation of D, E - not unique (and infinite)

Evaluate $Z_N = \langle W|(D+E)^N|V\rangle = \langle W|C^N|V\rangle$ having done this

Normal Order ($q = 0$)

Look at *Grand*-canonical normalization

$$\mathcal{Z}(z) = \sum_{N=0}^{\infty} Z_N z^N \ .$$

Use the expansion

$$\frac{1}{1 - zC} = \sum_{N=0}^{\infty} z^N C_N$$

Note that

$$(1 - \eta D)(1 - \eta E) = 1 - \eta(1 - \eta)C$$

Giving (Depken)

$$\frac{1}{1 - zC} = \frac{1}{1 - \eta E} \frac{1}{1 - \eta D}$$

Normal Order II ($q = 0$)

Take

$$\frac{1}{1 - zC} = \frac{1}{1 - \eta E} \frac{1}{1 - \eta D}$$

Sandwich

$$\mathcal{Z}(z) = \sum_{N=0}^{\infty} Z_N z^N = \sum_{N=0}^{\infty} \langle W | C^N z^N | V \rangle = \langle W | \frac{1}{1 - \eta E} \frac{1}{1 - \eta D} | V \rangle$$

Giving

$$\mathcal{Z}(z) = \frac{1}{(1 - \eta(z)/\alpha)} \frac{1}{(1 - \eta(z)/\beta)}$$

Where

$$\eta(z) = \frac{1}{2}(1 - \sqrt{1 - 4z})$$

Extracting Z_N from $\mathcal{Z}(z)$

Singularities of $\mathcal{Z}(z) \rightarrow$ *asymptotics* of the Z_N

In MC phase

$$Z_N \sim \frac{4^N}{\pi^{1/2} N^{3/2}} \left[\frac{1}{(2\alpha - 1)^2} - \frac{1}{(2\beta - 1)^2} \right]$$

In regions LD and HD poles dominate

$$Z_N \sim \frac{\alpha(1 - 2\beta)}{(\alpha - \beta)(1 - \beta)} \frac{1}{(\beta(1 - \beta))^N}$$

Comment I

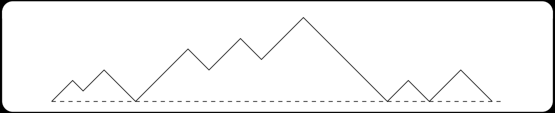
If we expand $\mathcal{Z}(z)$ we recover Z_N:

$$Z_N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{(1/\beta) - (1/\alpha)}$$

Comment II

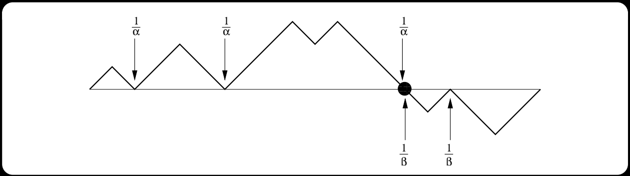
$\eta(z) = \frac{1}{2}(1 - \sqrt{1-4z})$ is the generating function for Catalan numbers

Also counts *Dyck* paths



Generating Function of What?

$\mathcal{Z}(z)$ is an elaboration of $\eta(z)$



Get generating function by iterating

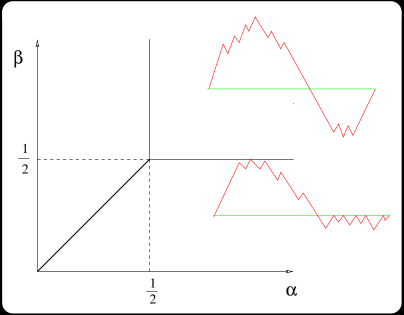
$$\eta(z) = z\left(1 + \eta(z) + [\eta(z)]^2 + \cdots\right) = \frac{z}{1 - \eta(z)}$$

Then iterate again with contact weights $1/\alpha$

$$\mathcal{Z}(z) = \frac{1}{(1 - \eta(z)/\alpha)} \frac{1}{(1 - \eta(z)/\beta)}$$

The Phase Diagram: (Sticky) Lattice Paths

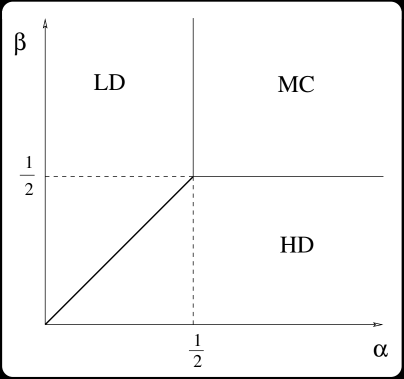
Large α, β: unbound paths



Smaller α or β: paths “stick”

The Phase Diagram (ASEP)

Large α, β: Maximal Current Phase



Smaller α or β: High and Low density phases

Half Time Conclusions

Non-equilibrium ASEP $==$ equilibrium lattice (Dyck) path model

Grand-canonical normalization (ASEP) is identical to partition function (paths)

PASEP?

Not so obvious, go the representation route....

(Transfer) Matrix Product Ansatz

Consider $D + E$ as a transfer matrix for paths

Vectors give.....

starting heights: $\langle W| = (1, 0, 0, \cdots)$

and finishing heights: $|V\rangle = (1, 0, 0, \cdots)^T$

Matrices for the ASEP (q=0)

$$
\begin{aligned}
D &= \begin{pmatrix} 1+\frac{1}{\beta} & \sqrt{\kappa} & 0 & \cdots \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} , \\
E &= \begin{pmatrix} 1+\frac{1}{\alpha} & 0 & 0 & \cdots \\ \sqrt{\kappa} & 1 & 0 & \\ 0 & 1 & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix} .
\end{aligned}
$$

$$
\langle W| = (1,0,0,\cdots) \qquad |V\rangle = (1,0,0,\cdots)^T
$$

$$
\kappa = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}
$$

And the PASEP? ($q \neq 0$)

$$D_q = \frac{1}{1-q} \begin{pmatrix} 1+\tilde{\beta} & \sqrt{c_1} & 0 & \cdots \\ 0 & 1+\tilde{\beta}q & \sqrt{c_2} & \\ 0 & 0 & 1+\tilde{\beta}q^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix},$$

$$E_q = \frac{1}{1-q} \begin{pmatrix} 1+\tilde{\alpha} & 0 & 0 & \cdots \\ \sqrt{c_1} & 1+\tilde{\alpha}q & 0 & \\ 0 & \sqrt{c_2} & 1+\tilde{\alpha}q^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

$$\langle W| = h_0^{1/2}(1,0,0,\cdots) \qquad |V\rangle = h_0^{1/2}(1,0,0,\cdots)^T$$

Definitions

$$\begin{aligned}\tilde{\alpha} &= \frac{1-q}{\alpha} - 1 \qquad \tilde{\beta} = \frac{1-q}{\beta} - 1 \\ c_n &= (1-q^n)(1-\tilde{\alpha}\tilde{\beta}q^{n-1}) \\ h_0 &= \frac{1}{(\tilde{\alpha}\tilde{\beta};q)_\infty}\end{aligned}$$

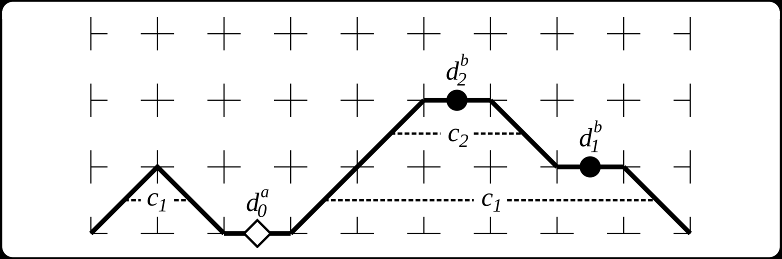
$$\begin{aligned}(a;q)_n &= \prod_{j=0}^{n-1}(1-aq^j) \\ (a,b,\ldots,c;q)_n &= (a;q)_n(b;q)_n\ldots(c;q)_n\end{aligned}$$

C as a transfer matrix...

$$C_q = \frac{1}{1-q}\begin{pmatrix} 2+\tilde{\alpha}+\tilde{\beta} & \sqrt{c_1} & 0 & \cdots \\ \sqrt{c_1} & 2+(\tilde{\alpha}+\tilde{\beta})q & \sqrt{c_2} & \\ 0 & \sqrt{c_2} & 2+(\tilde{\alpha}+\tilde{\beta})q^2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$\langle W| = h_0^{1/2}(1,0,0,\cdots) \qquad |V\rangle = h_0^{1/2}(1,0,0,\cdots)^T$$

Bicoloured *Motzkin* Paths



$$\tilde{d}_n = \frac{2 + (\tilde{\alpha} + \tilde{\beta})\, q^n}{1 - q} = \frac{1 + \tilde{\alpha} q^n}{1 - q} + \frac{1 + \tilde{\beta} q^n}{1 - q}$$

$$\tilde{c}_n = \frac{(1 - q^n)(1 - \tilde{\alpha}\tilde{\beta}\, q^{n-1})}{(1 - q)^2}$$

Generating Function as a Continued Fraction

$$\mathcal{Z}(z) = \sum_N Z_N z^N$$

$$\mathcal{Z}(\tilde{\alpha}, \tilde{\beta}, q, z) = \cfrac{1}{1 - \tilde{d}_0 z - \cfrac{\tilde{c}_1 z^2}{1 - \tilde{d}_1 z - \cfrac{\tilde{c}_2 z^2}{1 - \tilde{d}_2 z - \cfrac{\tilde{c}_3 z^2}{\dots}}}}$$

Recover ASEP

$$\mathcal{Z}(\tilde{\alpha},\tilde{\beta},0,z)=\cfrac{1}{1-(\alpha^{-1}+\beta^{-1})z-\cfrac{\kappa z^2}{1-2z-\cfrac{z^2}{1-2z-\cfrac{z^2}{1-2z-\cfrac{z^2}{\dots}}}}}$$

$$\mathcal{Z}(\tilde{\alpha},\tilde{\beta},0,z)=\frac{1}{(1-\eta(z)/\alpha)}\frac{1}{(1-\eta(z)/\beta)}$$

Why Motzkin rather than Dyck?: $D+E=DE$

Singularities of Continued Fractions in general - PASEP

generic, poles

Worpitzsky: A continued fraction of the form

$$\cfrac{1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{\ldots}}}}$$

converges if the partial numerators a_p satisfy

$$|a_p| < 1/4,\ p = 2, 3, 4, \ldots$$

Singularities of Continued Fractions II

Worpitzsky translates to

$$\frac{4\tilde{c}_n z_{cr}^2}{(1-\tilde{d}_{n-1}z_{cr})(1-\tilde{d}_n z_{cr})} = 1 \ \ \forall n$$

This the maximal current phase

$$z_{cr} \to (1-q)/4$$

Hunting Poles

Generic, poles
Look at $n^{\rm th}$ *convergent* of the continued fraction

$$\begin{aligned} K_0 &= \frac{1}{1-\tilde{d}_0 z}\,, \\ K_1 &= \frac{1}{1-\tilde{d}_0 z - \frac{\tilde{c}_1 z^2}{1-\tilde{d}_1 z}}\,, \end{aligned}$$

Continued fraction is given *exactly* by the convergent K_n if $\tilde{c}_{n+1} = 0$.

$$\tilde{\alpha}\tilde{\beta} = q^{-n}\,.$$

Hunting Poles II

Each convergent K_n can be expressed as

$$K_n = \frac{B_n}{A_n}$$

A_n and B_n both satisfy

$$\begin{aligned} A_n(z) &= (1 - \tilde{d}_n z)A_{n-1}(z) - \tilde{c}_n z^2 A_{n-2}(z) \, , \\ B_n(z) &= (1 - \tilde{d}_n z)B_{n-1}(z) - \tilde{c}_n z^2 B_{n-2}(z) \, , \end{aligned}$$

Different initial conditions: $A_{-2} = B_{-1} = 0$ and $A_{-1} = B_0 = 1$.

Finding Poles

Three term recurrence $\rightarrow$ orthogonal polynomials

Second nastiest q-orthogonal polynomials

Al-Salam-Chihara polynomials

$$A_n(z) = \frac{(\tilde{\alpha}\tilde{\beta};q)_{n+1} z^{n+1}}{(1-q)^{n+1}\tilde{\alpha}^{n+1}} \sum_{k=0}^{n+1} \frac{(q^{-(n+1)};q)_k (\tilde{\alpha}\, e^{i\theta};q)_k (\tilde{\alpha}\, e^{-i\theta};q)_k}{(\tilde{\alpha}\tilde{\beta};q)_k (q;q)_k} q^k$$

where $\cos(\theta) = (1-q)/(2z) - 1$.

Finding Poles II

When $\tilde{c}_{n+1} = 0$, *i.e.* $\tilde{\alpha}\tilde{\beta} = q^{-n}$.

$A_n(z) \to \mathcal{A}_n(z)$

$$\mathcal{A}_n(z) = \prod_{k=0}^{n} \left(1 - \frac{z}{z_k}\right)$$

$$z_k = \frac{1-q}{\left(1+\tilde{\alpha}q^k\right)\left(1+\frac{1}{\tilde{\alpha}q^k}\right)} .$$

Remarkable property: all poles of $\mathcal{A}_n(z)$ are poles of $\mathcal{A}_m(z)$ where $m > n$

Generically not the case

Counting alchohols

Counting alchohols II

Polya - chemical isomeres of $C_n H_{2n+1} OH$ without asymmetric carbon atoms

$$\cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z^2}{1 - \cfrac{z^4}{1 - \cfrac{z^6}{\cdots}}}}}$$

$$\begin{aligned} z &= 1 \\ & \quad 0.61803 \\ & \quad 0.5950892980 \\ & \quad \ldots \\ & \quad 0.594883.. \end{aligned}$$

Phases of the PASEP

Zeroth convergent

$$K_0 = \frac{1}{1 - \tilde{d}_0 z}$$

Pole at $z_0 = 1/\tilde{d}_0$

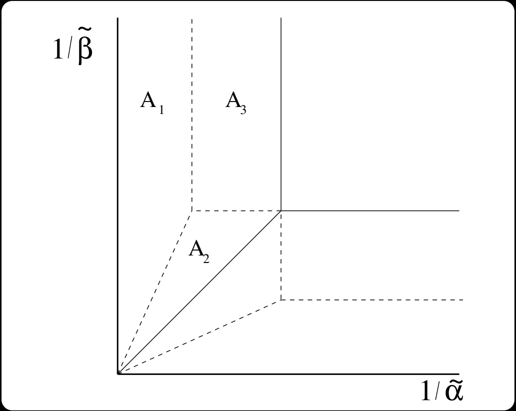
Stays a pole for all K_N (this is why mean field gets phase transition lines correct)

Gives HD, LD phases unless generic singularity is closer

Observations

z_0 and $\alpha \leftrightarrow \beta$ give $HD\ LD$, phases

Ratio of leading/subleading gives correlation lengths



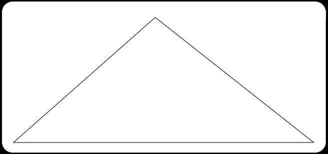
More Observations

$q \to 1$ is an inflation transition

Asymptotics for $q > 1$

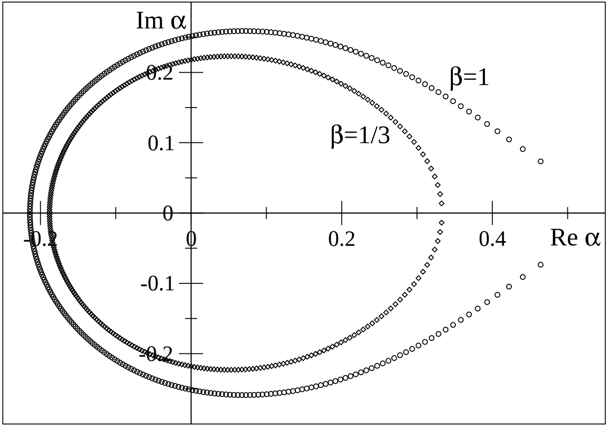
$$Z_N \sim Z_N \sim A(\tilde{\alpha}, \tilde{\beta}; q)\, (q^{-1}\tilde{\alpha}\tilde{\beta}, 1/\tilde{\alpha}\tilde{\beta}; q^{-1})_\infty \left(\frac{\sqrt{\tilde{\alpha}\tilde{\beta}}}{q-1} \right)^N q^{\frac{1}{4}N^2}$$

Given by “tent” paths



Yet More Observations

Lee-Yang/Fisher zeros for ASEP normalization



Even more Observations

$\tilde{\alpha}\tilde{\beta} = q^{-n}$ is condition for n-dimensional reps.

$$
\begin{aligned}
K_0 &= \frac{1}{1 - \tilde{d}_0 z} \; , \\
K_1 &= \frac{1}{1 - \tilde{d}_0 z - \frac{\tilde{c}_1 z^2}{1 - \tilde{d}_1 z}} \; ,
\end{aligned}
$$

No unbound paths $\rightarrow$ No MC phase

Full Time Conclusions

Non-equilibrium ASEP $==$ equilibrium lattice (Dyck) path model

PASEP and ASEP $==$ Motzkin path model

Casts light on the applicability of "partition function" zeroes, finite dimensional reps, why mean field works well....

Gets the combinatorialists interested

General feature?

References

The Grand-Canonical Asymmetric Exclusion Process and the One-Transit Walk, J. Stat. Mech. P06001 (2004)

Dyck Paths, Motzkin Paths and Traffic Jams, J. Stat. Mech. P10007 (2004)

Continued Fractions and the Partially Asymmetric Exclusion Process, J. Phys. A 325002 (2009)