Domino tilings, non-intersecting Random Motions and Critical Processes

Pierre van Moerbeke

Université de Louvain, Belgium & Brandeis University, Massachusetts

Institute of Physics, University of Edinburgh (Dec 2011)

1. The Airy process

2. The Pearcey process

3. The Tacnode process

4. Domino tilings of Aztec Diamonds

5. Domino tilings of Double Aztec Diamonds

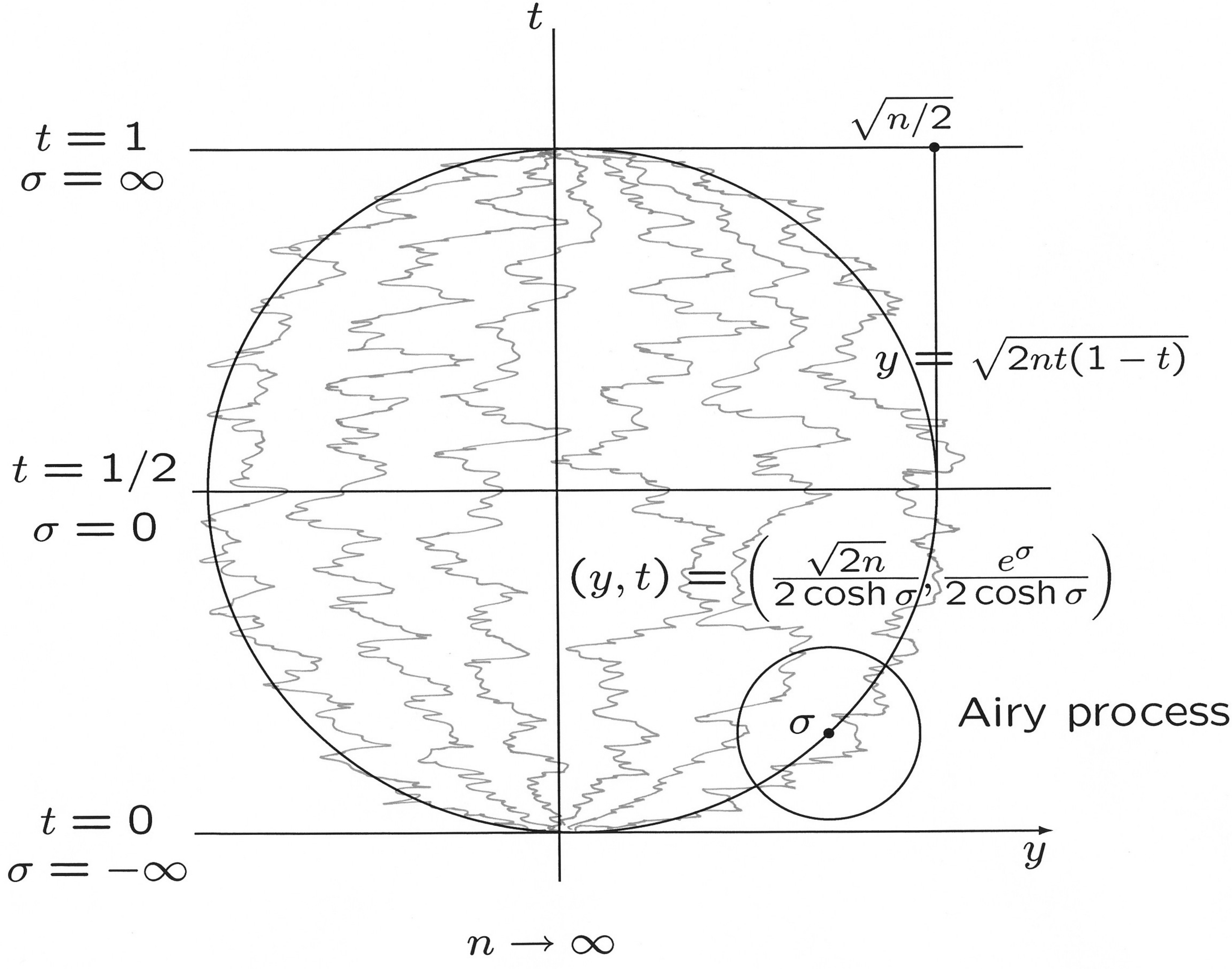
I. The Airy process $A(\tau)$

Non-intersecting Brownian motions $\begin{cases} \text{leaving from 0 at } t=0 \\ \text{returning to 0 at } t=1 \end{cases}$

(Equivalent with Dyson's Brownian motions)

Let $n \to \infty$: Infinite-dimensional diffusion !

(Prähofer-Spohn '02, Johansson '03, '05, Tracy-Widom '04, Adler-PvM '05),


t
$t = 1$
$\sigma = \infty$
$\sqrt{n/2}$
$y = \sqrt{2nt(1-t)}$
$t = 1/2$
$\sigma = 0$
$(y,t) = \left(\frac{\sqrt{2n}}{2\cosh\sigma}, \frac{e^{\sigma}}{2\cosh\sigma}\right)$
σ
Airy process
$t = 0$
$\sigma = -\infty$
y
$n \to \infty$

Look through an "Airy" microscope at **the fluctuations of the paths** about any point on the curve, for very large n:

$$(x,t) = \left(\frac{\sqrt{2n}}{2\cosh\sigma}, \quad \frac{e^{\sigma}}{2\cosh\sigma}\right) \in \quad \left\{\text{curve: } x = \sqrt{2nt(1-t)}\right\}$$

Markov cloud $\mathcal{A}(\tau)$ in the neighborhood of this point $(x,t) \in$ curve :

Pick times $0 < \tau_1 < \ldots < \tau_m < 1$ and points $\xi_1, \ldots, \xi_m \in \mathbb{R}$.

$$\lim_{n\to\infty} \mathbb{P}_{Br}\left(\bigcap_{k=1}^{m}\left\{\text{all } x_i\left(\frac{e^{\sigma+\tau_k n^{-1/3}}}{2\cosh(\sigma+\tau_k n^{-1/3})}\right) \le \frac{\sqrt{2n} + \frac{\xi_k}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau_k n^{-1/3})}\right\}\right)$$

$$=: \det\left(\mathbb{I} - (\chi_{(\xi_i,\infty)} K^{\mathcal{A}}(\tau_i,\xi_i;\tau_j,\xi_j)\chi_{(\xi_j,\infty)})_{1\le i,j\le m}\right), \qquad \text{independent of } \sigma!!!$$

$$= \mathbb{P}\left(\mathcal{A}(\tau_1) \le \xi_1, \ldots, \mathcal{A}(\tau_m) \le \xi_m\right) \qquad \text{(Universality!)}$$

Airy process

with kernel: (matrix Fredholm determinant !!)

$$K^{\mathcal{A}}(\tau_i,\xi_i;\tau_j,\xi_j) := \frac{1}{(2\pi i)^2} \int_{\nwarrow\nearrow} dU \int_{\searrow\nwarrow} dV \, \frac{e^{\frac{1}{3}U^3-\xi_i U}}{e^{\frac{1}{3}V^3-\xi_j V}} \; \frac{1}{(V+\tau_j)-(U+\tau_i)}$$

$$-\frac{\mathbb{1}_{(\tau_j>\tau_i)}}{\sqrt{4\pi(\tau_j-\tau_i)}} e^{-\frac{(\xi_j-\xi_i)^2}{4(\tau_j-\tau_i)}-\frac{1}{2}(\tau_j-\tau_i)(\xi_j+\xi_i)+\frac{1}{12}(\tau_j-\tau_i)^3}$$

In particular, for $\tau_i=\tau_j=\tau$, we have the Airy kernel

$$\begin{aligned} K^{\mathcal{A}}(\tau,x;\tau,y) = K_{\mathsf{Ai}}(x,y) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\nwarrow\nearrow} dU \int_{\searrow\nwarrow} \frac{dV}{V-U} \; \frac{e^{\frac{1}{3}U^3-xU}}{e^{\frac{1}{3}V^3-yV}} \\ &= \int_0^\infty \mathsf{Ai}(x+\lambda)\mathsf{Ai}(y+\lambda)d\lambda \end{aligned}$$

$$\mathsf{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3+xt\right)\, dt, \qquad \text{satisfying } \frac{d^2y}{dx^2}-xy=0,$$

- $\mathbb{P}(\mathcal{A}(\tau) \leq x) = e^{-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha} =: \mathcal{F}(x)$ (T-W distribution '95)

(time-independent: stationary process)

- $\mathbb{P}(\mathcal{A}(\tau) \leq x) = e^{-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha} =: \mathcal{F}(x)$ (T-W distribution '95)

(time-independent: stationary process)

- $\mathbb{Q}(s; x, y) := \log \mathbb{P}\left(\mathcal{A}(\tau_1) \leq x + y,\ \mathcal{A}(\tau_2) \leq x - y\right), \quad s = \tau_2 - \tau_1 > 0$

satisfies

$$2s \frac{\partial^3 \mathbb{Q}}{\partial s \partial x \partial y} = \left(\frac{s^2}{2} \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left(\frac{\partial^2 \mathbb{Q}}{\partial y^2} - \frac{\partial^2 \mathbb{Q}}{\partial x^2} \right) + \left\{ \frac{\partial^2 \mathbb{Q}}{\partial x \partial y}, \frac{\partial^2 \mathbb{Q}}{\partial x^2} \right\}_x .$$

with "initial" condition: for $s = \tau_2 - \tau_1 \to \infty$,

$\{\ ,\ \}_x$ is a Wronskian with regard to x

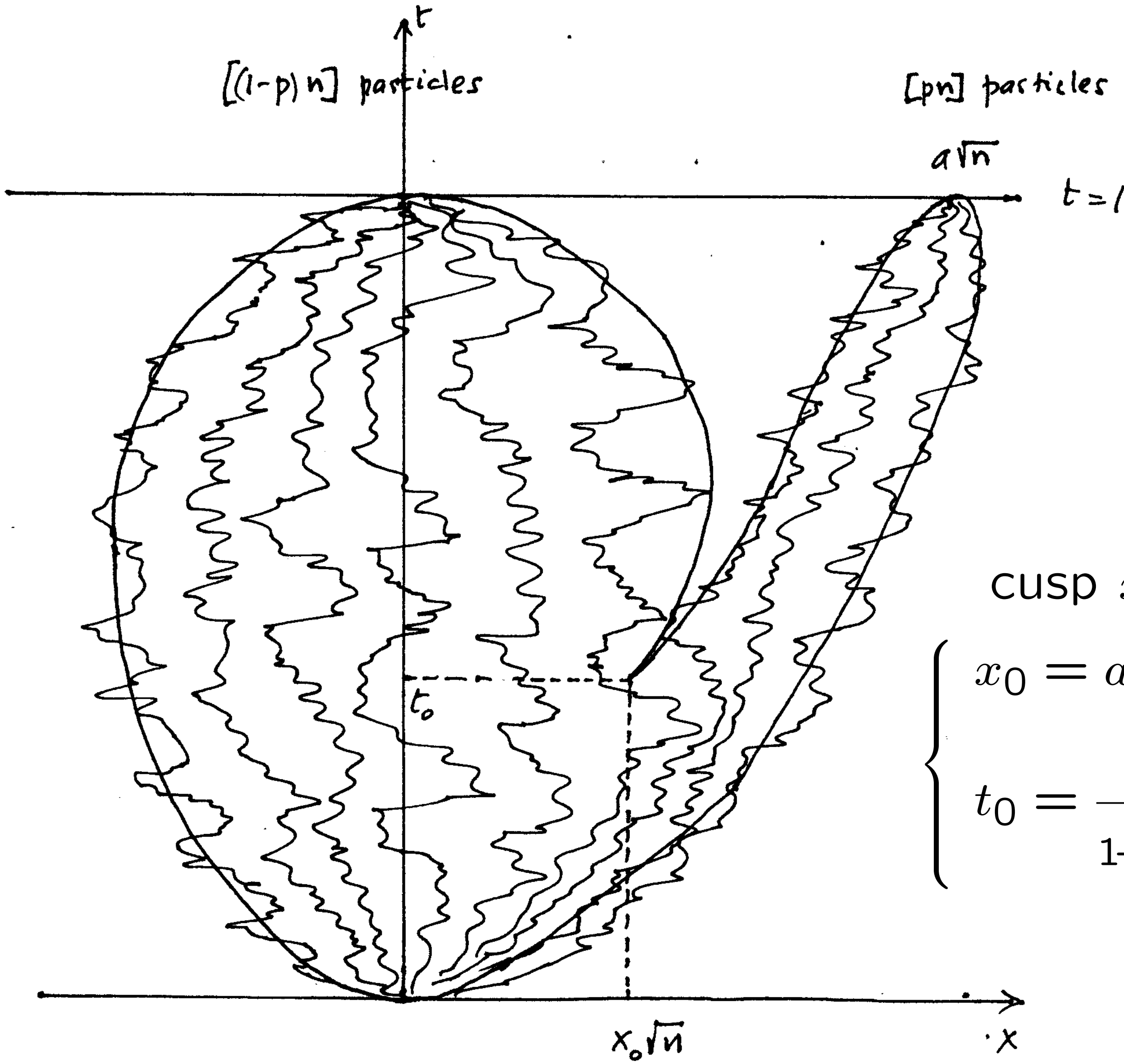
2. The Pearcey process $\mathcal{P}(\tau)$

Non-intersecting Brownian motions:

$$\begin{cases} \text{leaving from } 0 & \text{at } t=0 \\ \\ \begin{cases} [(1-p)n] \text{ paths forced to end at } 0 \\ \\ [pn] \text{ right paths forced to end at } a\sqrt{n} \end{cases} & \text{at } t=1 \qquad a>0, \;\; 0<p<1 \end{cases}$$

Let $n \to \infty$:

Pastur '72, Brézin-Hikami '96-98, Johansson '01, Bleher-Kuijlaars I, II, III '04, Tracy-Widom '05, Okounkov-Reshetikhin '05, Adler-PvM '05, Adler-Orantin-PvM '08



cusp $x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$ at

$$\begin{cases} x_0 = a t_0 \frac{2q-1}{q+1} \\ t_0 = \frac{1}{1+2\left(\frac{a(q^3-1)}{q+1}\right)^2}. \end{cases}$$

convenient parametrization:

$$p = \frac{1}{1+q^3}$$

Look through the "Pearcey" microscope at **the fluctuations of the paths** about the cusp, for very large n:

$$\textbf{cusp}: \qquad x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$$

New process $\mathcal{P}(\tau)$ **in the neighborhood of the cusp** : $[\xi_1, \xi_2] \subset \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}^{(n)}_{Br}\left(\text{all } x_j\left(t_0 + (c_0\mu)^2 \frac{2\tau}{n^{1/2}}\right) \in x_0 n^{1/2} + c_0 A\tau + c_0\mu \frac{[\xi_1,\xi_2]^c}{n^{1/4}}\right)$$

$$= \det\left(\mathbb{I} - K^{\mathcal{P}}\right)_{[\xi_1,\xi_2]} \qquad \text{(independent of } pn \text{ and } a\text{: universality!)},$$

$$=: \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\tau) \cap [\xi_1,\xi_2] = \emptyset\right) \qquad (\textit{Pearcey process})$$

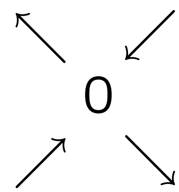
(via steepest descent method)

$$\mu^4 = \frac{q^2-q+1}{q}, \qquad c_0 := \frac{a t_0\sqrt{q^2-q+1}}{q+1}, \qquad A = \sqrt{q} - \frac{(2q-1)t_0}{\sqrt{q}}$$

with Pearcey kernel:

$$K^{\mathcal{P}}(\tau;\xi,\eta)=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}dU\int_{\Gamma_\times}dV\frac{1}{U-V}\;\frac{e^{-\frac{U^4}{4}+\tau\frac{U^2}{2}-\eta U}}{e^{-\frac{V^4}{4}+\tau\frac{V^2}{2}-\xi V}}$$

contour $\Gamma_\times =$



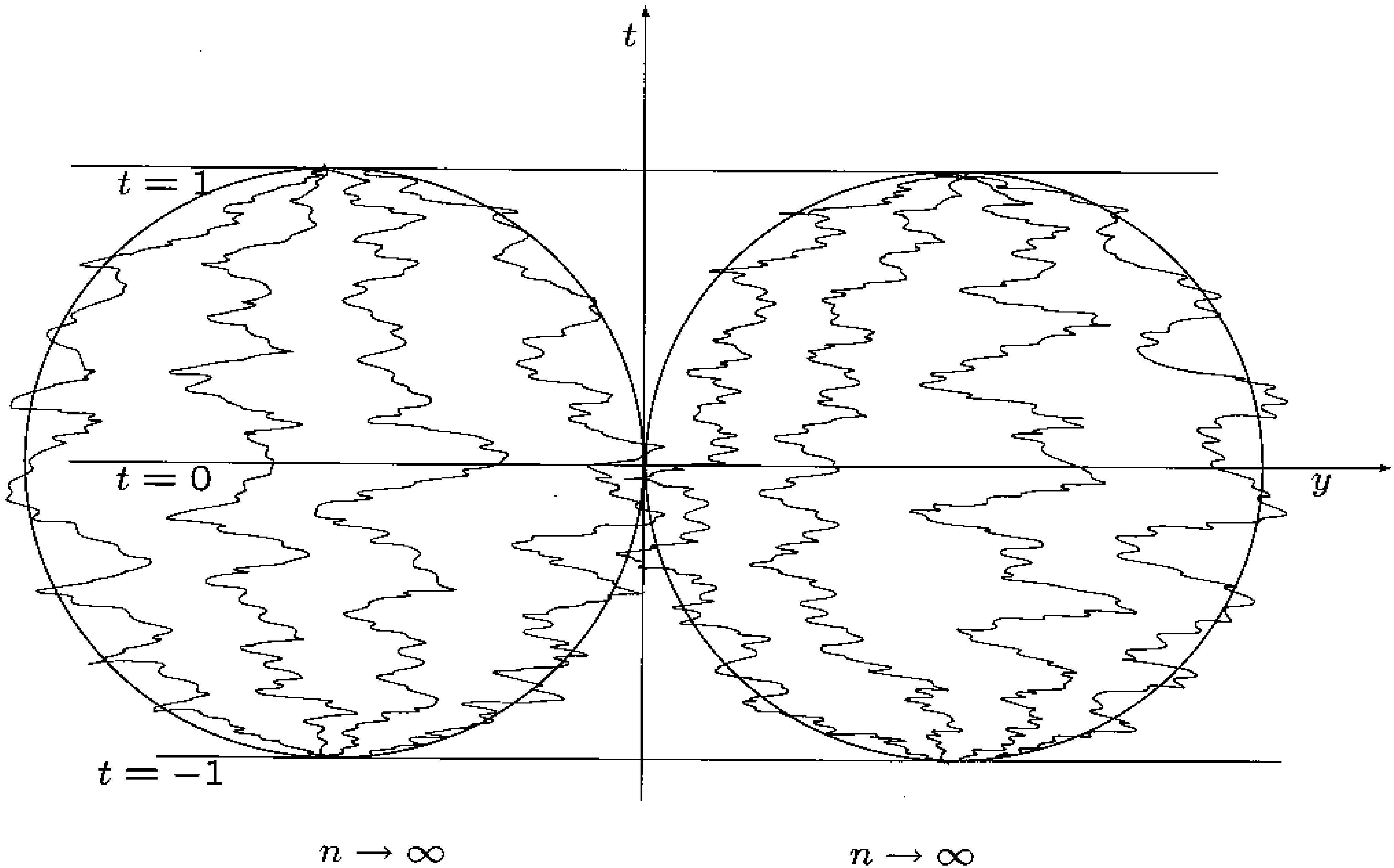
Then $\mathbb{Q}(t;x,y)=\log\mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(t)\cap[x,y]=\emptyset\right)$ satisfies a PDE:

$$\frac{\partial^3\mathbb{Q}}{\partial t^3}+\frac{1}{8}\left(\mathcal{E}-2t\frac{\partial}{\partial t}-2\right)\nabla^2\mathbb{Q}-\frac{1}{2}\left\{\nabla^2\mathbb{Q},\nabla\frac{\partial\mathbb{Q}}{\partial t}\right\}_\nabla=0.$$

$\nabla := \frac{\partial}{\partial x}+\frac{\partial}{\partial y}, \qquad \mathcal{E} := x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$

(Adler-PvM '06, Adler-Orantin-PvM '08, Adler-Cafasso-PvM)

3. The Tacnode process



Universality: three models, leading to the Tacnode process:

(1) continuous time and discrete space process: (random walks)

Mark Adler, Patrik Ferrari & PvM : *Non-intersecting random walks in the neighborhood of a symmetric tacnode, Annals of Prob 2011 (arXiv:1007.1163)*

(2) discrete time and discrete space process (Domino tilings): Mark Adler, Kurt Johansson & PvM: *Double Aztec Diamonds* (to appear)

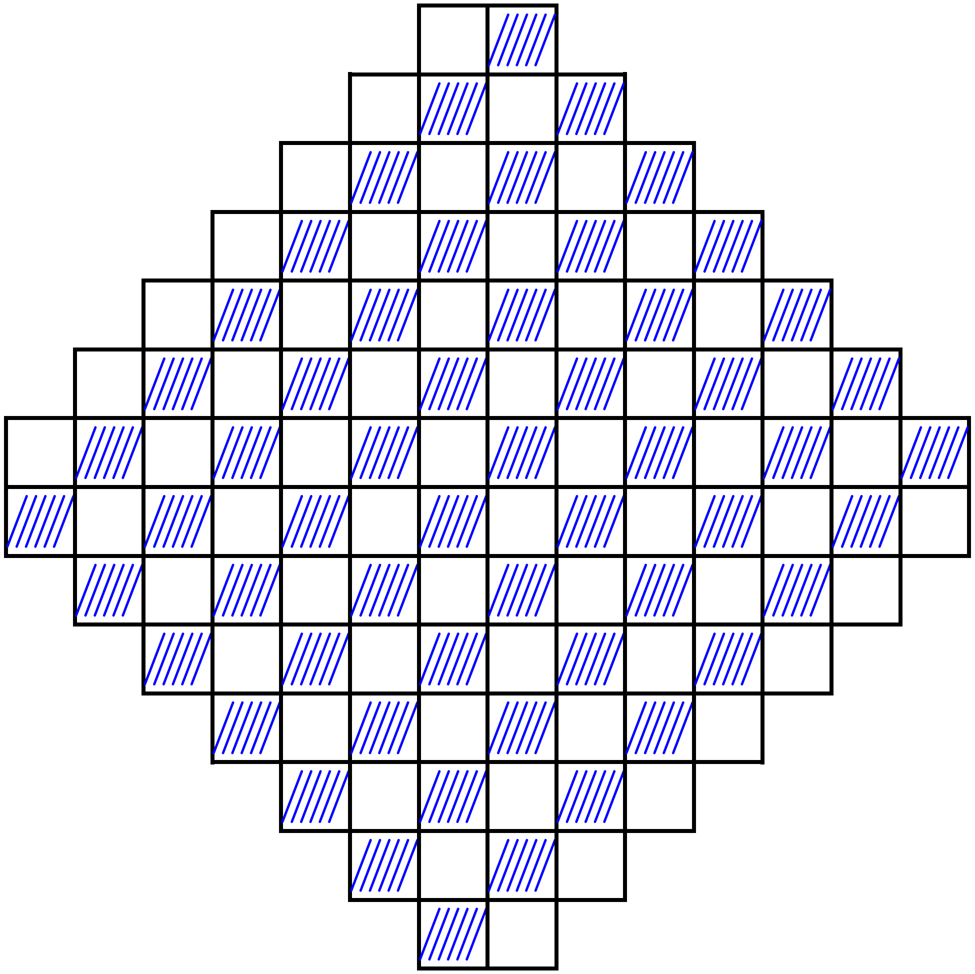
(3) continuous time and continuous space process

Delvaux-Kuijlaars-Zhang: *Critical behavior of non-intersecting Brownian motions at a Tacnode* (arXiv:1009.2457)

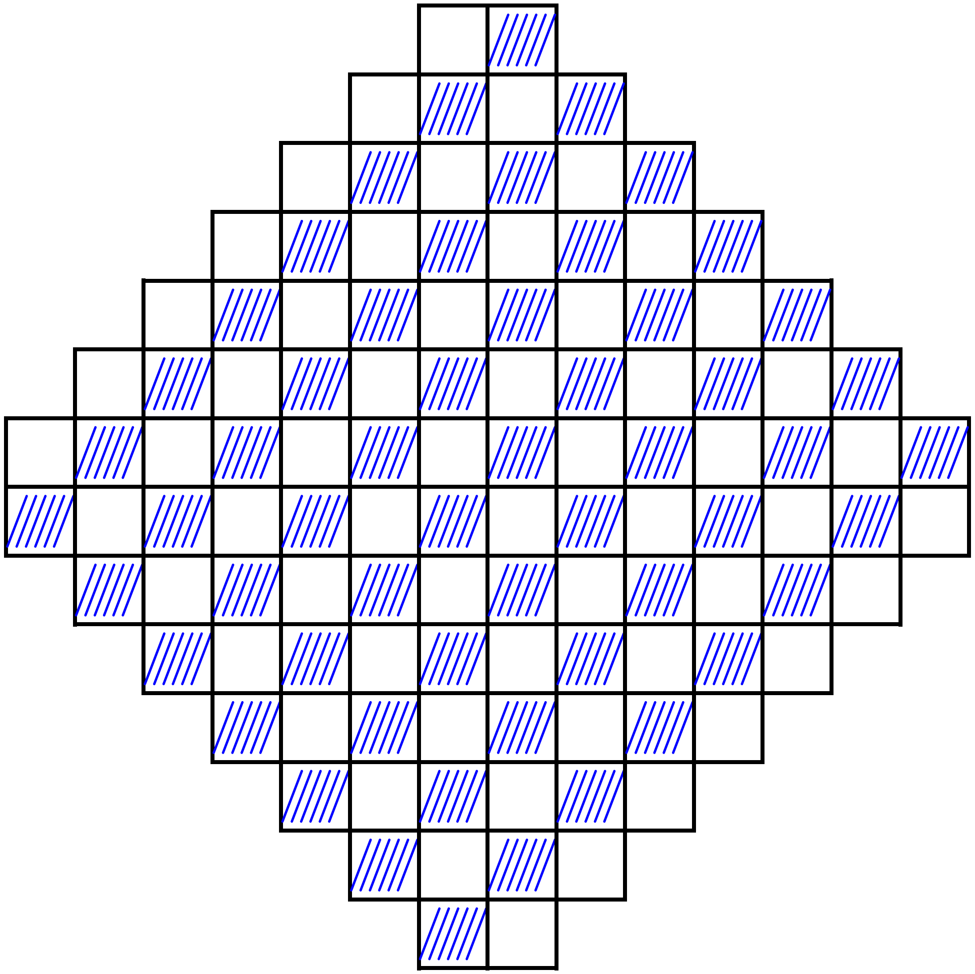
K. Johansson: *Non-colliding Brownian Motions and the extended tacnode process* (arXiv:1105.4027)

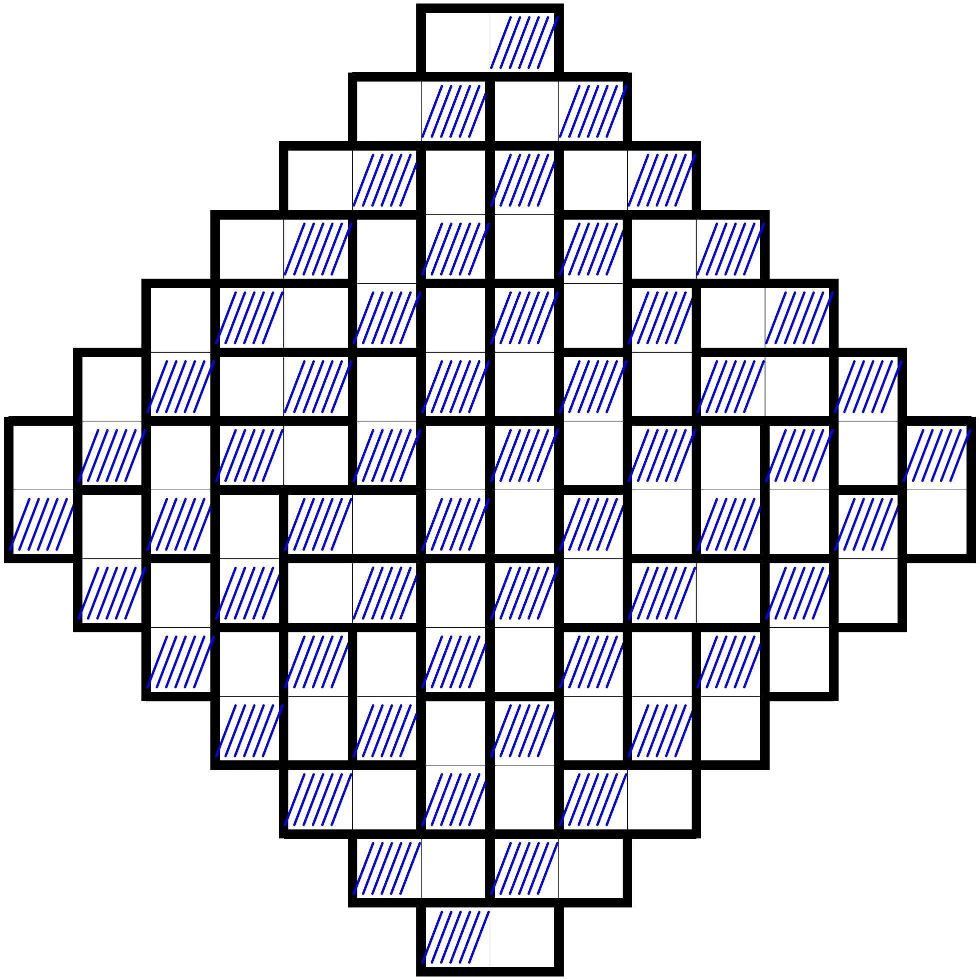
4. Domino tilings of Aztec Diamonds

An Aztec diamond (size n):



Domino tilings of an Aztec diamond :



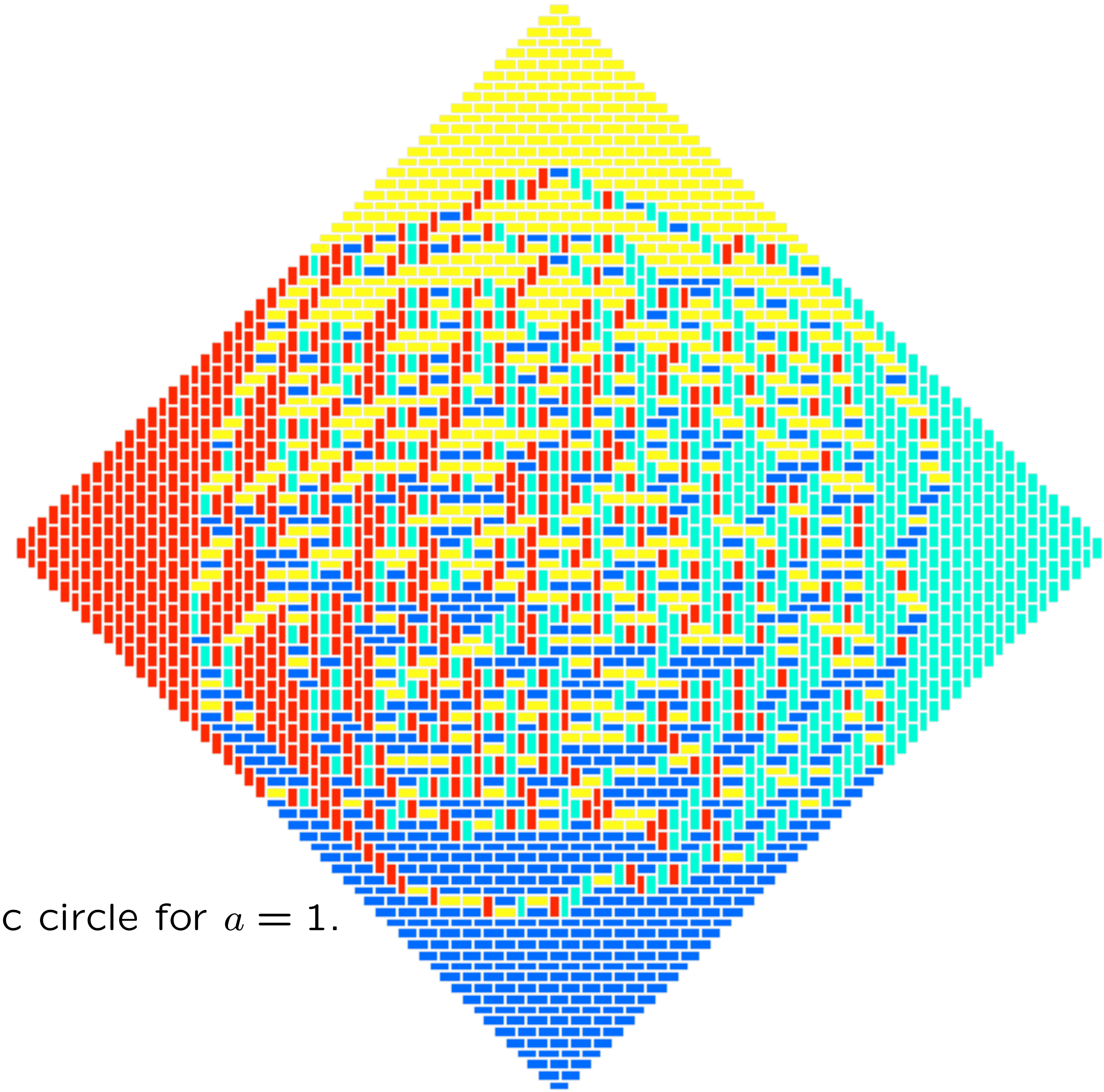


4 types of situations for the domino's!

Kasteleyn 1961, Elkies, Kuperberg, Larsen, Propp '92

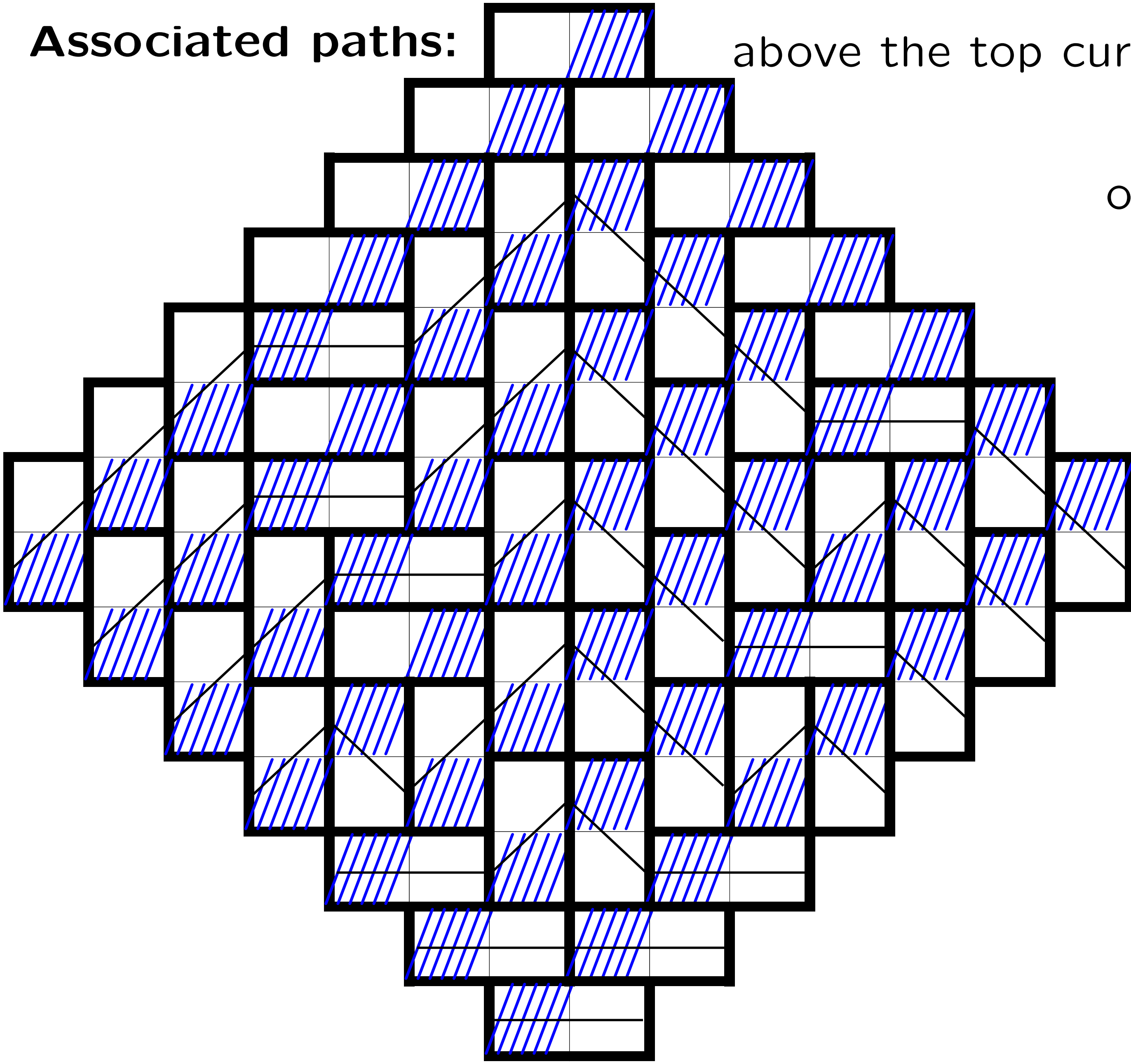
Cohn, Elkies, Propp '96, Johansson '00, '03, '05

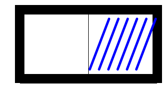
M. Fulmek, C. Krattenthaler '00, Johansson-Nordenstam '06

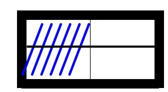


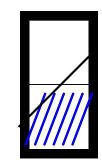
Arctic circle for $a = 1$.

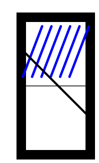
Associated paths: above the top curve $X_n(t)$ for $-n \leq t \leq n$



only  (North)

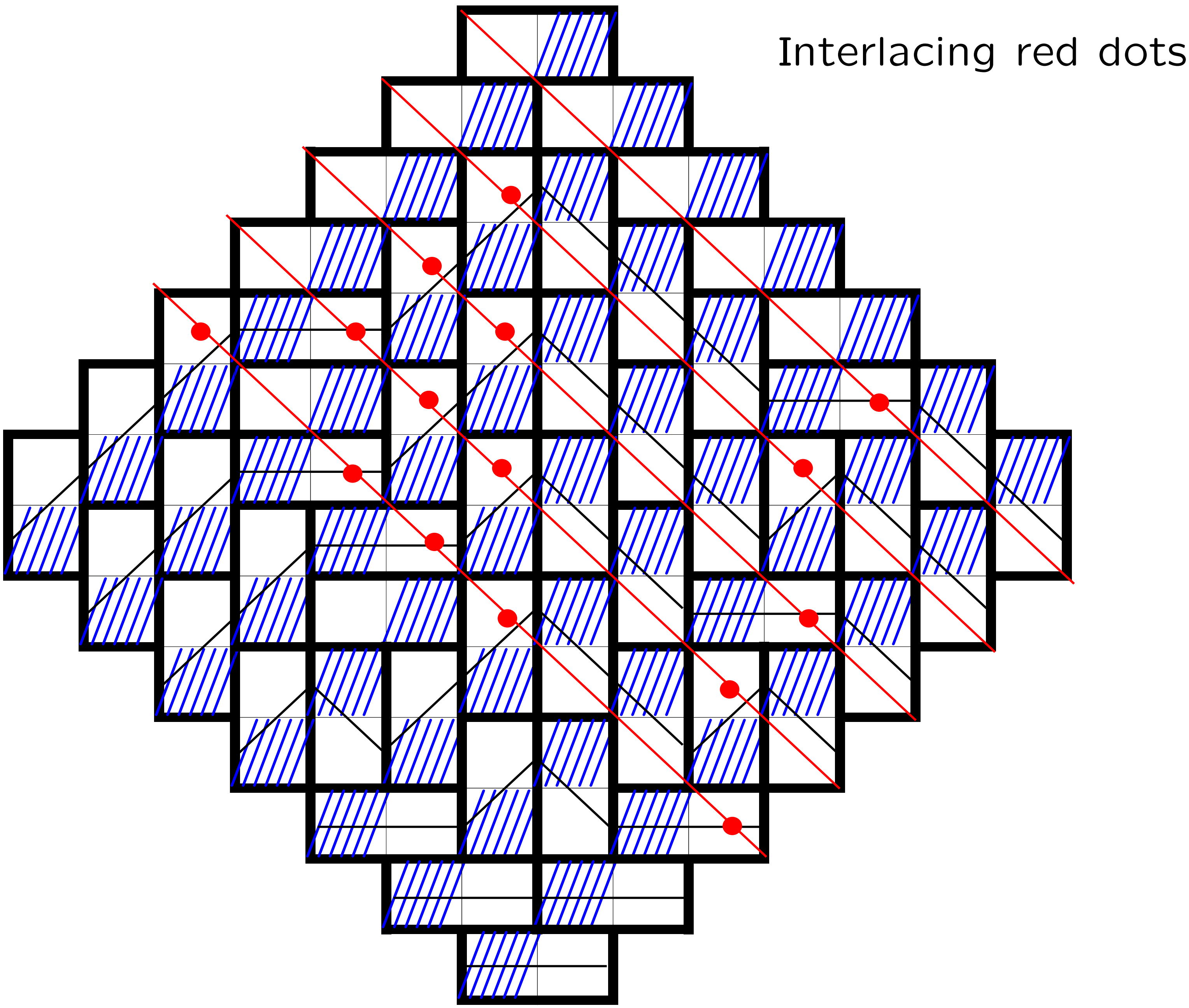
 (South)

 (West)

 (East)

The curves never penetrate North domino's

Associated point process: Interlacing red dots on the red lines



A weight on domino's $\Longrightarrow$ a probability on domino tilings :

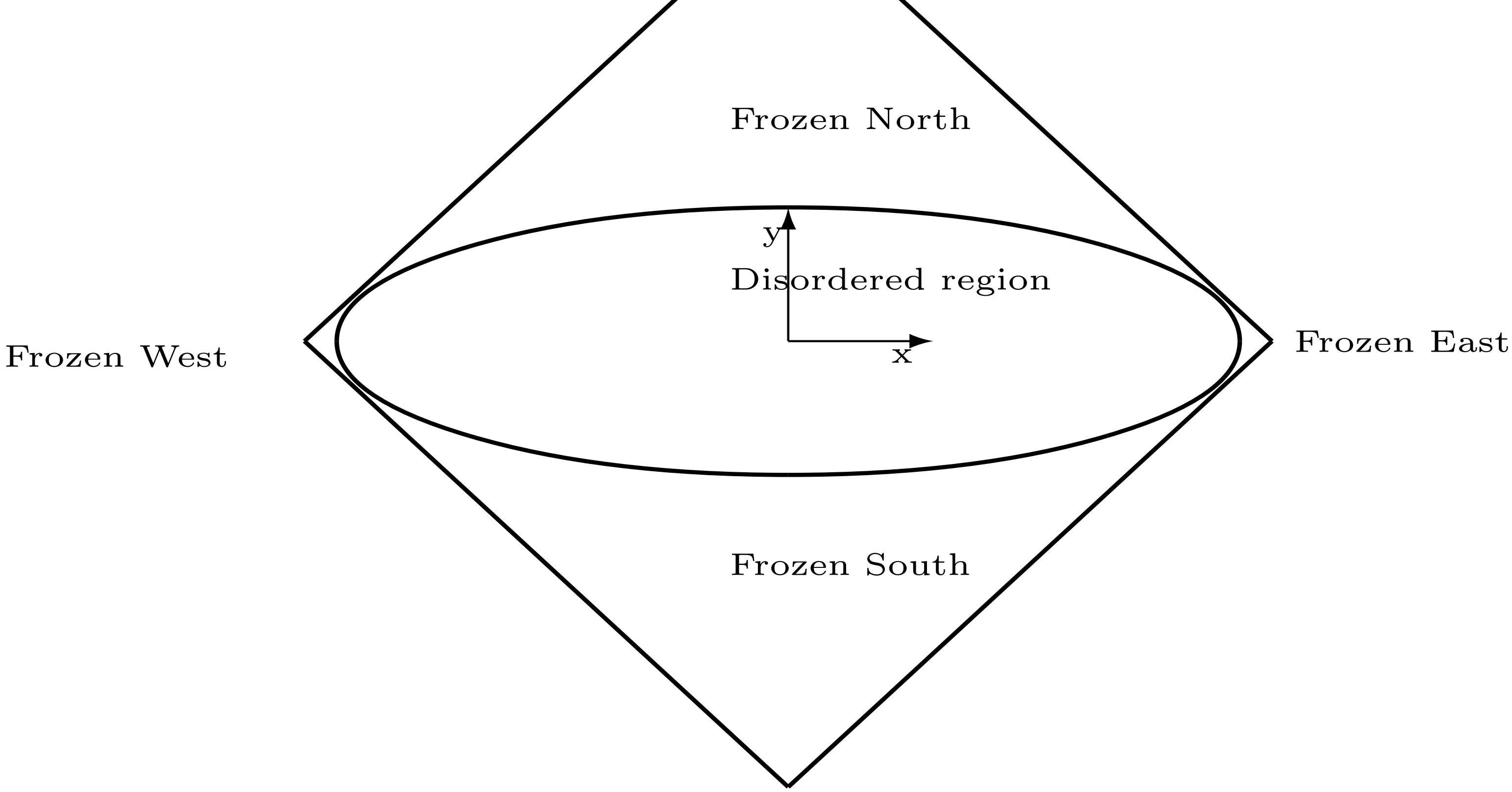
- put the weight $0 < a \leq 1$ on vertical dominoes

- put the weight 1 on horizontal dominoes,

Define:

$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}} \qquad (1)$$

Let $n \to \infty$:

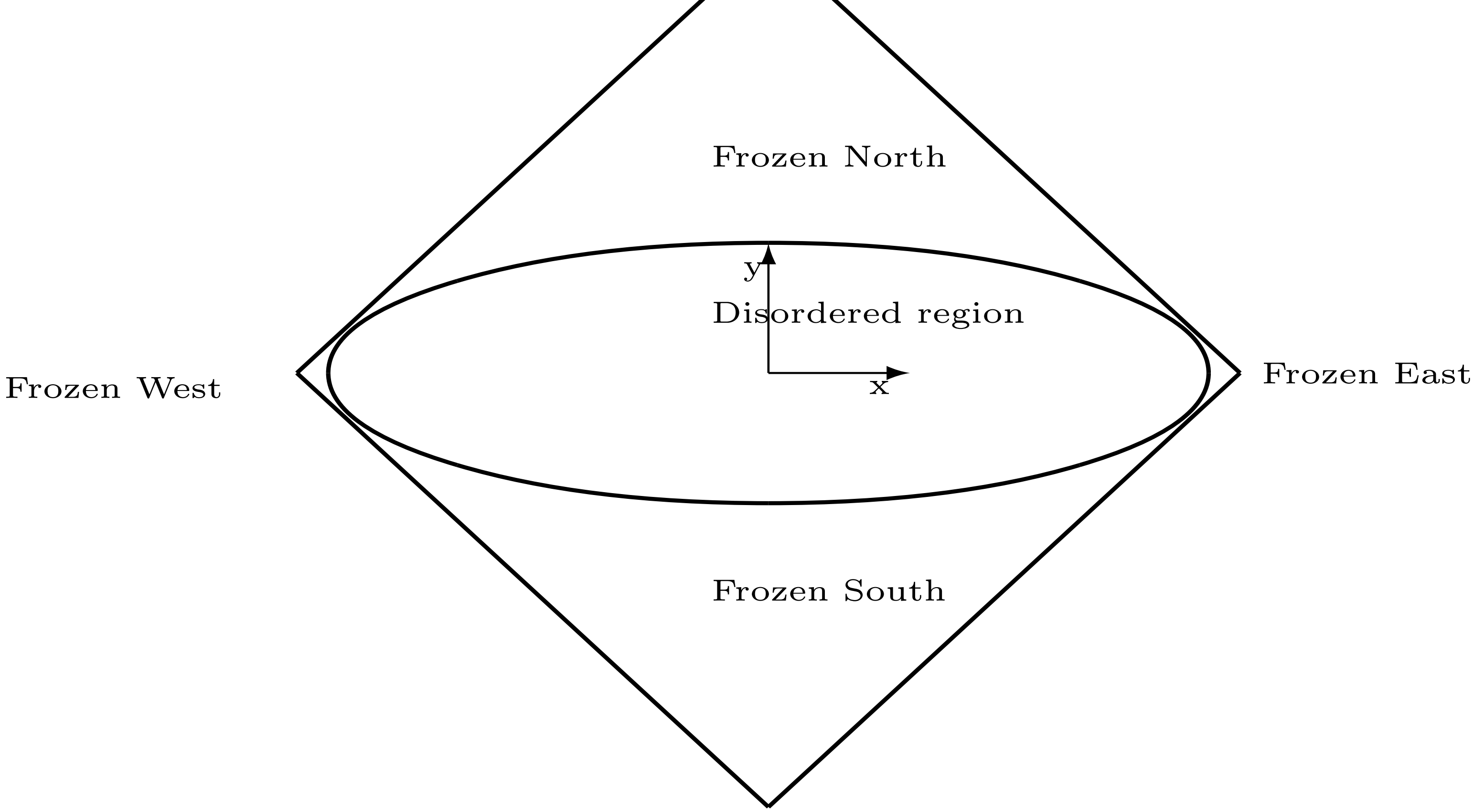


• Macroscopically: **arctic Ellipse** (N. Elkies, Kuperberg, Larsen and Propp '92, W. Jockush J. Propp and P. Shor '98)

$$\frac{x^2}{p} + \frac{y^2}{q} = 1, \quad \text{with } q = \frac{a}{a + a^{-1}} \text{ and } p = 1 - q = \frac{a^{-1}}{a + a^{-1}},$$

• Fluctuations about the generic points of the ellipse:
Airy process (Johansson '05) (for $a = 1$)

$$\frac{X_n(2^{-1/6}n^{2/3}t) - n/\sqrt{2}}{2^{-5/6}n^{1/3}} \to \mathcal{A}(t) - t^2$$



• Fluctuations about the generic points of the ellipse:

Airy process (Johansson '05) (for $a = 1$)

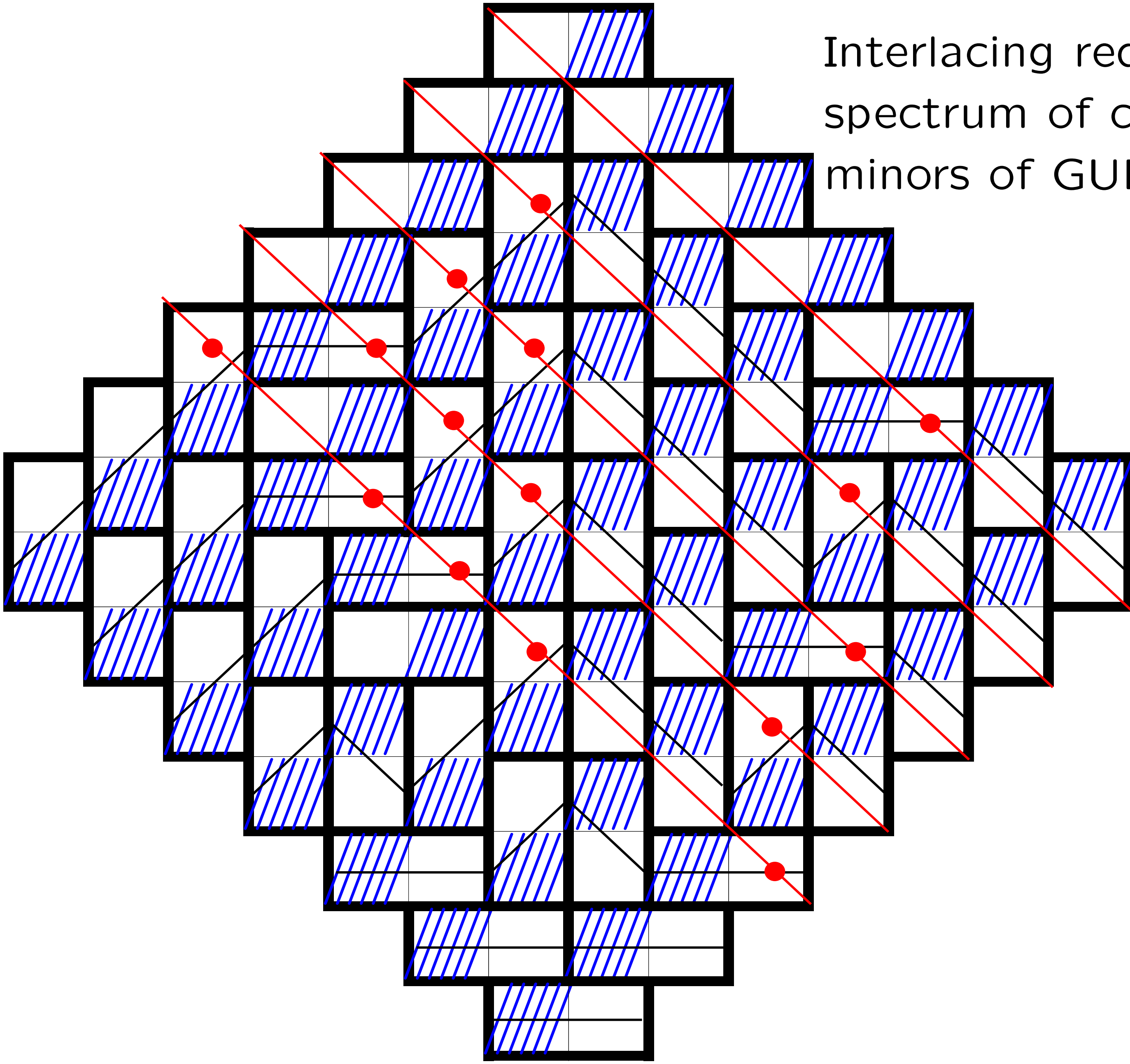
$$\frac{X_n(2^{-1/6}n^{2/3}t) - n/\sqrt{2}}{2^{-5/6}n^{1/3}} \to \mathcal{A}(t) - t^2$$

• Fluctuations near the points of osculation of the ellipse with the square:

GUE-minor process (Johansson-Nordenstam '06):

Spectrum of consecutive principal minors of GUE-matrices!

Associated point process: red dots on the red lines



Interlacing red dots for $n \to \infty \simeq$ spectrum of consecutive principal minors of GUE-matrices

Limits to the Airy process and to the GUE-minor process:

For the Aztec diamond, (Johansson '05, Johansson & Nordenstam '06)

$$\mathbb{P}\begin{pmatrix}\text{finding red dots at position } x_j \\ \text{along line } r_j \text{ for } j=1,\ldots,m\end{pmatrix} = \det\Big(K_n(r_i,x_i;r_j,x_j)\Big)_{1\le i,j\le m}$$

where (*Krawtchouk kernel*) (for $a=1$)

$$K_n(2r,x;2s,y) = \frac{1}{(2\pi \mathrm{i})^2}\int_\Gamma dz\int_\gamma \frac{dw}{z-w}\;\frac{w^{y-1}}{z^x}\frac{(1-w)^s(1+\frac{1}{w})^{n-s}}{(1-z)^r(1+\frac{1}{z})^{n-r}},\quad \text{for } r\le s$$

- **Limit to Airy process:**

$$\lim_{n\to\infty} n^{1/3}K_n(cn+2^{-1/6}\tau_1 n^{2/3}, c'n+2^{-5/6}\xi_1 n^{1/3}; \\ cn+2^{-1/6}\tau_2 n^{2/3}, c'n+2^{-5/6}\xi_2 n^{1/3}) = \mathbb{K}^{\mathcal{A}}(\tau_1,\xi_1;\tau_2,\xi_2)$$

- **Limit to GUE-minor process:**

$$\lim_{n\to\infty}\frac{g_n(r,\xi)}{g_n(s,\eta)}\sqrt{\frac{n}{2}}K_n\left(2r,\left[\frac{n}{2}+\xi\sqrt{\frac{n}{2}}\,\right];2s,\left[\frac{n}{2}+\eta\sqrt{\frac{n}{2}}\,\right]\right) = \mathbb{K}^{\text{GUEminor}}(r,\xi;s,\eta)$$

Johansson & Nordenstam prove: $(r, s \in \mathbb{Z}_{\geq 0})$

$$K^{\text{GUEminor}}(r, x; s, y) :=$$

$$\begin{cases} \sum_{j=-\infty}^{-1} \sqrt{\frac{(s+j)!}{(r+j)!}} h_{r+j}(x) h_{s+j}(y) e^{-(x^2+y^2)/2} & \text{for } r \leq s \\ -\sum_{j=0}^{\infty} \sqrt{\frac{(s+j)!}{(r+j)!}} h_{r+j}(x) h_{s+j}(y) e^{-(x^2+y^2)/2} & \text{for } r > s \end{cases}$$

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \{\text{spectrum of } i \times i \text{ minor}\} \cap E_i = \emptyset \right) \qquad E_i \subset \mathbb{R}$$

$$= \det \left[I - (\chi_{E_i}(x_i) K^{\text{GUEminor}}(i, x_i; j, x_j) \chi_{E_j}(x_j))_{1 \leq i,j \leq n} \right]$$

5. Domino tilings of Double Aztec Diamonds

Double Aztec diamond (Adler-Johansson-PvM '11)

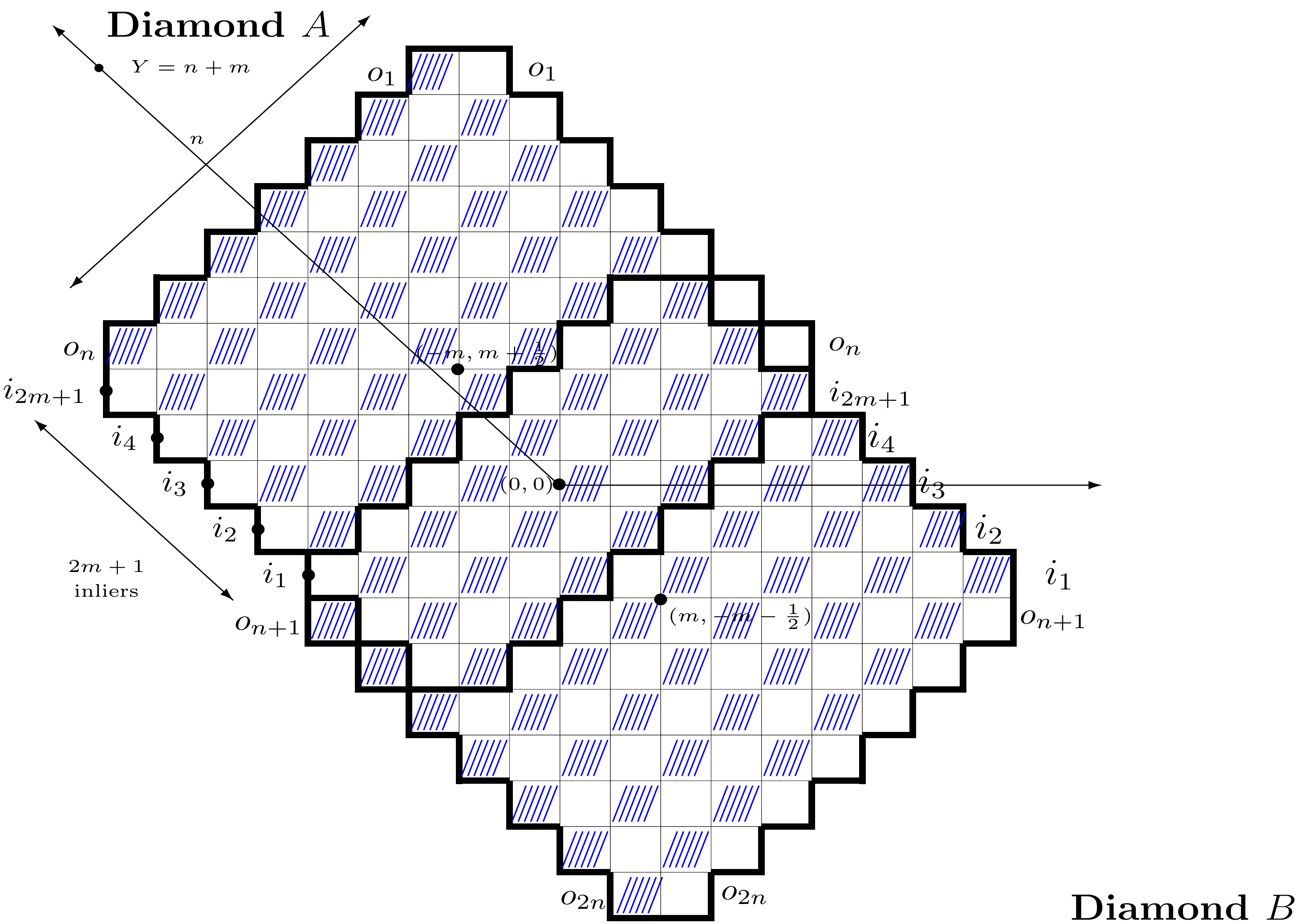


Figure 1: Double Aztec diamond of type $(n, m) = (7, 2)$ with $\#\{\text{inliers}\} = M = 2m + 1 = 5$.

A weight on domino's, a probability on domino tilings and on non-intersecting random walks:

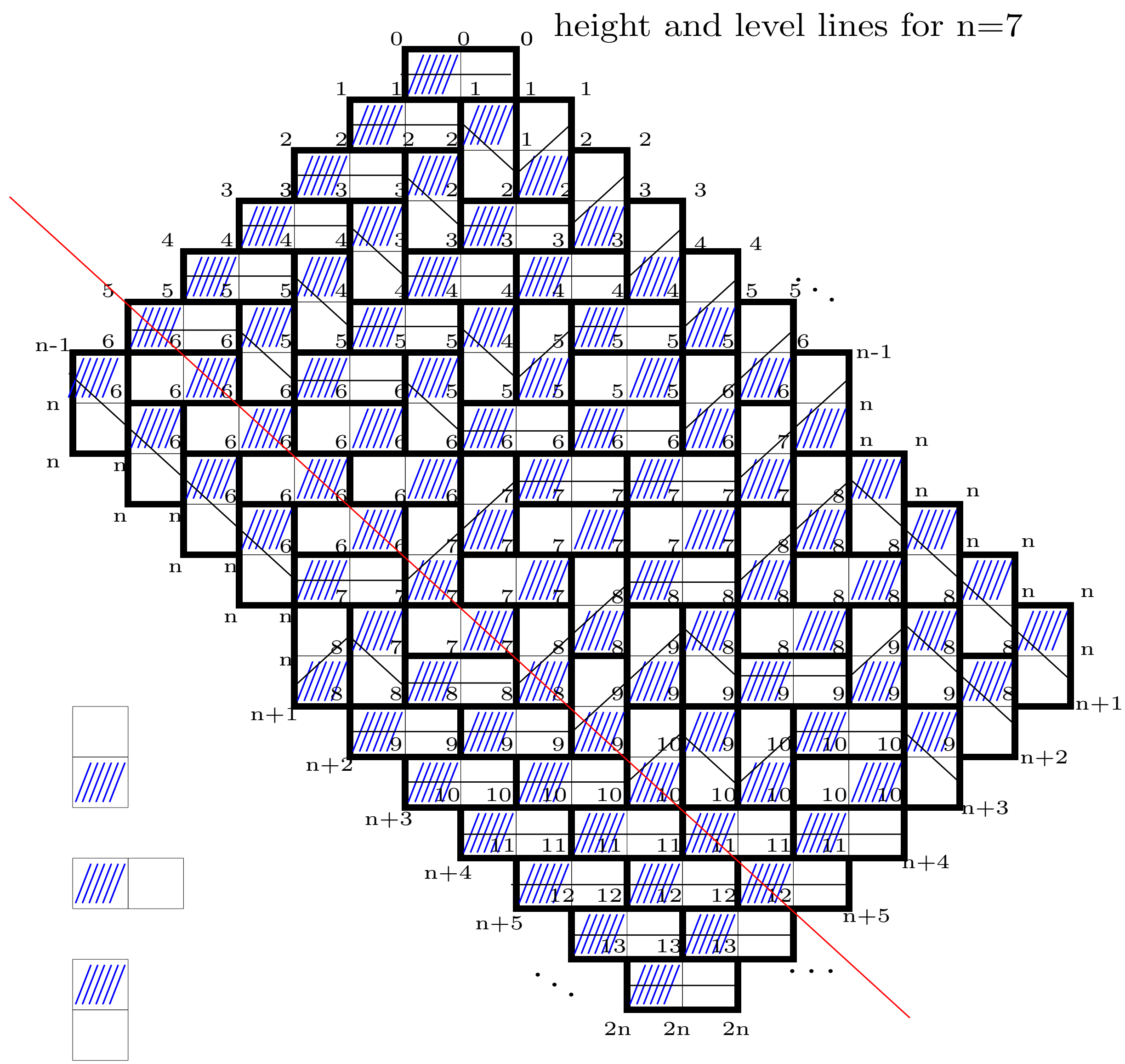
- put the weight $0 < a < 1$ on vertical dominoes

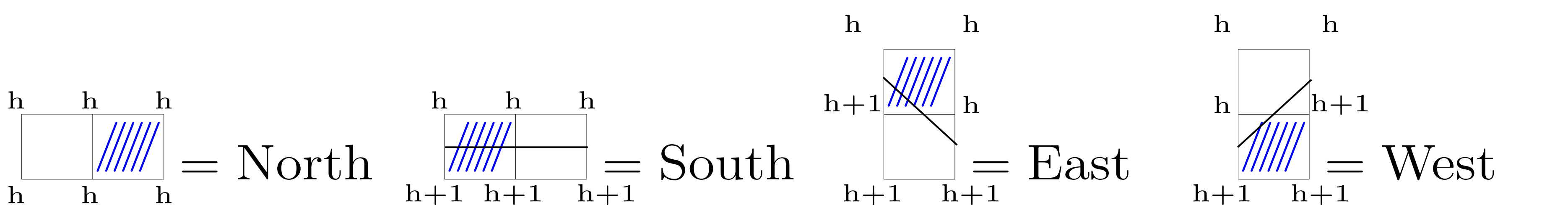
- put the weight 1 on horizontal dominoes,

Define:

$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}} \qquad (2)$$

Cover randomly with domino's, leading to non-intersecting paths:





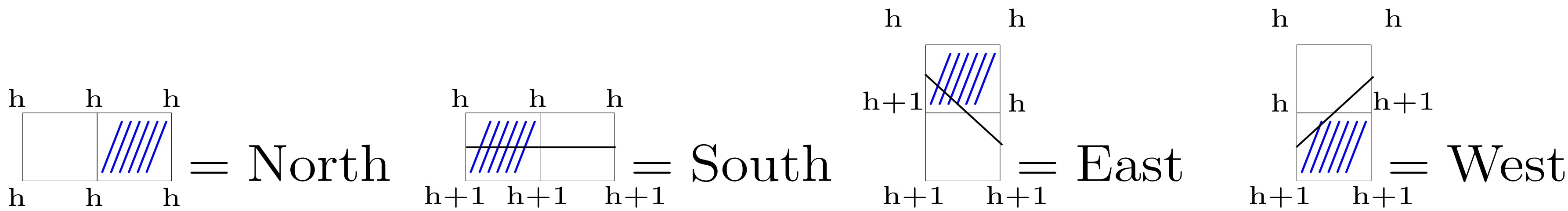


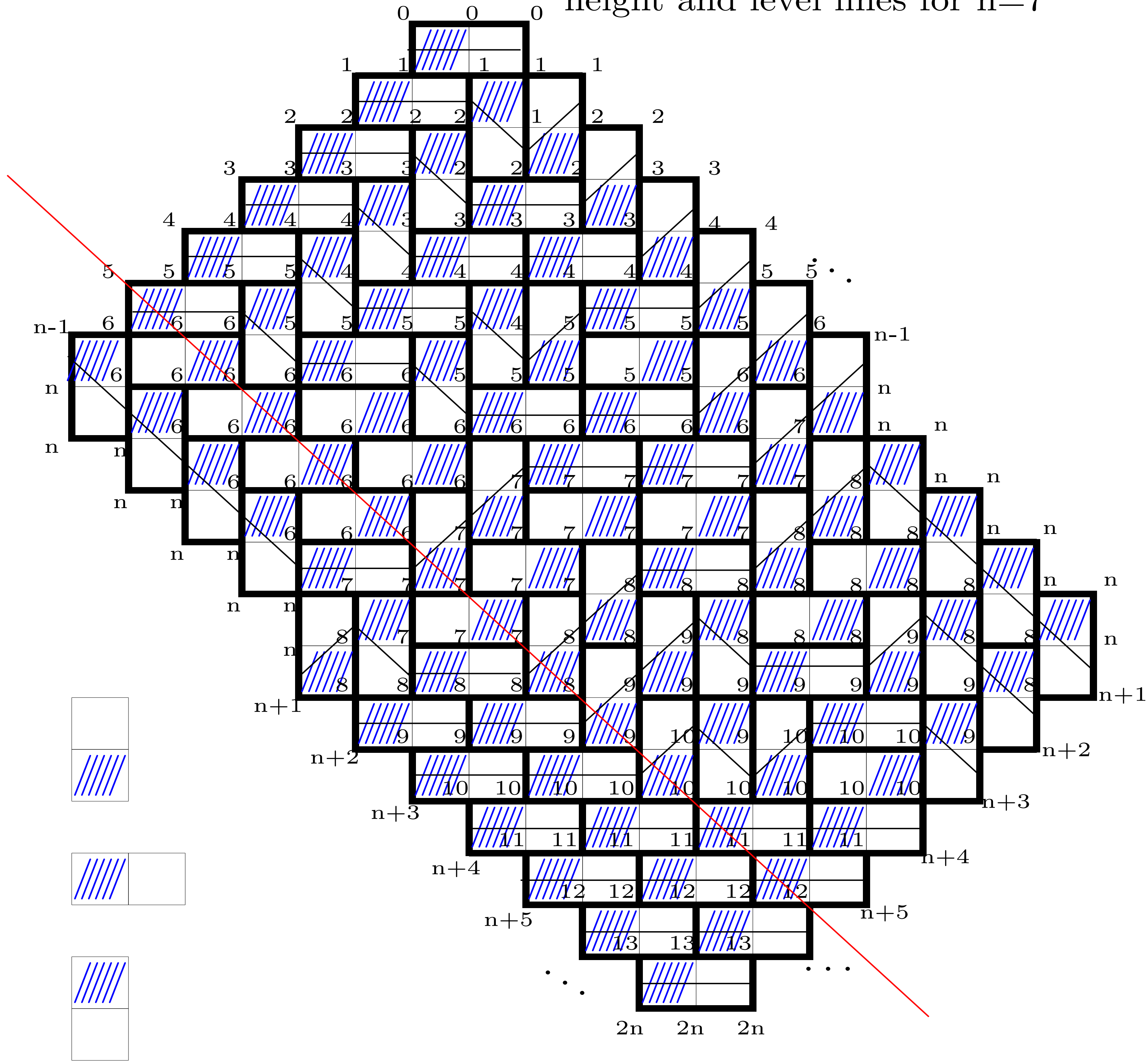
Figure 4. Height function on domino's and level-lines.

Domino-tiling ⇔ Random Surface

(piecewise-linear)

- **The curves are level curves for this Random Surface**
- **Implies fixed heights along the boundary**

height and level lines for n=7



Question:

For an axis Y_{2r} going through black squares only, and an interval $[k, \ell] \subset \mathbb{Z}$,

$$\mathbb{P}(\text{height function is flat along the interval } [k, \ell]) = ?$$

$\Longleftrightarrow$

$\mathbb{P}$(domino's are pointing to the left of Y_{2r} along the points of interval $[k, \ell]$)

$\Longleftrightarrow$

$$\mathbb{P}(\text{no dots along the points of interval } [k, \ell]) = ?$$

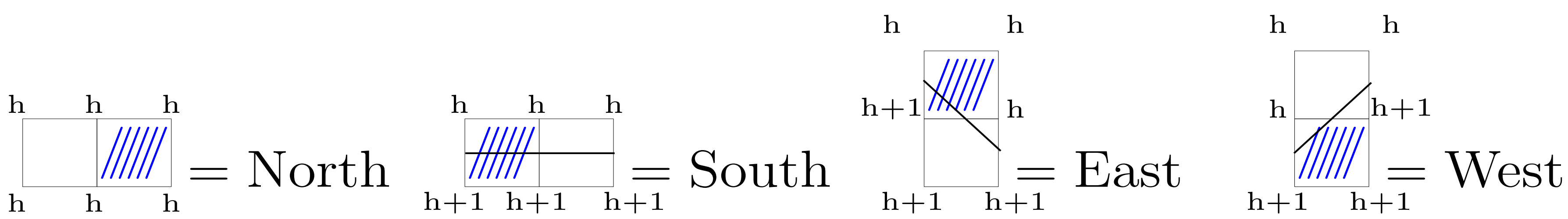
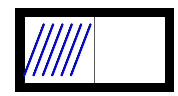


Figure 4. Height function on domino's and level-lines.

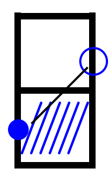
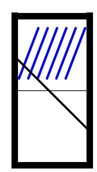
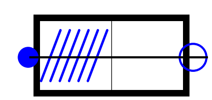
How does one turn this into a determinantal process?

In other terms: How does one turn this into non-intersecting random walks, with synchronized time?

(i) Complete the double Aztec diamond with South domino's (in a trivial way).

 (South)

(ii) Then add dot-particles and circle-particles according to the recipe:

West  East  South 

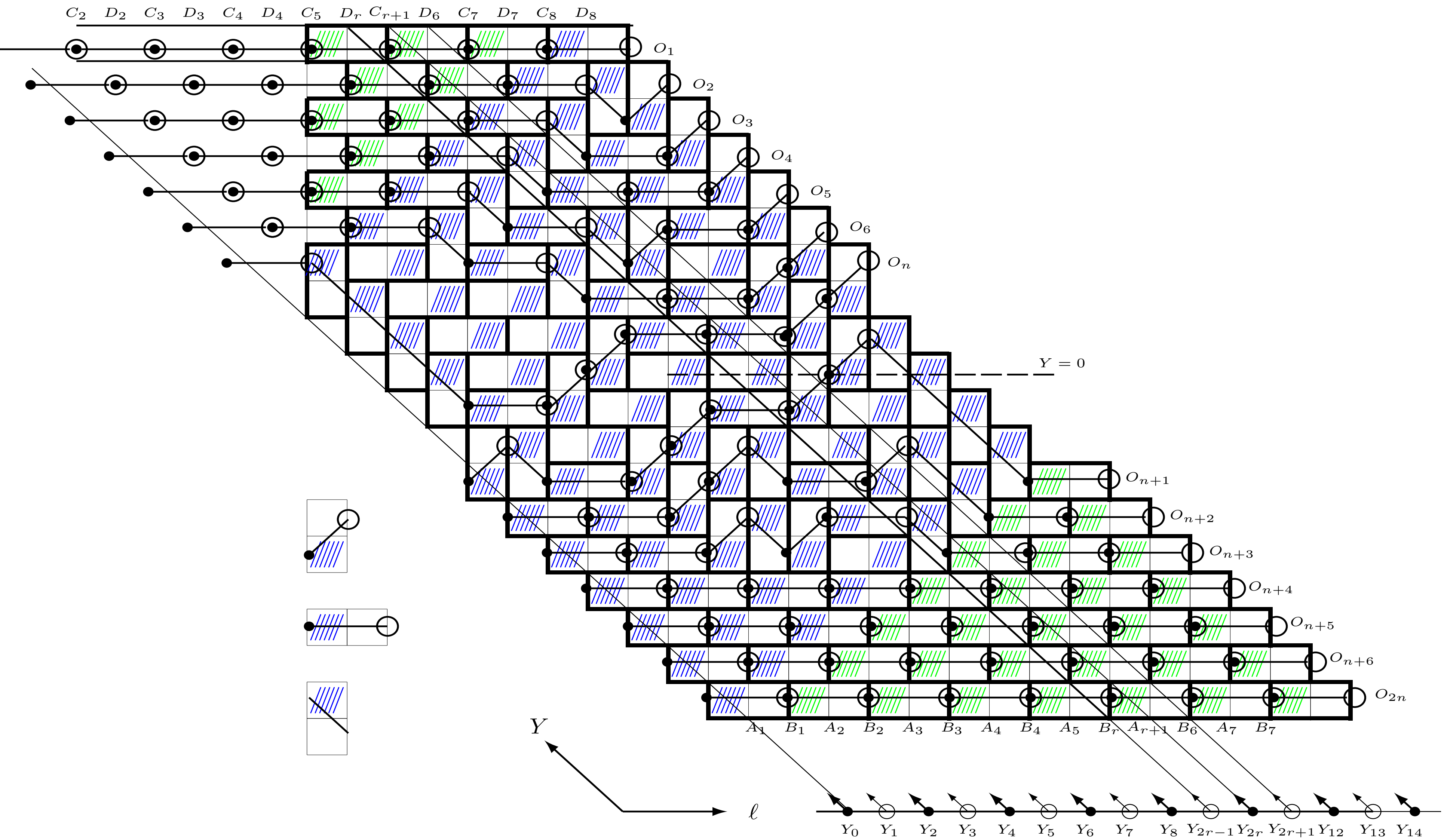
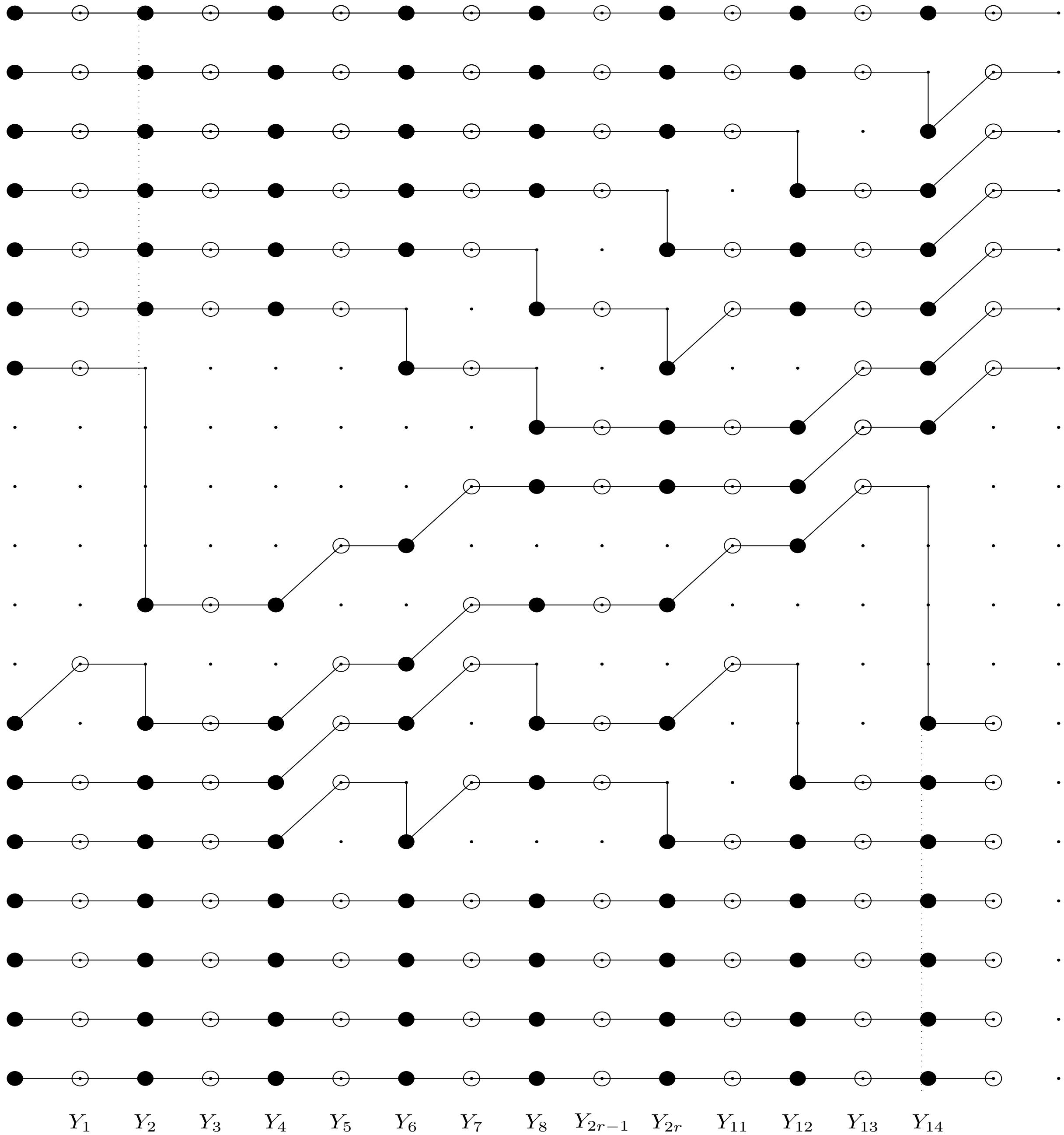
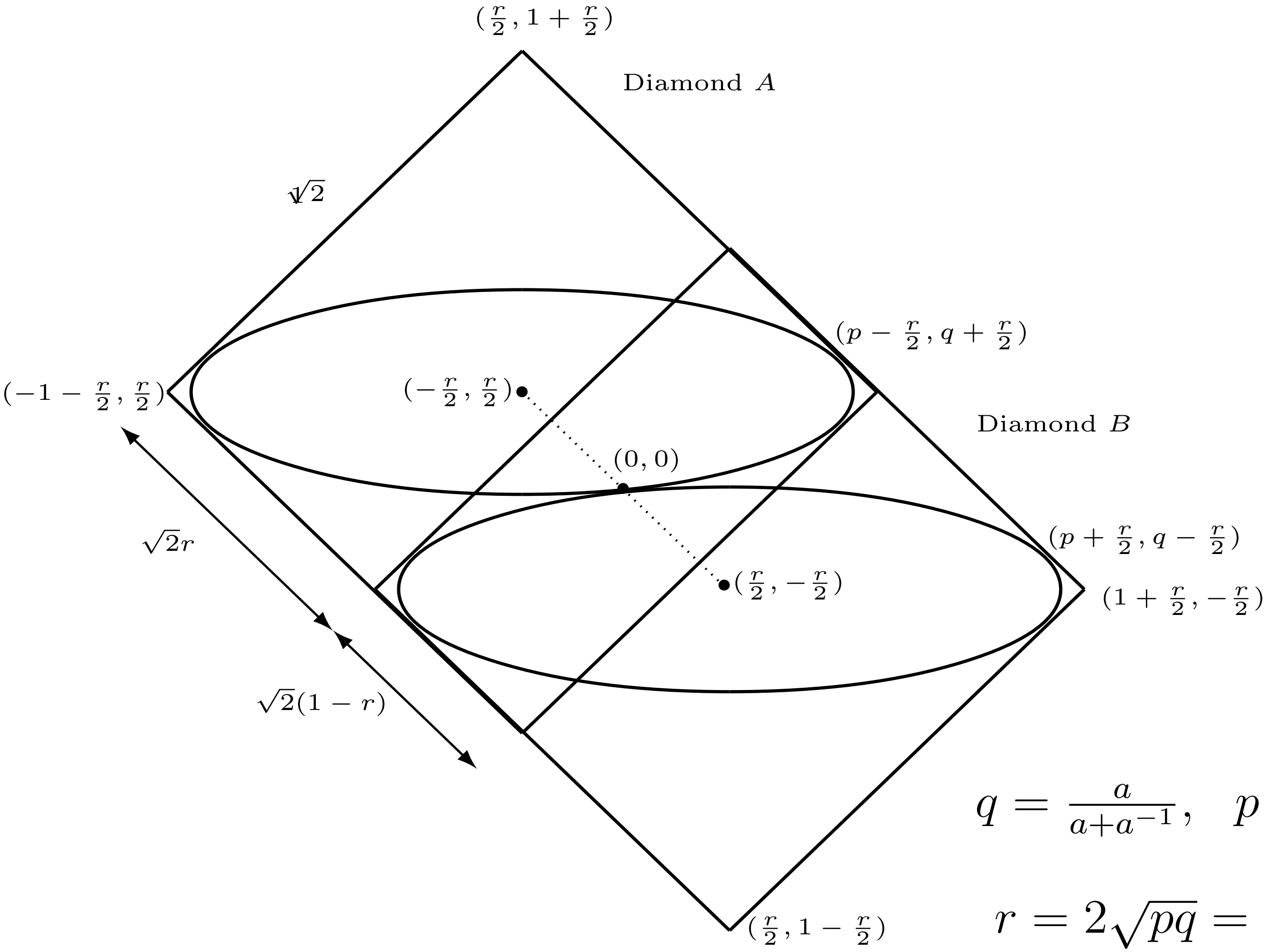


Figure 8. **Outlier** non-intersecting domino paths, circles and dots. The coordinate Y_{2r} (resp. Y_{2r-1}) records the location of the dots (resp. circles), obtained by moving the dots (resp. circles) in between Y_{2r} and Y_{2r-1} to the axis Y_{2r} (resp. Y_{2r-1}). Gaps at $-4,\ -2,\ 0,\ 4,\ 6$.



$$\mathbb{P}\left(\{\text{non-intersecting random walks}\} \cap [k, \ell] = \emptyset \text{ along } Y_{2r}\right) = ?$$

A little preview: Let $n \to \infty$:



$$q = \frac{a}{a+a^{-1}}, \quad p = \frac{a^{-1}}{a+a^{-1}}$$

$$r = 2\sqrt{pq} = \frac{2}{a+a^{-1}}$$

Figure 10: The macroscopic description of the double Aztec diamonds and their frozen region outside the ellipses.

In order to compute the kernel for the determinantal process, it is easier to first consider **Inliers**, instead of the walks before (**outliers**).

Therefore remember the height function and introduce a dual heigth:

Figure 4. Height function h on domino's and level line.

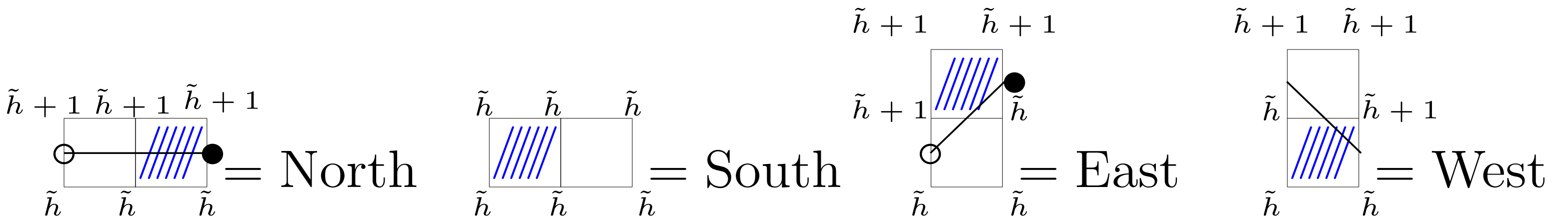
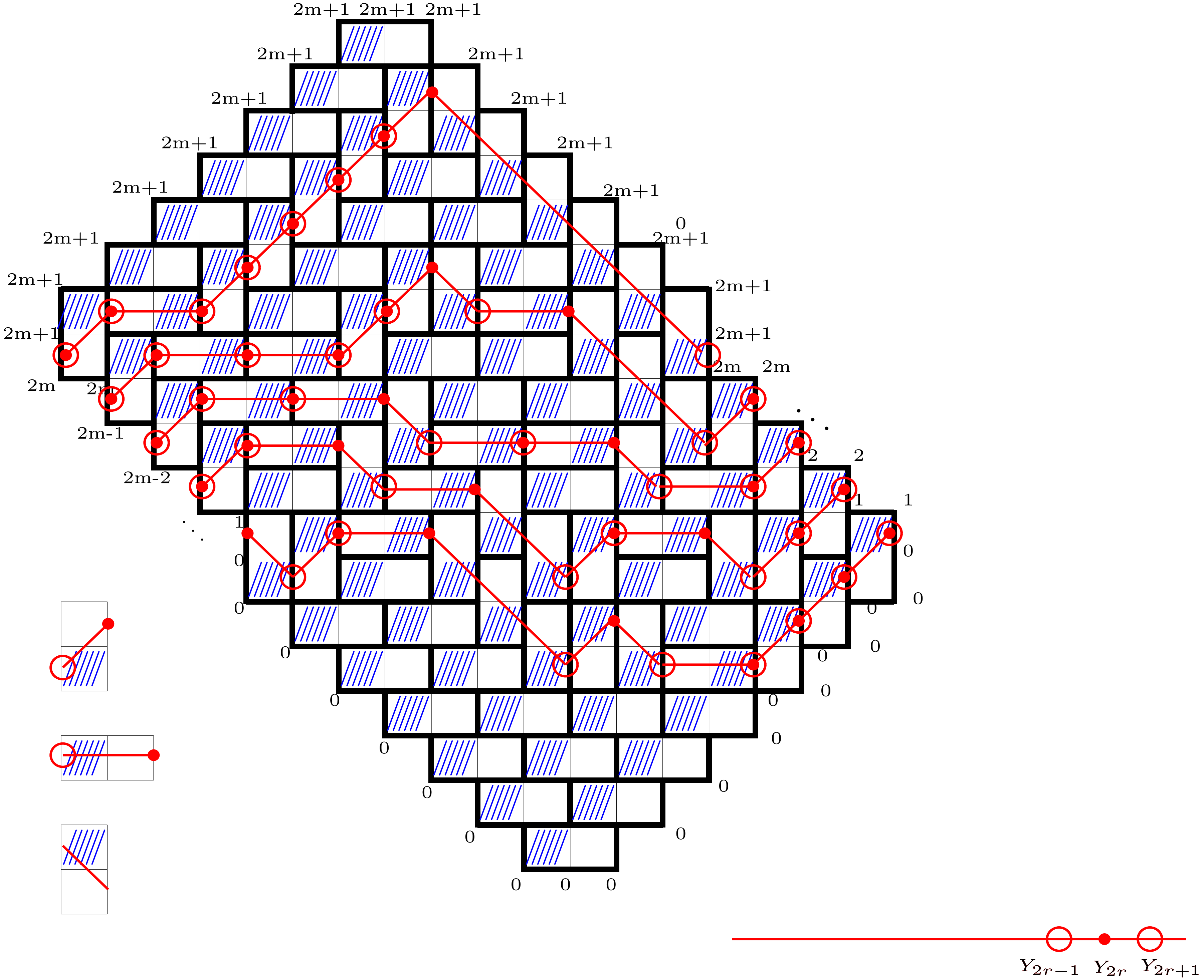
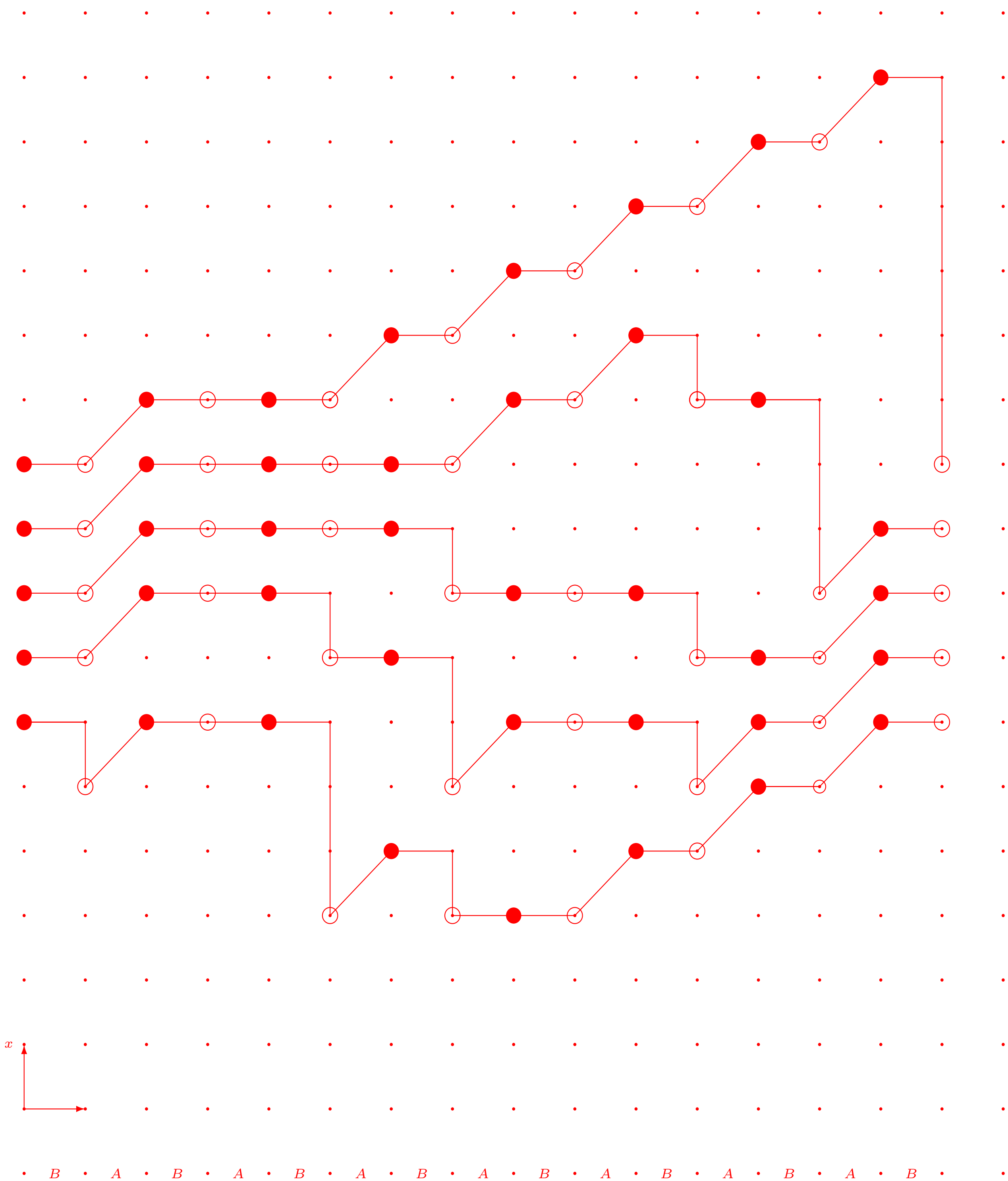


Figure 5. Dual height function $\tilde{h}$ on domino's and level line.



2m+1
2m
2m-1
2m-2
2
1
0
Y_{2r-1}
Y_{2r}
Y_{2r+1}


x
B A B A B A B A B A B A B A B A B

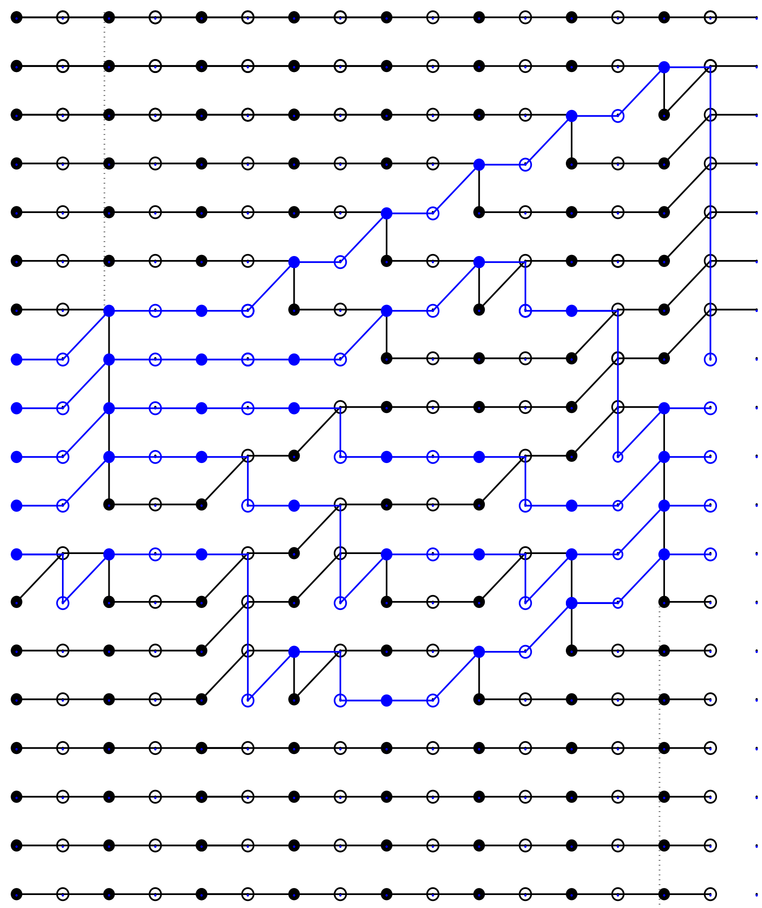
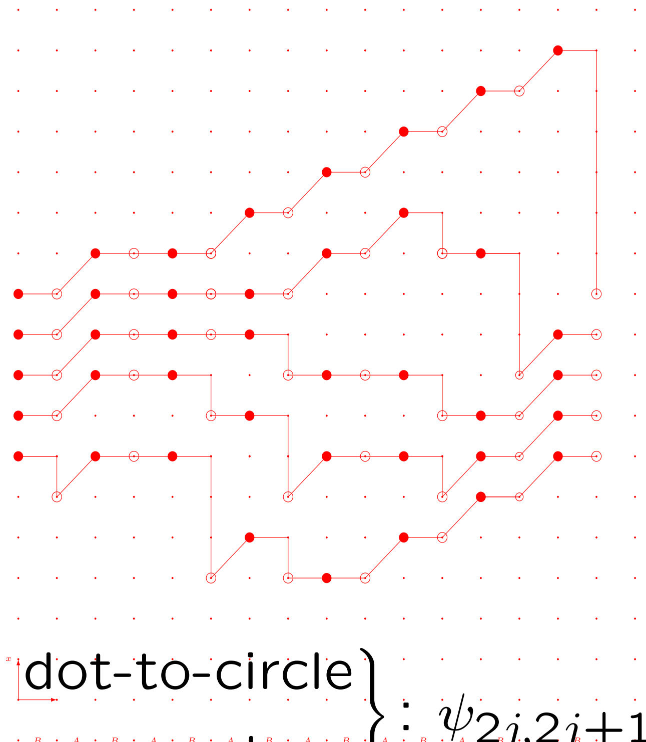


Figure 17: Lattice paths: superimposing in- and outliers.



$$\left.\begin{array}{r}\text{dot-to-circle}\\ \text{-steps}\end{array}\right\}: \psi_{2j,2j+1}(x,y) := \left\{\begin{array}{l} a^{x-y}, \ \text{if } y-x \leq 0 \\ 0 \ \text{otherwise}\end{array}\right\} = \int_{\Gamma_{0,a}} \frac{dz}{2\pi \mathrm{i} z} \frac{z^{x-y}}{1-\frac{a}{z}}$$

$$\left.\begin{array}{r}\text{circle-to-dot}\\ \text{-steps}\end{array}\right\}: \psi_{2j+1,2j+2}(x,y) := \left\{\begin{array}{l} a, \ \text{if } y-x = 1 \\ 1, \ \text{if } y-x = 0 \\ 0, \ \text{otherwise}\end{array}\right\} = \int_{\Gamma_0} \frac{dz}{2\pi \mathrm{i} z} z^{x-y}(1+az).$$

Then define in general: $\left\{\begin{array}{l} \psi_{r,s} = \psi_{r,r+1} * \ldots * \psi_{s-1,s}, \ \text{if } s > r \\ \qquad = 0, \text{if } s \leq r. \end{array}\right.$

where

$$f(x) * g(x) := \sum_{x \in \mathbb{Z}} f(x) g(x)$$

Lindström-Gessel-Viennot (Karlin-McGregor)

$$\mathbb{P}(r; x_1, \ldots, x_{2m+1})$$

$$= \frac{1}{Z_{n,m}} \det(\psi_{0,r}(-m+i-1, x_j))_{1\le i,j\le 2m+1} \det(\psi_{r,2n+1}(x_j, -m+i-1))_{1\le i,j$$

$$= \det\left(\mathbb{K}_{n,m}(r, x_i; r, y_j)\right)_{1\le i,j\le m}$$

Then (Eynard-Mehta formula)

$$\mathbb{K}_{n,m}(r, x; s, y)$$
$$= \sum_{i,j=1}^{2m+1} \psi_{r,2n+1}(x, -m+i-1)([A^{-1}]_{ij})\psi_{0,s}(-m+j-1, y) - \mathbb{1}_{r<s}\psi_{r,s}(x,y),$$

Lindström-Gessel-Viennot (Karlin-McGregor)

$$\mathbb{P}(r; x_1, \ldots, x_{2m+1}) = \frac{1}{Z_{n,m}} \det(\psi_{0,r}(-m+i-1, x_j))_{1\le i,j\le 2m+1} \det(\psi_{r,2n+1}(x_j, -m+i-1))_{1\le i,j}$$

Then (Eynard-Mehta formula)

$$\mathbb{K}_{n,m}(r,x;s,y) = \sum_{i,j=1}^{2m+1} \psi_{r,2n+1}(x,-m+i-1)([A^{-1}]_{ij})\psi_{0,s}(-m+j-1,y) - \mathbb{1}_{r<s}\psi_{r,s}(x,y),$$

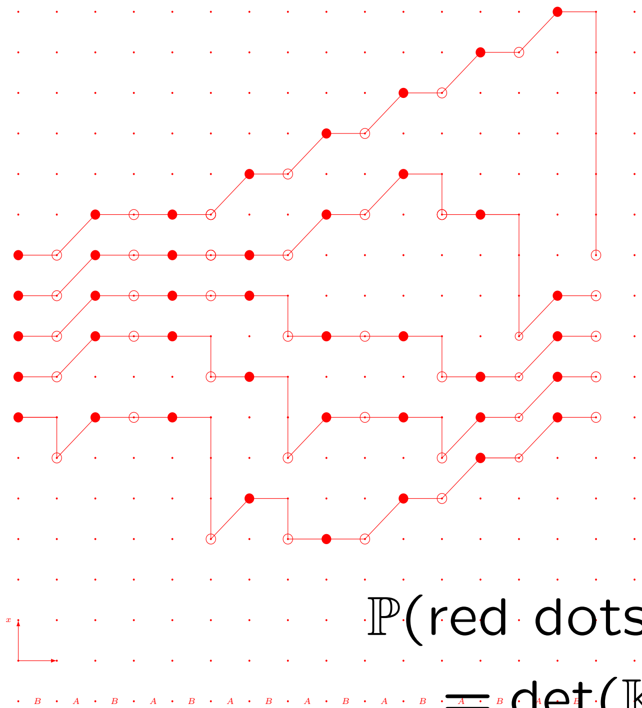
with the entries of the $(2m+1)\times(2m+1)$ matrix A,

$$A_{ij} = \psi_{0,2n+1}(-m+i-1, -m+j-1) = f_i * g_j,$$

defined in terms of the functions

$$f_j(y) = \psi_{0,n}(-m+j-1,y) = \oint_{\Gamma_{0,a}} \frac{dz}{2\pi \mathrm{i} z} \frac{z^{j-1}}{z^{m+y}} \overbrace{\left(\frac{1+az}{1-\frac{a}{z}}\right)^{n/2}}^{\rho_a^L(z)}$$

$$g_k(x) = \psi_{n,2n+1}(x,-m+k-1) = \oint_{\Gamma_{0,a}} \frac{dz}{2\pi \mathrm{i} z} \frac{z^{m+x}}{z^{k-1}} \underbrace{\frac{(1+az)^{n/2}}{\left(1-\frac{a}{z}\right)^{n/2+1}}}_{\rho_a^R(z)},$$



$$\mathbb{P}(\text{red dots at } x_1, \ldots, x_k, \text{ along vertical line } Y_n)$$
$$= \det(\mathbb{K}_{n,m}(n, x_i; n, x_j))_{1 \le i,j \le k}$$

Consider the kernel at a single (half-way even) time n:

$$\mathbb{K}_{n,m}(n,x;n,y) := \mathbb{K}_{n,m}(r,x;s,y)\Big|_{r=s=n} = \sum_{i,j=1}^{2m+1} g_i(x)([A^{-1}]_{ij})f_j(y).$$

where

$$A_{k\ell} = f_k * g_\ell,$$

Find linear combinations $\tilde{f}_i = \sum_{1 \le j \le i} c_{ij} f_j$ and $\tilde{g}_i = \sum_{1 \le j \le i} c'_{ij} g_j$ such that

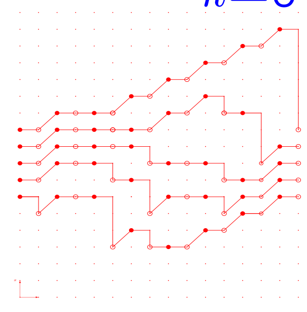
$$A_{k\ell} = \delta_{k,\ell}$$

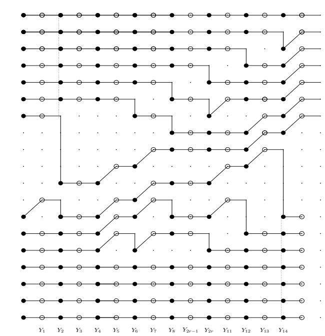
• **Kernel (inlier paths)**: Expression in terms of **orthogonal polynomials** P_k **and** $\widehat{P}_k$ **on the circle for the weight** $\rho_a^L(z)\rho_a^R(z)$:

$$\mathbb{K}_{n,m}(n,x;n,y) = \sum_{k,\ell=1}^{2m+1} \tilde{g}_k(x)([\tilde{A}^{-1}]_{k\ell})\tilde{f}_\ell(y) \text{ with } \tilde{A} = \mathbb{1}, \qquad \text{(inlier paths)}$$

$$= \sum_{i,j=1}^{2m+1} \tilde{g}_i(x)\tilde{f}_j(y)$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma_{r_2}} \frac{dw}{w}\rho_a^R(w) \oint_{\gamma_{r_1}} \frac{dz}{z}\rho_a^L(z)\frac{w^{x+m}}{z^{y+m}} \sum_{k=0}^{2m} P_k(z)\widehat{P}_k(w^{-1}).$$





• **Kernel (outlier paths)** $\tilde{\mathbb{K}}_{n,m}(n,x;n,y) = \delta_{x,y} - \mathbb{K}_{n,m}(n,x;n,y)$.

• **Kernel for the outlier paths at different levels:** (Extended kernel)

$$\tilde{\mathbb{K}}_{n,m}(2r,x;2s,y) \qquad \text{(outlier paths)}$$

$$= -\mathbb{1}_{s<r}\psi_{(2s-2r)}(x,y) + \psi_{n-2r}(x,\cdot) * \tilde{\mathbb{K}}_{n,m}(n,\cdot\ ;n,\circ) * \psi_{(2s-n)}(\circ,y),$$

Then the kernel for the Double Aztec Diamond reads:

$$\widetilde{\mathbb{K}}_{n,m}(2r,x;2s,y) = \{\text{Kernel for the single Aztec diamond}\}$$

$$+ (-1)^{x-y} \sum_{k=2m+1}^{\infty} b_{-x,r}(k)[(\mathbb{1}-\mathcal{K})^{-1}a_{-y,s}](k)$$

SCALING:

$$\frac{n}{2} = t, \qquad m = \frac{2t}{a + a^{-1}} + \sigma \rho t^{1/3}$$

$$x = 2a^2\theta\tau_1 t^{2/3} + \xi_1 \rho t^{1/3}, \qquad y = 2a^2\theta\tau_2 t^{2/3} + \xi_2 \rho t^{1/3}$$

$$r = t + (1 + a^2)\theta\tau_1 t^{2/3}, \qquad s = t + (1 + a^2)\theta\tau_2 t^{2/3},$$

Given the weight $0 < a < 1$ on vertical dominoes, define:

$$v_0 := -\frac{1 - a}{1 + a} < 0, \quad A^3 := \frac{a(1 + a)^5}{(1 - a)(1 + a^2)},$$

$$\rho := -Av_0 > 0, \quad \theta := \sqrt{\rho(a + a^{-1})} > 0,$$

$$\lim_{t\to\infty}(-v_0)^{y-x+r-s}(-1)^{y-x}\widetilde{\mathbb{K}}_{n,m}(2r,x;2s,y)\rho t^{1/3}=\mathbb{K}^{\text{tac}}(\tau_1,\xi_1;\tau_2,\xi_2)$$

For $\tau_2>\tau_1$, one has the **tacnode process**, depending on a pressure σ:

$$\mathbb{K}^{\text{tac}}(\tau_1,\xi_1;\tau_2,\xi_2)=K^{\mathcal{A}}(\tau_1,\xi_1;\tau_2,\xi_2)$$

$$+\frac{g(\tau_1,\xi_1)}{g(\tau_2,\xi_2)}2^{1/3}\int_{\tilde\sigma}^{\infty}\left((\mathbb{1}-K_{\text{Ai}})_{\tilde\sigma}^{-1}\mathcal{A}^{\tau_1}_{\xi_1-\sigma}\right)(\lambda)\mathcal{A}^{(-\tau_2)}_{\xi_2-\sigma}(\lambda)d\lambda.$$

where

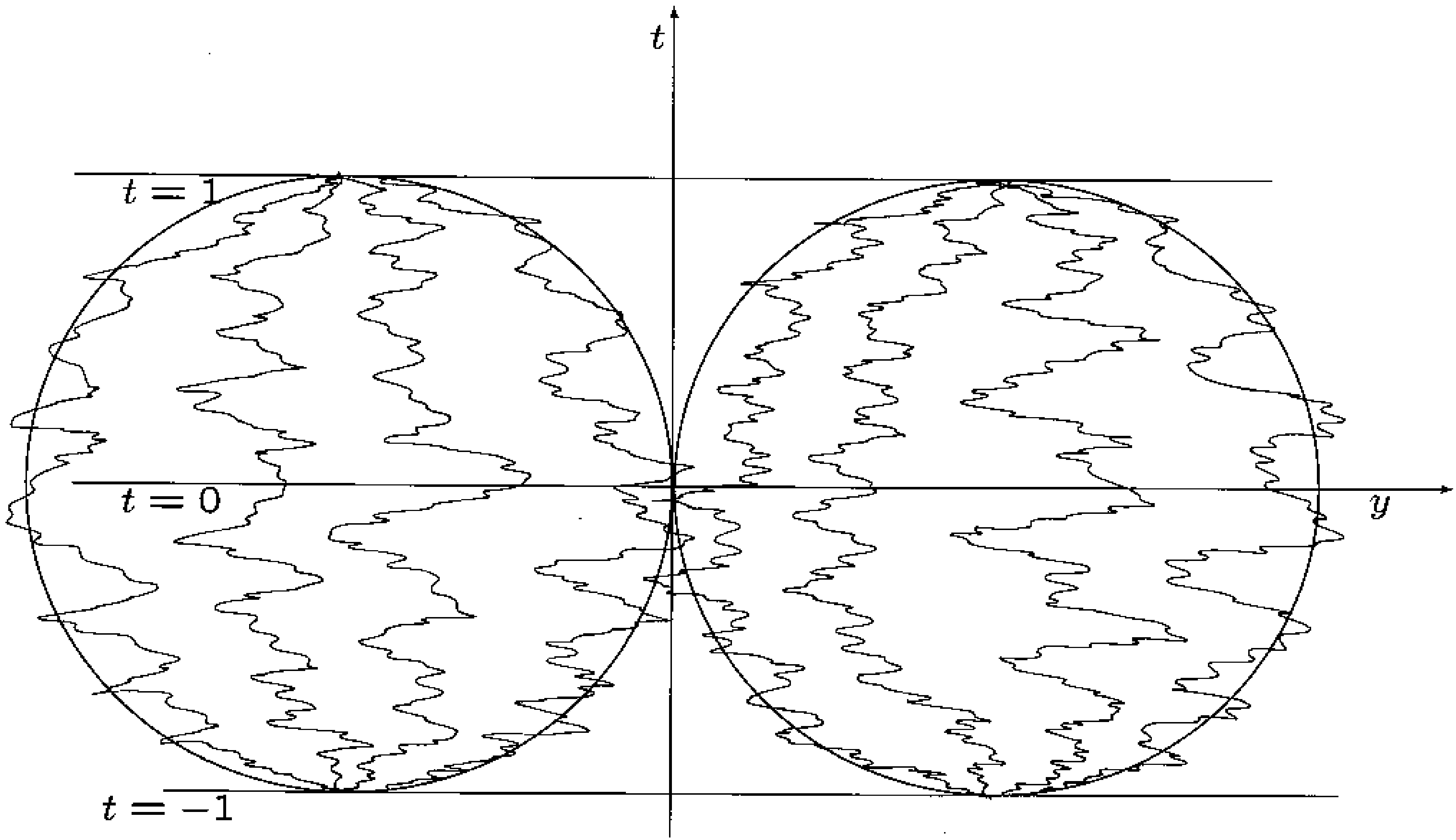
$$\mathcal{A}^{\tau}_{\xi}(\kappa):=\text{Ai}^{\tau}(\xi+2^{1/3}\kappa)-\int_0^{\infty}\text{Ai}^{\tau}(-\xi+2^{1/3}\beta)\text{Ai}(\kappa+\beta)d\beta$$

$$\text{Ai}^{(s)}(x):=e^{sx+\frac{2}{3}s^3}\text{Ai}(x+s^2).$$

(Adler-Ferrari-PvM ’09 , Adler-Johansson-PvM ’11)

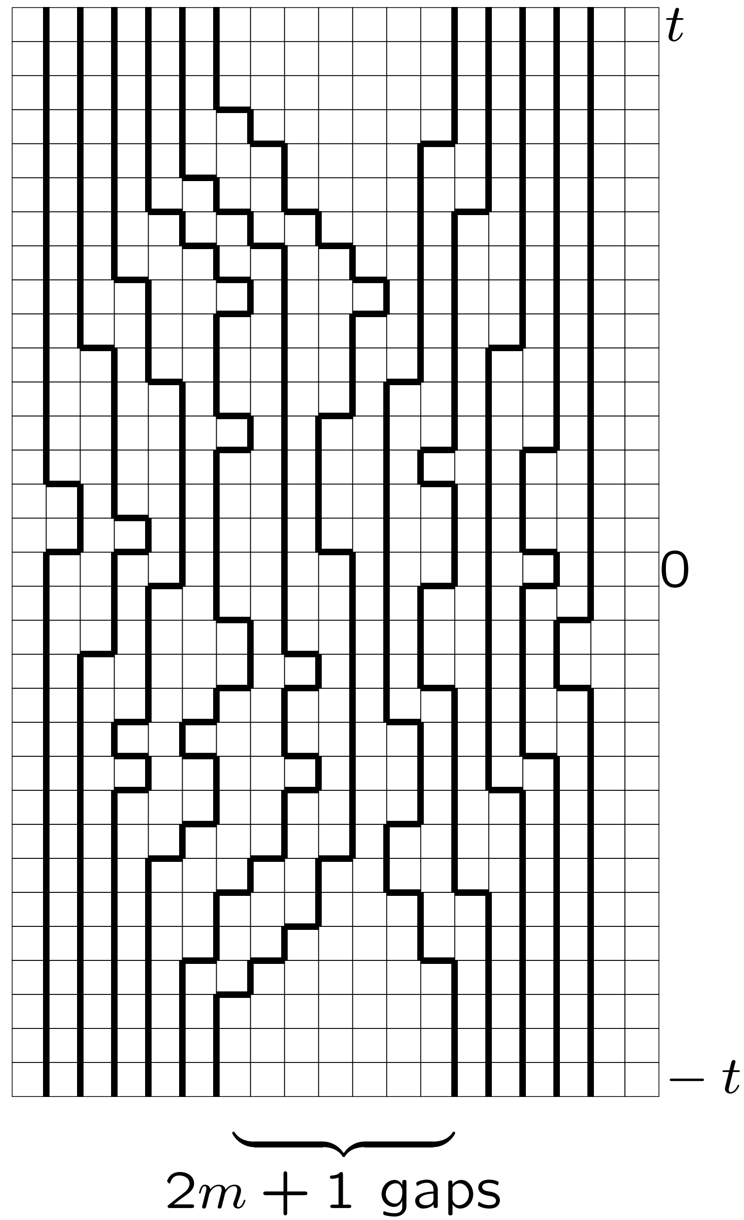
Same tacnode process for two groups

of non-intersecting Brownian motions



$n \to \infty$ $n \to \infty$

Same tacnode process for two groups of non-intersecting random walkers $\mathbf{x}(t)$ on $\mathbb{Z}$ with exponential holding time



OPEN PROBLEMS:

- Find PDE's for the transition probability for the tacnode process ?

- Limiting behavior for more general dimer models ?

- Can one find other singularities at the macroscopic level, leading to other statistical fluctuations (and other kernels) ?

- Relation with the KPZ equation ?

Thank you !