Current Fluctuations in the Exclusion Process

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Introduction

The statistical mechanics of a system at thermal equilibrium is encoded in the Boltzmann-Gibbs canonical law:

$$P_{\mathrm{eq}}(\mathcal{C}) = \frac{\mathrm{e}^{-E(\mathcal{C})/kT}}{Z}$$

the Partition Function Z being related to the Thermodynamic Free Energy F:

$$F = -kTLog\, Z$$

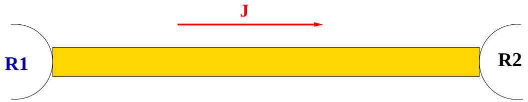
This provides us with a well-defined prescription to analyze systems *at equilibrium*:
(i) Observables are mean values w.r.t. the canonical measure.
(ii) Statistical Mechanics predicts fluctuations (typically Gaussian) that are out of reach of Classical Thermodynamics.

Systems far from equilibrium

No fundamental theory is yet available.

- What are the relevant macroscopic parameters?
- Which functions describe the state of a system?
- Do Universal Laws exist? Can one define Universality Classes?
- Can one postulate a general form for the microscopic measure?
- What do the fluctuations look like ('non-gaussianity')?

Example: Stationary driven systems in contact with reservoirs.



Rare Events and Large Deviations

Let $\epsilon_1, \ldots, \epsilon_N$ be N independent binary variables, $\epsilon_k = \pm 1$, with probability $1/2$. Their sum is denoted by $S_N = \sum_1^N \epsilon_k$.

- The Law of Large Numbers implies that $S_N/N \to 0$ a.s.
- The Central Limit Theorem implies that $S_N/\sqrt{N}$ converges towards a Gaussian Law.

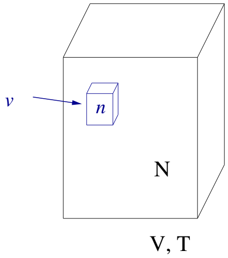
One can show that for $-1 < r < 1$, in the large N limit,

$$\mathrm{Pr}\left(\frac{S_N}{N} = r\right) \sim \mathrm{e}^{-N\,\Phi(r)}$$

where the positive function $\Phi(r)$ vanishes for $r = 0$.

The function $\Phi(r)$ is a Large Deviation Function: it encodes the probability of rare events.

Density fluctuations in a gas



Mean Density $\rho_0 = \frac{N}{V}$

In a volume v s. t. $1 \ll v \ll V$
$\langle \frac{n}{v} \rangle = \rho_0$

The local density varies around ρ_0 . Typical fluctuations scale as $\sqrt{v/V}$.

The probability of observing large fluctuations is given by

$$\Pr\left(\frac{n}{v} = \rho\right) \sim \mathrm{e}^{-v\,\Phi(\rho)} \text{ with } \quad \Phi(\rho_0) = 0$$

Thermodynamic Free Energy as a L. D. F.

The Large Deviation Function for density fluctuations is given by

$$\Phi(\rho) = f(\rho, T) - f(\rho_0, T) - (\rho - \rho_0)\frac{\partial f}{\partial \rho_0}$$

where $f = -\log Z(\rho, T)$ is the *free energy per unit volume* in units of kT.

The Free Energy of Thermodynamics can be viewed as a Large Deviation Function

Conversely, large deviation functions *may* play the role of potentials in non-equilibrium statistical mechanics.

Large deviation functions obey remarkable identities that remain valid far from equilibrium: *Fluctuation Theorem of Gallavotti and Cohen.*
In the vicinity of equilibrium the Fluctuation Theorem yields the fluctuation-dissipation relation (Einstein), Onsager's relations and linear response theory (Kubo).

Outline

1. Spectral Properties of the Exclusion Process
(O. Golinelli)

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2. Large deviations of the current in a closed ring: Bethe Ansatz
(S. Prolhac)

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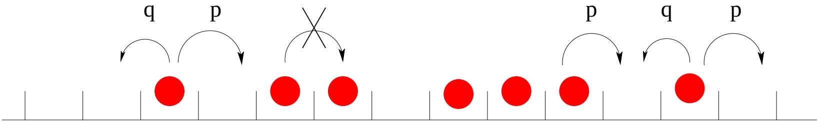
2. Large deviations of the current in a closed ring: Bethe Ansatz
(S. Prolhac)

3. Fluctuations of the current in an open system
(A. Lazarescu)

1. The Exclusion Process:

Spectral Properties

ASEP



Asymmetric Exclusion Process. A paradigm for non-equilibrium Statistical Mechanics.

- EXCLUSION: Hard core-interaction; at most 1 particle per site.
- ASYMMETRIC: External driving; breaks detailed-balance
- PROCESS: Stochastic Markovian dynamics; no Hamiltonian

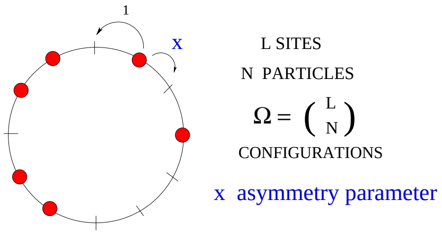
ORIGINS

- Interacting Brownian Processes (Spitzer, Harris, Liggett).
- Driven diffusive systems (Katz, Lebowitz and Spohn).
- Transport of Macromolecules through thin vessels.
  Motion of RNA templates.
- Hopping conductivity in solid electrolytes.
- Directed Polymers in random media. Reptation models.

APPLICATIONS

- Traffic flow.
- Sequence matching.
- Brownian motors.

Markov Equation for the ASEP on a ring



Master Equation for the Probability $P_t(x_1, \ldots, x_N)$ of being in configuration $1 \leq x_1 < \ldots < x_N \leq L$ at time t.

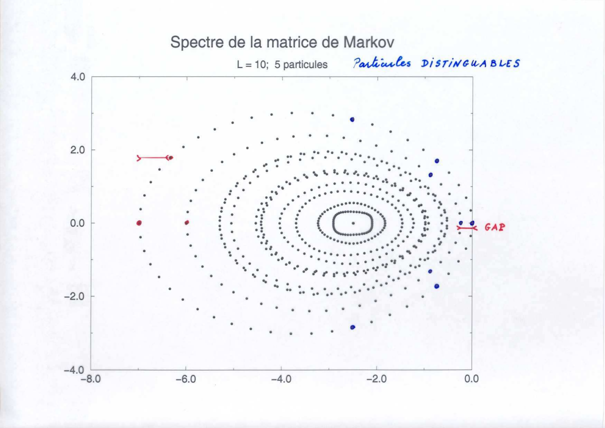
$$\begin{aligned} \frac{\mathrm{dP_t}}{\mathrm{d}t} &= \sum_i{}' \left[P_t(x_1, \ldots, x_i - 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N)\right] \\ &+ x \sum_i{}' \left[P_t(x_1, \ldots, x_i + 1, \ldots, x_N) - P_t(x_1, \ldots, x_i, \ldots x_N)\right] . \end{aligned}$$

The sum being restricted to admissible configurations.

Spectrum

Complex Eigenvalues $M\psi = E\psi$ with $\Re(E) \leq 0$ (Perron-Frobenius)

- Ground State $E = 0$ corresponds to the stationary state.
- Excited States $\rightarrow$ relaxation times.



ASEP: An Integrable System

MAPPING TO A NON-HERMITIAN SPIN CHAIN

$$M = \sum_{l=1}^{L} \left(\mathbf{S}_l^+ \mathbf{S}_{l+1}^- + x \mathbf{S}_l^- \mathbf{S}_{l+1}^+ + \frac{1+x}{4} \mathbf{S}_l^z \mathbf{S}_{l+1}^z - \frac{1+x}{4} \right)$$

Complex Eigenvalues $M\psi = E\psi$:

- Ground State : $E = 0$, $P = \Omega^{-1}$ (non-degenerate).
- Excited States : $\Re(E) < 0$ (Perron-Frobenius).

Excitations correspond to relaxation times.

TASEP : $x = 0$

Integrability of ASEP: Bethe Ansatz

Eigenvector ψ of M written as a linear combination of plane waves, with pseudo-momenta given by $z_1, \ldots z_N$:

$$\psi(x_1, \ldots, x_N) = \sum_{\sigma \in \Sigma_N} \mathcal{A}_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

The Bethe Equations provide us with the quantification of the z_i's:

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x z_i z_j - (1+x) z_i + 1}{x z_i z_j - (1+x) z_j + 1}$$

The corresponding eigenvalue is given by

$$E(z_1, z_2 \ldots z_N) = \sum_{i=1}^{N} \frac{1}{z_i} + x \sum_{i=1}^{N} z_i - N(1+x)\,.$$

The special case of TASEP

Eigenvectors of M as linear combinations of plane waves, with pseudo-momenta given by $z_1, \dots z_N$:

$$\psi(x_1, \dots, x_N) = \det\left(\frac{2^{x_j}(z_i+1)^{j-x_j}}{(z_i-1)^j}\right) \quad \text{for} \quad 1 \le i, j \le N$$

- ψ is an eigenfunction with eigenvalue $\mathbf{E} = \frac{1}{2}(-\mathbf{N} + \sum_j \mathbf{z_j})$.
- Cancellation of the two-particle collision terms $(x_{k-1} = x_k - 1)$.
- Bethe Equations

$$(1 - z_i)^N (1 + z_i)^{L-N} = -2^L \prod_{j=1}^{N} \frac{z_j - 1}{z_j + 1} \quad \text{for} \quad i = 1, \dots N$$

Note that the r.h.s. is a constant independent of i: DECOUPLING.

Procedure for solving the TASEP Bethe Equations

- For any given value of Y, *SOLVE* $(1 - z_i)^N (1 + z_i)^{L-N} = Y$. The roots are located on Cassini Ovals
- *CHOOSE* N *roots* $z_{c(1)}, \dots z_{c(N)}$ amongst the L available roots, with a *choice set* $c : \{c(1), \dots, c(N)\} \subset \{1, \dots, L\}$.
- *SOLVE* the self-consistent equation $\mathbf{A_c(Y) = Y}$ where

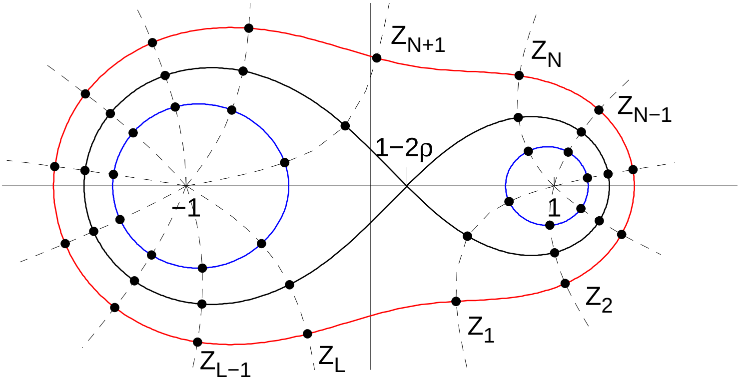
$$A_c(Y) = -2^L \prod_{j=1}^{N} \frac{z_{c(j)} - 1}{z_{c(j)} + 1} .$$

- *DEDUCE* from the value of Y, the $z_{c(j)}$'s and the energy corresponding to the choice set c :

$$2E_c(Y) = -N + \sum_{j=1}^{N} z_{c(j)} .$$

Labelling the roots of the Bethe Equations

The loci of the roots (for $x = 0$) are remarkable curves: **The Cassini Ovals**



Calculation of the GAP

The first excited state is solution of a transcendental equation. For a density ρ:

$$E_1 = \underbrace{-2\sqrt{\rho(1-\rho)}\frac{6.509189337\ldots}{L^{3/2}}}_{\text{RELAXATION}} \pm \underbrace{\frac{2i\pi(2\rho-1)}{L}}_{\text{OSCILLATIONS}}.$$

• Non-diffusive: Largest relaxation time $T \sim L^z$ with $z = 3/2$ *(D. Dhar, L.H. Gwa and H. Spohn, D. Kim)*.

• Oscillations $\rightarrow$ Traveling waves probed by dynamical correlations *(M. Barma, S. Majumdar, P. Krapivsky)*.

• Classification of higher excitations *(J. de Gier and F.H.L. Essler, 2006)*.

2. Current Fluctuations

on a ring

Large Deviations of the Current

Statistics of the total current Y_t: total distance covered by all the N particles, hopping on a ring of size L, between time 0 and time t.

Let $P_t(\mathcal{C}, Y)$ be the joint probability of being at time t in configuration $\mathcal{C}$ with $Y_t = Y$. The time evolution of this joint probability can be deduced from the original Markov equation, by splitting the Markov operator

$$M = M_0 + M_+ + M_-$$

The Laplace transform of $P_t(\mathcal{C}, Y)$ with respect to Y, defined as $\hat{P}_t(\mathcal{C}, \mu) = \sum_Y \mathrm{e}^{\mu Y} P_t(\mathcal{C}, Y)$, satisfies a dynamical equation governed by the deformation of the Markov Matrix M, obtained by adding a jump-counting *fugacity* μ:

$$\frac{d\hat{P}_t}{dt} = M(\mu)\hat{P}_t$$

with

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

Cumulant generating function

In the long time limit, $t \to \infty$

$$\langle \mathrm{e}^{\mu Y_t} \rangle \simeq \mathrm{e}^{E(\mu)t}$$

where $E(\mu)$ is the eigenvalue of $M(\mu)$ with maximal real part.
Equivalently, $\Phi(j)$, the *large deviation function* of the current

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

is related to $E(\mu)$ by a Legendre transform

$$\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$$

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz.

Bethe Ansatz for current statistics

The Bethe Equations are given by

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_i + \mathrm{e}^{\mu}}{x\mathrm{e}^{-\mu} z_i z_j - (1+x) z_j + \mathrm{e}^{\mu}}$$

The eigenvalues of $M(\mu)$ are

$$E(\mu; z_1, z_2 \ldots z_N) = \mathrm{e}^{\mu} \sum_{i=1}^{N} \frac{1}{z_i} + x\mathrm{e}^{-\mu} \sum_{i=1}^{N} z_i - N(1+x)\,.$$

The Bethe equations do not decouple unless $x = 0$
(This case was solved by B. Derrida and J. L. Lebowitz, 1998).

TASEP CASE (Derrida Lebowitz 1998)

$E(\mu)$ is calculated by Bethe Ansatz to all orders in μ, thanks to the decoupling property of the Bethe equations.
The structure of the solution is given by a parametric representation of the cumulant generating function $E(\mu)$:

$$\mu = -\frac{1}{L}\sum_{k=1}^{\infty}\frac{[kL]!}{[kN]!\,[k(L-N)]!}\frac{B^k}{k}\,,$$

$$E = -\sum_{k=1}^{\infty}\frac{[kL-2]!}{[kN-1]!\,[k(L-N)-1]!}\frac{B^k}{k}\,.$$

Mean Total current:

$$J = \lim_{t\to\infty}\frac{\langle Y_t\rangle}{t} = \frac{N(L-N)}{L-1}$$

Diffusion Constant:

$$D = \lim_{t\to\infty}\frac{\langle Y_t^2\rangle - \langle Y_t\rangle^2}{t} = \frac{LN(L-N)}{(L-1)(2L-1)}\frac{C_{2L}^{2N}}{\left(C_L^N\right)^2}$$

Exact formula for the large deviation function.

Functional Bethe Ansatz for the General Case

After a change of variable, $y_i = \frac{1-\mathrm{e}^{-\mu} z_i}{1-x\mathrm{e}^{-\mu} z_i}$, the Bethe equations read

$$\mathrm{e}^{L\mu}\left(\frac{1-y_i}{1-xy_i}\right)^L = -\prod_{j=1}^{N}\frac{y_i - xy_j}{xy_i - y_j} \quad \text{for} \quad i = 1\ldots N\,.$$

Let T be auxiliary variable playing a symmetric role w.r.t. all the y_i:

$$\mathrm{e}^{L\mu}\left(\frac{1-T}{1-xT}\right)^L = -\prod_{j=1}^{N}\frac{T - xy_j}{xT - y_j} \quad \text{for} \quad i = 1\ldots N\,.$$

$$i.e.\, P(T) = \mathrm{e}^{L\mu}(1-T)^L\prod_{j=1}^{N}(xT - y_j) + (1-xT)^L\prod_{j=1}^{N}(T - xy_j) = 0.$$

But $P(y_i) = 0$ (Bethe Eqs.). Thus, $Q(T) = \prod_{i=1}^{N}(T - y_i)$ divides $P(T)$:

$$Q(T) \;\; \text{DIVIDES} \;\; e^{L\mu}(1-T)^L Q(xT) + (1-xT)^L x^N Q(T/x).$$

Functional Bethe Ansatz

There exist two polynomials $Q(T)$ and $R(T)$ such that

$$\boxed{Q(T)R(T) = e^{L\mu}(1-T)^L Q(xT) + x^N(1-xT)^L Q(T/x)}$$

where $Q(T)$ of degree N vanishes at the Bethe roots.
Functional Bethe Ansatz (Baxter's TQ equation): Restatement of the Bethe Ansatz as a purely algebraic problem. This equation is solved perturbatively w.r.t. μ.

Knowing $Q(T)$, we obtain an expansion of $E(\mu)$. This provides the full statistics of the current and its large deviations.

Cumulants of the Current

- Mean Current: $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$

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$$D \sim 4\phi L\rho(1-\rho)\int_0^\infty du \frac{u^2}{\tanh \phi u} e^{-u^2}$$

when $L \to \infty$ and $x \to 1$ with *fixed value* of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

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when $L \to \infty$ and $x \to 1$ with *fixed value* of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

- Third cumulant (Skewness):

$$\frac{E_3}{\phi(\rho(1-\rho))^{3/2}L^{5/2}} \simeq -\frac{4\pi}{3\sqrt{3}} +$$
$$12\int_0^\infty dudv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh\phi u \tanh\phi v}$$

$\to$ Non Gaussian fluctuations. TASEP limit for $\phi \to \infty$:

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

$$
\begin{aligned}
\frac{E_3}{6L^2} &= \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j}\\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j}\\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N-i}C_L^{N-j}C_L^{N+i+j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j}\\
&- \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2\\
&+ (1-x)\frac{N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2}\\
&- (1-x)\frac{N(L-N)}{6(L-1)(3L-1)}\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}
$$

The weakly symmetric case

For large system sizes, $L \to \infty$, in the scaling limit $x = 1 - \frac{\nu}{L}$, the cumulant generating function is given by

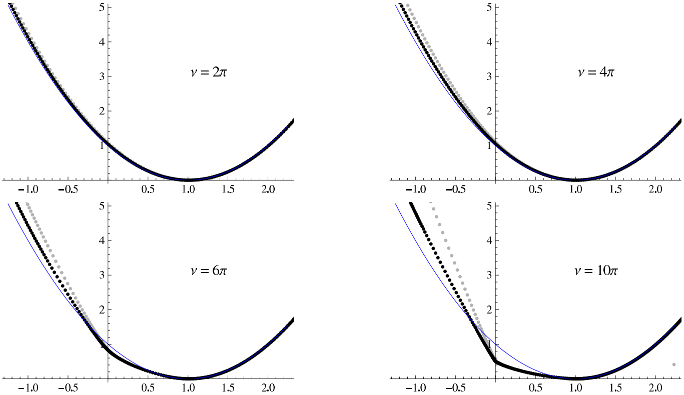
$$E\left(\frac{\mu}{L}\right) \simeq \frac{\rho(1-\rho)(\mu^2+\mu\nu)}{L} - \frac{\rho(1-\rho)\mu^2\nu}{2L^2} + \frac{1}{L^2}\psi[\rho(1-\rho)(\mu^2+\mu\nu)]$$

$$\text{with} \quad \psi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$$

- The B_j's are Bernoulli Numbers.
- Leading order (in $1/L$): Gaussian fluctuations.
- Subleading (in $1/L^2$) : Non-Gaussian correction.
- Phase transition (*predicted by T. Bodineau and B. Derrida*) when

$$\nu \geq \nu_c = \frac{2\pi}{\sqrt{\rho(1-\rho)}}$$

Behaviour of the large deviation function



The General Case (S. Prolhac, 2010)

The function $E(\mu)$ is again obtained in a parametric form:

$$\mu = -\sum_{k\geq 1} C_k \frac{B^k}{k} \quad \text{and} \quad E = -(1-x)\sum_{k\geq 1} D_k \frac{B^k}{k}$$

C_k and D_k are combinatorial factors enumerating some tree structures. There exists an auxiliary function

$$W_B(z) = \sum_{k\geq 1} \phi_k(z) \frac{B^k}{k}$$

such that C_k and D_k are given by complex integrals along a small contour that encircles 0 :

$$C_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_{\mathcal{C}} \frac{dz}{2\,i\,\pi} \frac{\phi_k(z)}{(z+1)^2}$$

The function $W_B(z)$ contains the full information about the statistics of the current.

The function $W_B(z)$ is the solution of a functional Bethe equation:

$$W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)$$

where

$$F(z) = \frac{(1+z)^L}{z^N}$$

The operator X is a integral operator

$$X[W_B](z_1) = \oint_{\mathcal{C}} \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2) K(z_1, z_2)$$

with the kernel

$$K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}$$

Solving this Functional Bethe Ansatz equation to all orders enables us to calculate cumulant generating function. For $x = 0$, the TASEP result is readily retrieved.

The function $W_B(z)$ also contains information on the 6-vertex model associated with the ASEP.

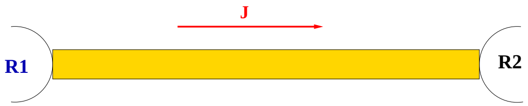
From the Physics point of view, the solution allows one to

- Classify the different universality classes (KPZ, EW).
- Study the various scaling regimes.
- Investigate the hydrodynamic behaviour.

3. Current Fluctuations in the open TASEP

The total Current in the Open System

The fundamental paradigm



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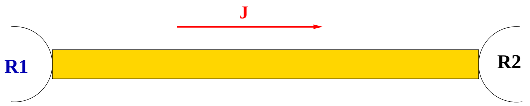
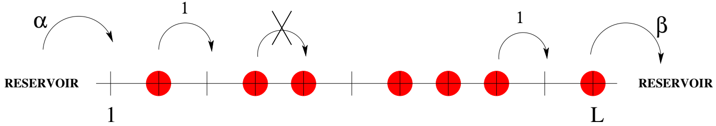


The totally asymmetric exclusion model with open boundaries



The Matrix Ansatz (DEHP, 1993)

The stationary probability of a configuration $\mathcal{C}$ is given by

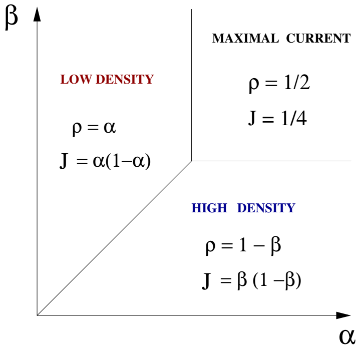
$$P(\mathcal{C}) = \frac{1}{Z_L} \langle\alpha| \prod_{i=1}^{L} (\tau_i D + (1-\tau_i) E) |\beta\rangle \, .$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).
The normalization constant is $Z_L = \langle\alpha| (D+E)^L |\beta\rangle$

The operators D and E, the vectors $\langle\alpha|$ and $|\beta\rangle$ satisfy

$$\begin{aligned} D\,E &= D + E \\ D\,|\beta\rangle &= \frac{1}{\beta}|\beta\rangle \\ \langle\alpha|\,E &= \frac{1}{\alpha}\langle\alpha| \end{aligned}$$

Phase Diagram



Large Deviations of the Current: Framework

Let N_t be the TOTAL (time-integrated) current through the system between 0 and t. When a particle enters the system:

$$N_t = N_t + 1$$

- Expectation value: $\lim_{t\to\infty} \frac{\langle N_t \rangle}{t} = J(\alpha, \beta, L) = \frac{Z_{L-1}}{Z_L}$
- Variance: $\lim_{t\to\infty} \frac{\langle N_t^2 \rangle - \langle N_t \rangle^2}{t} = \Delta(\alpha, \beta, L)$
- Cumulant Generating Function: $\langle \exp(\mu\, N_t) \rangle \simeq \exp(E(\mu) t)$

The **Large-Deviation Function** $\Phi(j)$ of the total current

$$P\left(\frac{N_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

is the *Legendre transform* of the Cumulant Generating Function $E(\mu)$.

Full Current Statistics

In the case $\alpha = \beta = 1$, a parametric representation of the cumulant generating function $E(\mu)$:

$$
\begin{aligned}
\mu &= -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)]!}{[k(L+1)]!\,[k(L+2)]!} \frac{B^k}{2k} \,, \\
E &= -\sum_{k=1}^{\infty} \frac{(2k)!}{k!} \frac{[2k(L+1)-2]!}{[k(L+1)-1]!\,[k(L+2)-1]!} \frac{B^k}{2k} \,.
\end{aligned}
$$

First cumulants of the current

- Mean Value : $J = \frac{L+2}{2(2L+1)}$
- Variance : $\Delta = \frac{3}{2} \frac{(4L+1)![L!(L+2)!]^2}{[(2L+1)!]^3(2L+3)!}$
- Skewness :
  $E_3 = 12 \frac{[(L+1)!]^2[(L+2)!]^4}{(2L+1)[(2L+2)!]^3} \left\{ 9 \frac{(L+1)!(L+2)!(4L+2)!(4L+4)!}{(2L+1)![(2L+2)!]^2[(2L+4)!]^2} - 20 \frac{(6L+4)!}{(3L+2)!(3L+6)!} \right\}$
  For large systems: $E_3 \to \frac{2187-1280\sqrt{3}}{10368} \pi \sim -0.0090978...$

Full Current Statistics

For arbitrary (α, β), the parametric representation of $E(\mu)$ is

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha,\beta)\frac{B^k}{2k}$$

$$E = -\sum_{k=1}^{\infty} D_k(\alpha,\beta)\frac{B^k}{2k}$$

with

$$C_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{z} \quad \text{and} \quad D_k(\alpha,\beta) = \oint_{\{0,a,b\}} \frac{dz}{2i\pi}\frac{F(z)^k}{(1+z)^2}$$

where

$$F(z) = \frac{-(1+z)^{2L}(1-z^2)^2}{z^L(1-az)(z-a)(1-bz)(z-b)}, \qquad a = \frac{1-\alpha}{\alpha}, \qquad b = \frac{1-\beta}{\beta}$$

Some explicit expressions

- Mean Current: (Same expression as in DEHP)

$$J = \frac{D_1(\alpha, \beta)}{C_1(\alpha, \beta)}$$

- Fluctuations: (an expression more compact than the one of 1995)

$$\Delta = \frac{D_1\, C_2 - D_2\, C_1}{C_1^3}$$

- Saddle point analysis in the low density phase: $(\rho = \alpha)$

$$\begin{aligned}
E_1 &= \rho(1-\rho) \\
E_2 &= \rho(1-\rho)(1-2\rho) \\
E_3 &= \rho(1-\rho)(1-6\rho+6\rho^2) \\
E_4 &= \rho(1-\rho)(1-2\rho)(1-12\rho+12\rho^2) \\
E_5 &= \rho(1-\rho)(1-30\rho+150\rho^2-240\rho^3+120\rho^4) \ \ldots
\end{aligned}$$

An old Formula for Δ:

$$\Delta = \frac{\langle\alpha|\,C^{N-1}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle} - \frac{\langle\alpha|\,C^{N-1}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle} \sum_{n=1}^{N-1} \left[\frac{\langle\alpha|\,C^{n-1}DC^{N-n}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle} - \frac{\langle\alpha|\,C^{n-1}DC^{N-1-n}\,|\beta\rangle}{\langle\alpha|\,C^{N-1}\,|\beta\rangle} \right]$$

$$+ \sum_{n=0}^{N-1} \frac{(2n)!}{n!\,(n+1)!} \left[(N-n-1) \frac{\langle\alpha|\,C^{N-n-2}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle} - (N-n+1) \frac{\langle\alpha|\,C^{N-1}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle} \frac{\langle\alpha|\,C^{N-n-1}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle} \right]$$

$$+ 2 \frac{\langle\alpha|\,C^{N-1}\,|\beta\rangle}{\langle\alpha|\,C^{N}\,|\beta\rangle^{3}} \sum_{n=1}^{N} \langle\alpha|\,C^{N-n}\,|\beta\rangle \times [\langle\alpha|\,C^{n-1}DC^{N}\,|\beta\rangle - \langle\alpha|\,C^{N}DC^{n-1}\,|\beta\rangle]$$

$$- \frac{1}{\langle\alpha|\,C^{N}\,|\beta\rangle^{2}} \sum_{n=1}^{N-1} \langle\alpha|\,C^{N-n}\,|\beta\rangle \times [\langle\alpha|\,C^{n-1}DC^{N-1}\,|\beta\rangle - \langle\alpha|\,C^{N-1}DC^{n-1}\,|\beta\rangle]$$

$$- \frac{1}{\langle\alpha|\,C^{N}\,|\beta\rangle^{2}} \sum_{n=1}^{N-1} \langle\alpha|\,C^{N-n-1}\,|\beta\rangle \times [\langle\alpha|\,C^{n-1}DC^{N}\,|\beta\rangle - \langle\alpha|\,C^{N-1}DC^{n}\,|\beta\rangle]$$

(

where the matrix elements involved are given by expressions derived in ref. 14:

$$\langle\alpha|\,C^{N}\,|\beta\rangle = \frac{\alpha\beta}{\alpha-\beta} \left[R_N\left(\frac{1}{\beta}\right) - R_N\left(\frac{1}{\alpha}\right) \right] \langle\alpha\,|\,\beta\rangle \tag{59}$$

$$\langle\alpha|\,C^{m}DC^{n}\,|\beta\rangle = \sum_{p=0}^{n-1} \frac{(2p)!}{p!\,(p+1)!} \langle\alpha|\,C^{n+m-p}\,|\beta\rangle + R_n\left(\frac{1}{\beta}\right) \langle\alpha|\,C^{m}\,|\beta\rangle \tag{60}$$

where

$$R_n\left(\frac{1}{\beta}\right) = \sum_{p=1}^{n} \frac{p(2n-1-p)!}{n!\,(n-p)!} \frac{1}{\beta^{p+1}} \quad \text{for} \quad n \geqslant 1$$

$$R_0\left(\frac{1}{\beta}\right) = \frac{1}{\beta} \tag{61}$$

Behaviour in the TASEP Phase Diagram

In the limit $L \to \infty$ of systems of large size, we have

- Maximal Current phase $\alpha > 1/2$ and $\beta > 1/2$: Cumulants are independent from α and β and are the same as for $\alpha = \beta = 1$:

$$E_k \sim \pi(\pi L)^{k/2-3/2} \text{ for } \quad k \geq 2$$

- Low Density phase $\alpha < \min(\beta, 1/2)$: Use saddle-point and Lagrange Inversion Formula to obtain

$$E(\mu) = \frac{a}{a+1}\frac{\mathrm{e}^{\mu}-1}{\mathrm{e}^{\mu}+a}$$

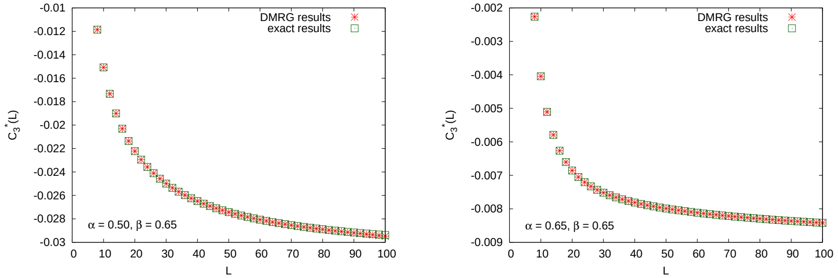
Agrees with Bethe Ansatz (Essler and de Gier) and with Macroscopic Fluctuation Theory.

- High Density phase is symmetrical to Low Density via $\alpha \leftrightarrow \beta$.
- Along the shock line $\alpha = \beta \leq 1/2$:

$$E_k \simeq \epsilon_k \alpha(1-\alpha)(1-2\alpha)^{k-1} L^{k-2} \text{ for } \; k \geq 2$$

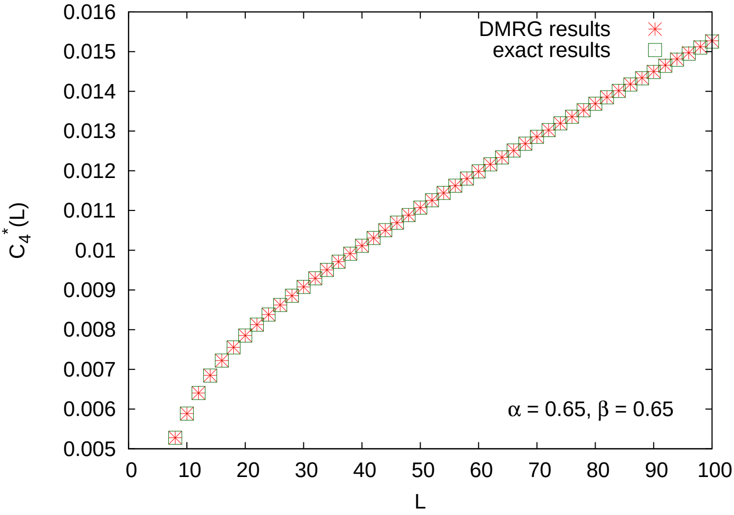
The coefficients $\epsilon_2 = 2/3$, $\epsilon_3 = -1/30$, $\epsilon_4 = 2/315$, $\epsilon_5 = -1/1890$..., can be calculated by *Domain Wall Theory.*

DMRG Results (M. Gorissen, C. Vanderzande)



SKEWNESS

Fourth Cumulant (DMRG)



Relation to the Macroscopic Fluctuation Theory

By Legendre Transform of the cumulant generating function, the Large Deviation Function of the current is found. In the limit of very large systems (Low Density Phase):

$$\Phi(j) = \alpha - r + r(1-r)\ln\left(\frac{1-\alpha}{\alpha}\frac{r}{1-r}\right)$$

where the current j is parametrized as $j = r(1-r)$. This expression is consistent with the one derived from the Macroscopic Fluctuation Theory of Jona-Lasinio et al. (cf T. Bodineau and B. Derrida).

- In the Macroscopic Fluctuation Theory, the hydrodynamic limit of the ASEP is a stochastic Burgers equation (in the weakly asymmetric regime).
- This allows one to define a probability measure on density profiles and currents.
- The optimal profile that generates the atypical current j is found by solving a variational problem.
- The probability of occurrence of this optimal profile allows one to calculate $\Phi(j)$.

Outline of the calculation

- The function $E(\mu)$ is expressed as the dominant eigenvalue of a deformation of the Markov Matrix: $M(\mu) = M + (\mathrm{e}^{\mu} - 1)M_1$
- $E(\mu)$ and its corresponding eigenvector are developed perturbatively w.r.t. μ.
- Construction of a Matrix Ansatz at each order k with $(2k+1)$ Tensor Products of quadratic algebras as in the multispecies exclusion process.
- k-th Matrix Ansatz $\rightarrow$ k-th term in the expansion of $E(\mu)$.

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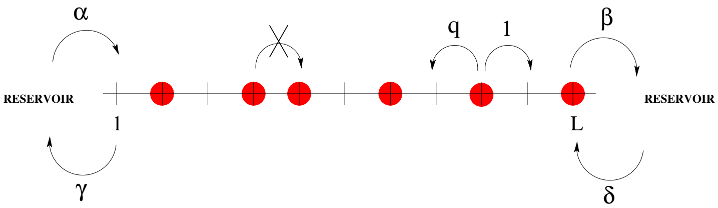
- k-th Matrix Ansatz $\rightarrow$ k-th term in the expansion of $E(\mu)$.

- The formula is checked against exact calculations on systems of sizes ≤ 10 for arbitrary rational values of (α, β).
- Large system size limits and known special cases are recovered.
- DMRG results of M. Gorissen and C. Vanderzande.

Current fluctuations in the general ASEP



For arbitrary values of $(\alpha, \beta, \gamma, \delta)$, and for any system size L the parametric representation of $E(\mu)$ is given by

$$\mu = -\sum_{k=1}^{\infty} C_k(\alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k} \quad \text{and} \quad E = -\sum_{k=1}^{\infty} D_k(\alpha, \beta, \gamma, \delta, L) \frac{B^k}{2k}$$

The coefficients C_k and D_k are generated by contour integrals in the complex plane.

The key function $F(z)$ is now given by

$$\frac{(1+z)^L(1+z^{-1})^L(z^2)_\infty(z^{-2})_\infty}{(a_+z)_\infty(a_+z^{-1})_\infty(a_-z)_\infty(a_-z^{-1})_\infty(b_+z)_\infty(b_+z^{-1})_\infty(b_-z)_\infty(b_-z^{-1})_\infty}$$

where $(x)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$ and $a_\pm$, $b_\pm$ are simple functions of the boundary rates.

The complex integrals are taken along a small contour in the complex plane that encircles the points 0, $q^k a_+, q^k a_-, q^k b_+$ and $q^k b_-$ for all integers $k \geq 0$.

The function $F(z)$ must be convoluted with the same kernel as in the closed ASEP problem

$$K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \left\{ \left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k \right\}$$

- This structure was obtained using the Tensor Matrix Ansatz. These results are of combinatorial nature. They are valid for arbitrary values of the parameters and for any system size with no restrictions.
- At first order, the formula for the mean current in the open ASEP (Sasamoto et al.) is retrieved.
- At second order, an expression of the Diffusion Constant that generalizes the old TASEP result is obtained.
- In the limit of large system sizes, the expression of the cumulant generating function becomes

$$E(\mu) = (1-q)\frac{a_+}{a_+ + 1}\,\frac{\mathrm{e}^{\mu} - 1}{\mathrm{e}^{\mu} + a_+}$$

  Agrees with Bethe Ansatz results of Essler and de Gier and with Macroscopic Fluctuation Theory.

- Scaling limits, Weakly Asymmetric regime are under investigation.
- There is an underlying Functional Bethe Ansatz structure. The relevant $W_B(z)$ has been found.

Conclusion

Exact solutions of the asymmetric exclusion process are paradigms for the behaviour of systems far from equilibrium in low dimensions: Dynamical phase transitions, Non-Gibbsean measures, Large deviations, Fluctuations Theorems...

Tensor products of quadratic algebras provides us with an efficient tool to solve very challenging problems: multispecies models; current fluctuations in the open TASEP.

The large deviation functions (LDF) appear as the right generalization of the thermodynamic potentials: convex, optimized at the stationary state, and non-analytic features can be interpreted as phase transitions. Besides, the LDF's satisfy remarkable identities (Gallavotti-Cohen) valid far away from equilibrium. The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.