Topological order

from quantum loops and nets

Paul Fendley

There are two-dimensional quantum lattice models with non-abelian topological order in the ground state.

Finding and understanding them requires exploiting properties of two-dimensional classical integrable models and conformal field theory.

Here I'll describe the simplest (so far!) such models which

1. require only interactions around a face (e.g. four-spin interactions on the square lattice)
2. are naturally expressed in terms of loops and nets simultaneously
3. possess "quantum self-duality"

Recent paper on the archive;

also essential collaborations with J. Jacobsen, V. Krushkal, E. Fradkin and N. Read

Why quantum loops?

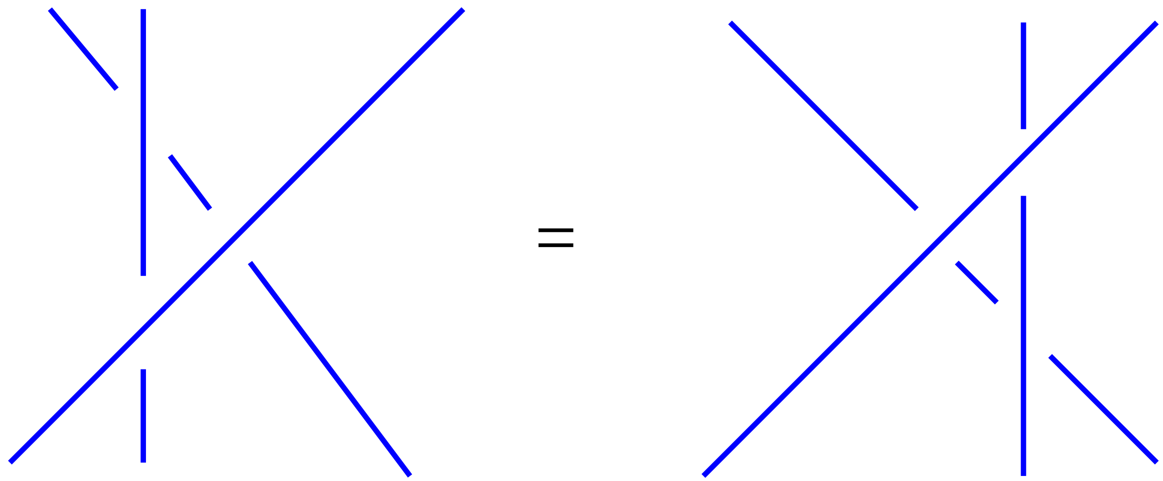
A convenient way of describing non-abelian anyons in 2+1 dimensions is in terms of their wordlines.

Then their statistics is the behavior of the wavefunction under braiding of the worldlines.

Brading is a purely topological property, and so if realizable, might prove the basis for a fault-tolerant quantum computer.

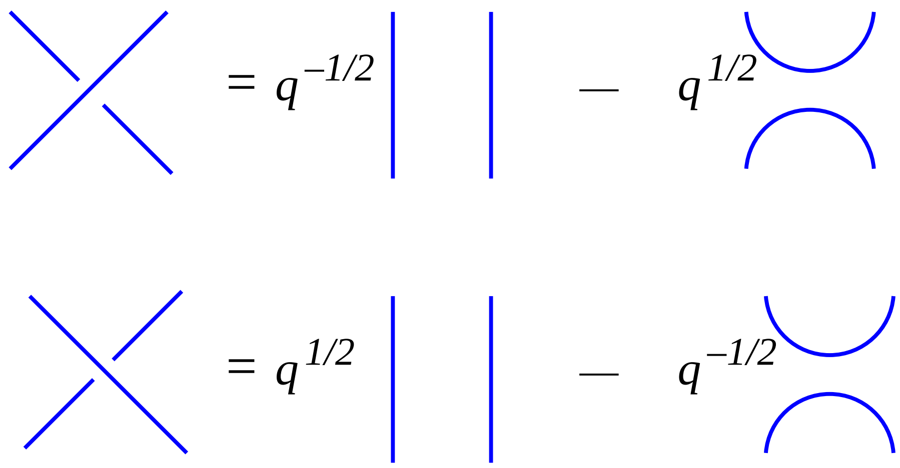
It is convenient to project the world line of the particles onto the plane. Then the braids become overcrossings and undercrossings

The braids must satisfy the consistency condition



i.e. the Yang-Baxter equation. The general rules governing braiding and fusion of non-abelian anyons are those of a chiral rational conformal field theory.

A simple way of satisfying the consistency conditions ($SU(2)_k$) leads to the Jones polynomial in knot theory. Replace the braid with the linear combination



so that the lines no longer cross. q is a parameter which is a root of unity in the cases of interest: Fibonacci anyons corresponds to $q = e^{i\pi/5}$.

In integrable language, the braid comes from taking the rapidity/spectral parameter to infinity in the S/R matrix.

This gives a representation of the braid group if the resulting loops satisfy d-isotopy.

- isotopy: Configurations related by deforming without making any lines cross receive the same weight.

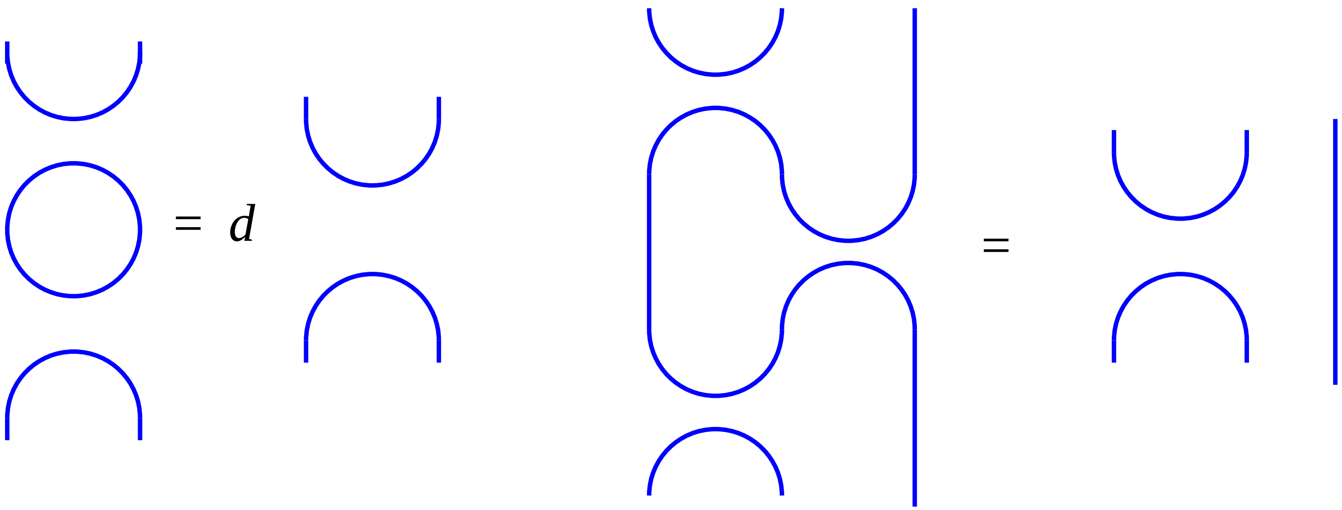
- d: A configuration with a closed loop receives weight

$$d = q + q^{-1}$$

relative to the configuration without the loop.

d is the quantum dimension of the anyon. The dimension of the $\mathcal{N}$-anyon Hilbert space grows as $d^{\mathcal{N}}$; think of it as the number of anyons created and annihilated in the loop. In terms of CFT, $d_a = S_{a0}/S_{00}$.

If you like algebras, the proper framework to analyze this is the Temperley-Lieb algebra, which graphically is



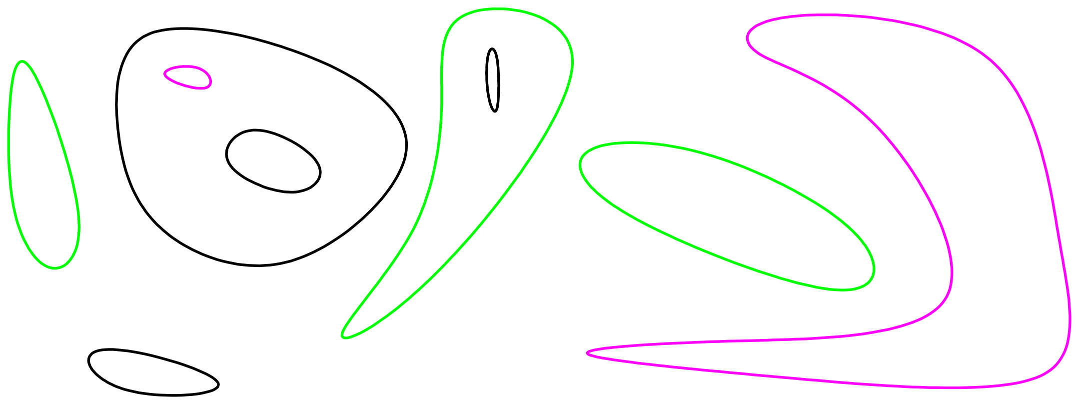
The task is now to find a lattice model whose quasiparticles have such braiding.

The clever idea of the the quantum loop model is to use these pictures to build the model:

1. find a 2d classical loop model which has a critical point
2. use each loop configuration as a basis element of the quantum Hilbert space
3. find a Hamiltonian whose ground state a sum over loop configurations with the appropriate weighting, so that
4. if you "cut" a loop, you end up with two deconfined anyonic excitations

Kitaev; Moessner and Sondhi; Freedman

In quantum loop models, each loop in the ground state gets a weight d ($= \tau$ for Fibonacci)

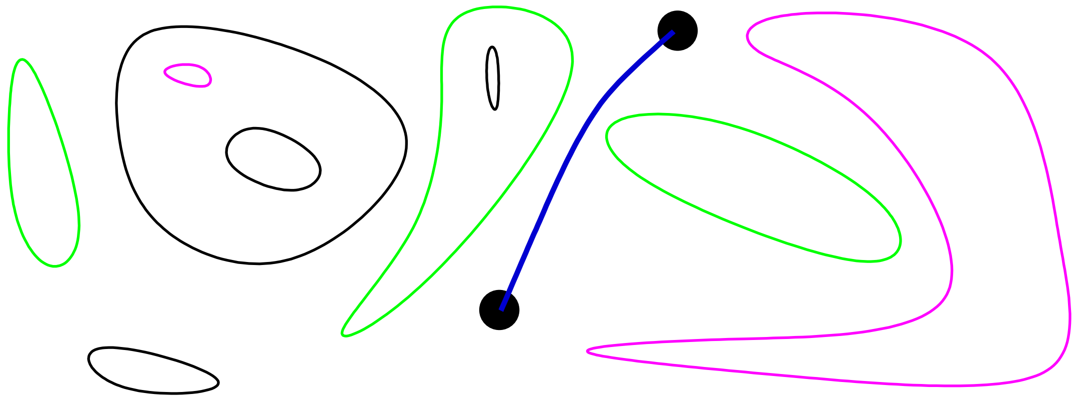


i.e. the ground state Ψ is the sum over all loop configurations

$$|\Psi\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle$$

where $n_{\mathcal{L}}$ is the number of loops in configuration $\mathcal{L}$.

The excitations with non-abelian braiding are defects in the sea of loops.



When the defects are deconfined, they will braid with each other like the loops in the ground state.

When

$$d = 2\cos[\pi/(k+2)] \qquad \text{i.e.} \quad q = e^{i\pi/(k+2)},$$

these are the statistics of Wilson loops in $SU(2)_k$ Chern-Simons theory. These are correlators in (chiral) conformal field theory.

Witten; Freedman, Nayak, Shtengel, Walker, and Wang

To have non-abelian braiding, the quantum loop models need to be gapped and have topological order.

However, for this all to work, the classical loop model needs to have a critical point.

In a little more detail...

The classical models being discussed have partition functions of the form

$$Z = \sum_{\mathcal{L}} w(\mathcal{L}) K^{L(\mathcal{L})}$$

where

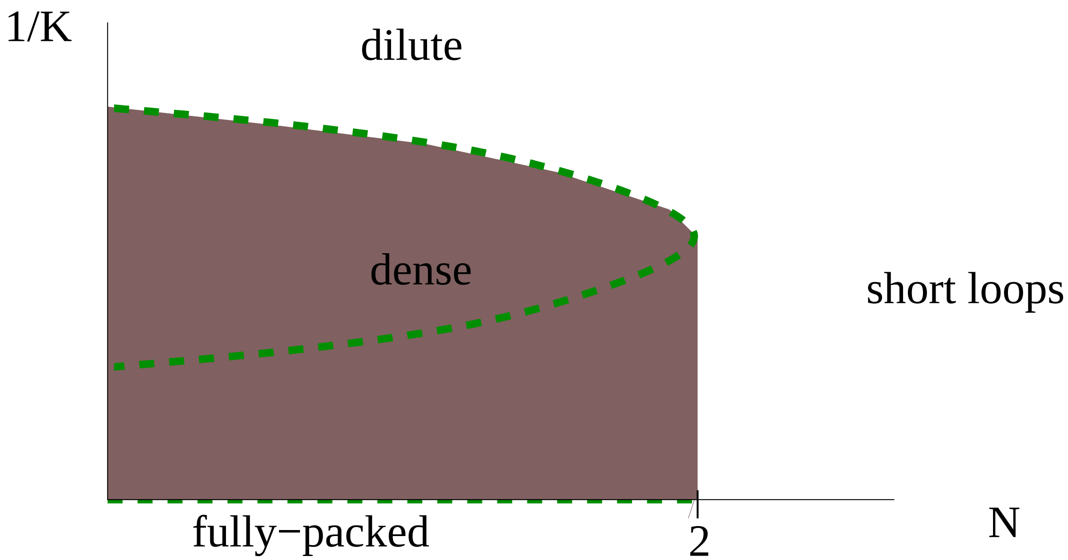
$w(\mathcal{L})$	is the topological weight of configuration $\mathcal{L}$,
$L(\mathcal{L})$	is the length of all the loops in $\mathcal{L}$,
K	is the weight per unit length

For closed loops which do not touch or cross, we have

$$w(\mathcal{L}) = N^{n_{\mathcal{L}}}$$

for some parameter N. This is usually called the $O(N)$ loop model.

For the $O(N)$ loop model in two dimensions, the phase diagram is



Typically, a critical point can occur when $K \approx 1$ (for the honeycomb lattice, the dilute-dense critical line occurs at $K = K_c = [2 + \sqrt{2 - N}]^{-1/2}$). The dense critical line is stable throughout the shaded region.

For $N > 2$, the model is not critical for any K – the partition function is dominated by short loops and so is not scale-invariant.

Important point:

At a critical point, loops of all sizes contribute to the partition function in the long-distance limit. This behavior is necessary to get topological order – otherwise a length scale appears.

This length scale physically is the confinement length.

Thus to build a quantum loop model from the classical $O(N)$ loop model, we must have $N \leq 2$.

In the wave function, each loop has weight $d = q + q^{-1}$. When q is a root of unity, $d \leq 2$.

However: This is quantum mechanics!

In any correlation function, each configuration is weighted by the probability amplitude squared. Thus with the naive inner product that all loop configurations are orthonormal,

$$N = d^2$$

We must have $d \leq \sqrt{2}$ for this construction to work!

Fibonacci anyons have $d = \tau = 2\cos(\pi/5) > \sqrt{2}$.

There are two ways of crashing through the $d = \sqrt{2}$ barrier to find quantum loop models whose deconfined excitations are Fibonacci anyons:

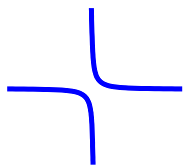
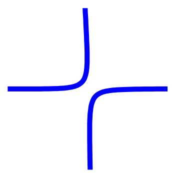
- Allow the loops to branch, so that they are not really loops, but rather nets.

- Change the inner product in the quantum-mechanical model.

It turns out that the two are essentially the same!

In the completely packed loop model, every link of the lattice is covered by a loop.

The only degrees of freedom are therefore the two choices of how the loops avoid each other at each vertex:

$|1\rangle =$  $|\widehat{1}\rangle =$ 

There is thus a quantum two-state system at every vertex.

If we set $\langle 1|\widehat{1}\rangle = 0$, then we have the $d = \sqrt{2}$ barrier.

So instead, don't make them orthogonal!

$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|\widehat{1}\rangle \\ \langle \widehat{1}|1\rangle & \langle \widehat{1}|\widehat{1}\rangle \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda^* & 1 \end{pmatrix}$$

For this to be positive definite, $|\lambda| < 1$.

Keep the ground state

$$|\Psi\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle$$

so that now

$$\langle\Psi|\Psi\rangle = \sum_{\mathcal{L}} \sum_{\mathcal{M}} d^{(n_{\mathcal{L}}+n_{\mathcal{M}})/2} \lambda^{n_X}$$

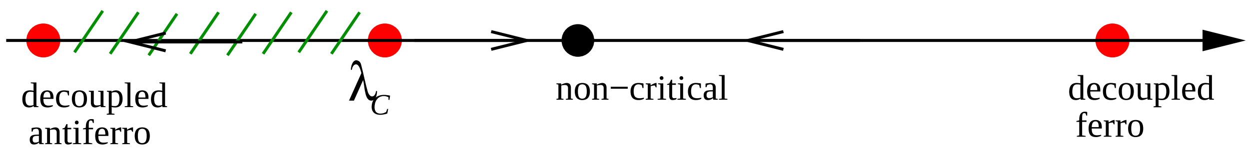
is a sum over two flavors of loops $\mathcal{L}$ and $\mathcal{M}$, which are different at n_X vertices.

Good news #1:

The classical loop model with $d = 2\cos(\pi/(k+2))$ has a critical point at

$$\lambda_c = -\sqrt{2}\sin\left(\frac{\pi(k-2)}{4(k+2)}\right)$$

following from level-rank duality of the BMW algebra. The CFT description is $SU(2)_k \times SU(2)_k/SU(2)_{2k}$. The classical loops have a critical phase for $\lambda < \lambda_c$.



Fendley and Jacobsen

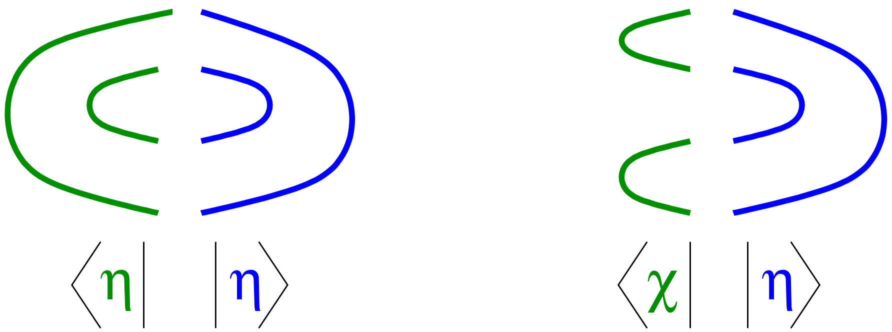
The ground state of the quantum model therefore is a sum over loops of all length scales.

The excitations should be deconfined!

Good news #2:

This inner product has nice topological properties.

Consider two four-anyon states with inner products:



$|\eta\rangle$ and $|\chi\rangle$ are topologically equivalent to $|1\rangle$ and $|\widehat{1}\rangle$, and $\langle\chi|\eta\rangle$ is topologically equivalent to a single loop. Thus we indeed want $\langle\widehat{1}|1\rangle \neq 0$.

In fact, maybe

$$\begin{aligned}\lambda &= \frac{\langle \widehat{1} | 1 \rangle}{\sqrt{\langle 1 | 1 \rangle \langle \widehat{1} | \widehat{1} \rangle}} = \frac{\langle \chi | \eta \rangle}{\sqrt{\langle \chi | \chi \rangle \langle \eta | \eta \rangle}} \\ &= \pm \frac{1}{d}\end{aligned}$$

???

Good news #1 means we should choose λ negative.

Setting $\lambda = -1/d$ leads to...

Good news #3:

Loops are nets!

Two natural orthonormal bases:

- $(|0\rangle, |1\rangle)$, where

$$|0\rangle = \frac{1}{\sqrt{d^2-1}}\left(d|\widehat{1}\rangle + |1\rangle\right)$$

- $(|\widehat{0}\rangle, |\widehat{1}\rangle)$, where

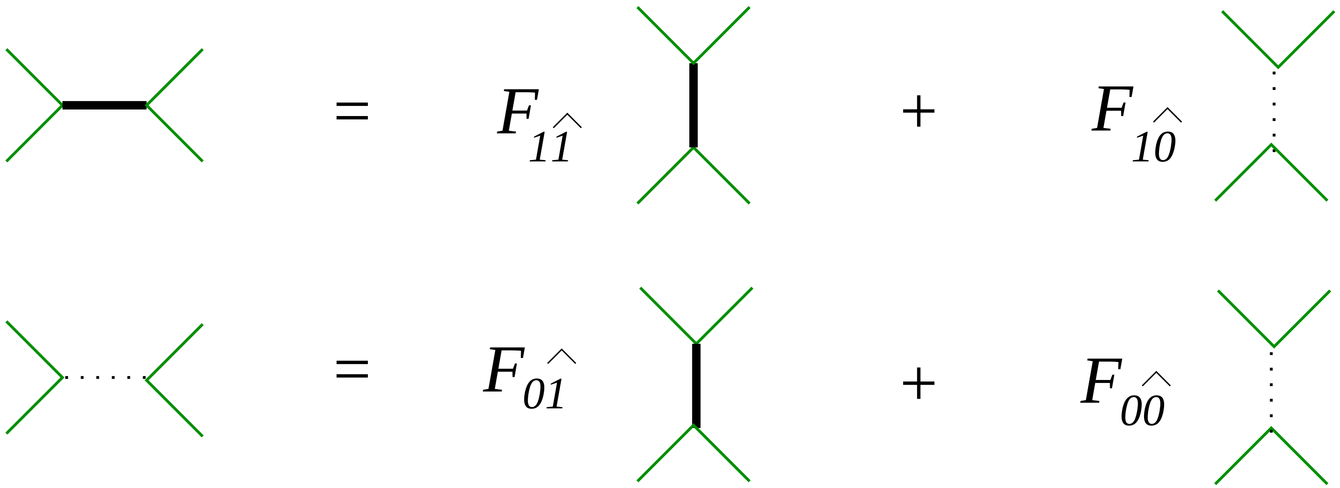
$$|\widehat{0}\rangle = \frac{1}{\sqrt{d^2-1}}\left(d|1\rangle + |\widehat{1}\rangle\right)$$

This indeed yields $\langle 0|1\rangle = \langle\widehat{0}|\widehat{1}\rangle = 0$ and $\langle 1|1\rangle = \langle\widehat{1}|\widehat{1}\rangle = 1$.

The unitary transformation relating the two bases is

$$F = \begin{pmatrix} \langle\widehat{0}|0\rangle & \langle\widehat{0}|1\rangle \\ \langle\widehat{1}|0\rangle & \langle\widehat{1}|1\rangle \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & \sqrt{d^2-1} \\ \sqrt{d^2-1} & -1 \end{pmatrix}$$

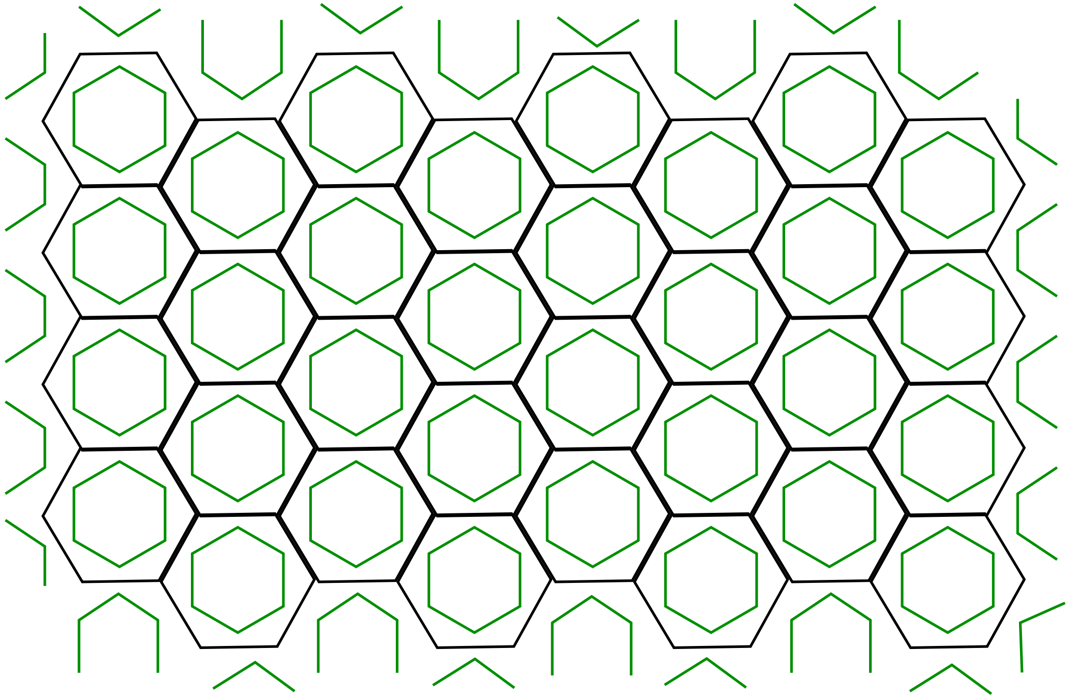
This F is the $SU(2)_k$ fusion matrix, describing anyons from quantum loops!



When lines meet at a vertex, they fuse to one of two states:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

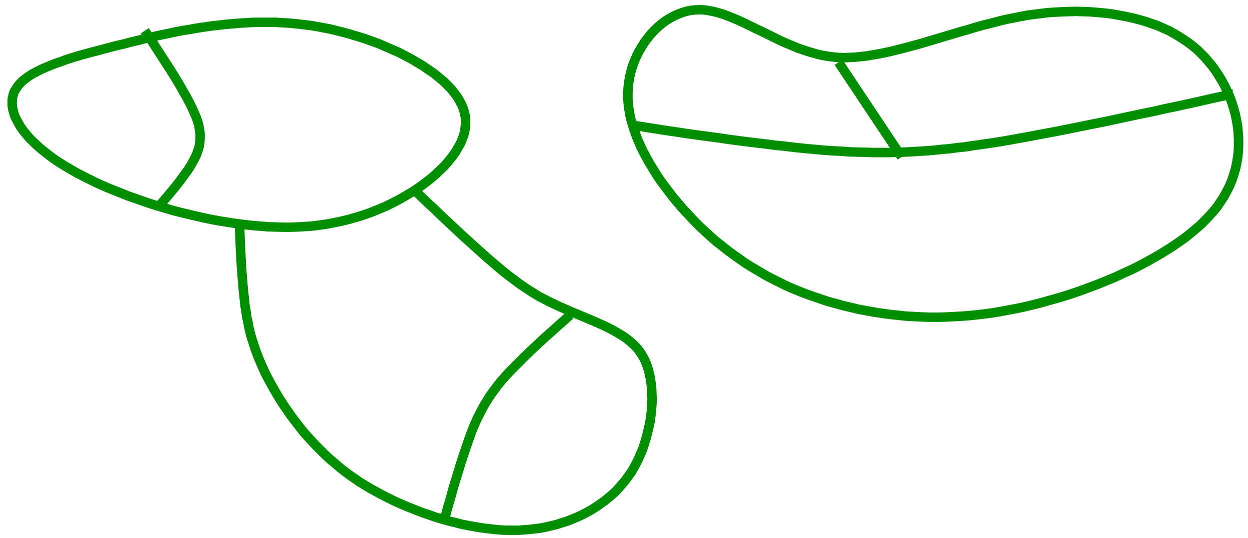
This suggests that we represent the state $|1\rangle$ as a filled link on the net lattice, e.g. if all vertices are in state $|1\rangle$:



Vertices of the loop lattice correspond to edges of the net lattice, so loops on Kagome correspond to nets on the honeycomb.

I call these nets because when the ground state $|\Psi\rangle$ is written in this orthonormal basis, there cannot be a single state $|1\rangle$ touching a vertex!

States which do contribute to $|\Psi\rangle$ look like



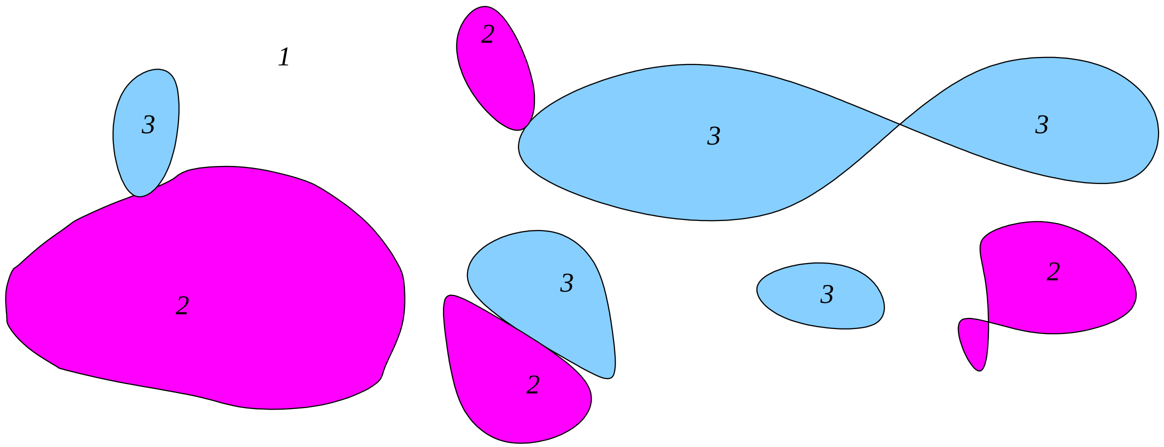
The weight of each loop configuration in the ground state is still $d^{n_{\mathcal{L}}}$.

Going to the orthonormal basis gives the weight of each net $|N\rangle$ to be

$$\langle N|\Psi\rangle = \left(\frac{1}{\sqrt{d^2-1}}\right)^{L_N} \chi_{\widehat{N}}(d^2)$$

where $\chi_{\widehat{N}}(d^2)$ is the chromatic polynomial, and L_N is the length of the net (the number of links covered).

The chromatic polynomial only depends on the topology of N. When Q is an integer, $\chi(Q)$ is the number of ways of coloring each region with Q colors such that adjacent regions have different colors.



Clasically, think of these loops as domain walls in the low-temperature expansion of the Q-state Potts model.

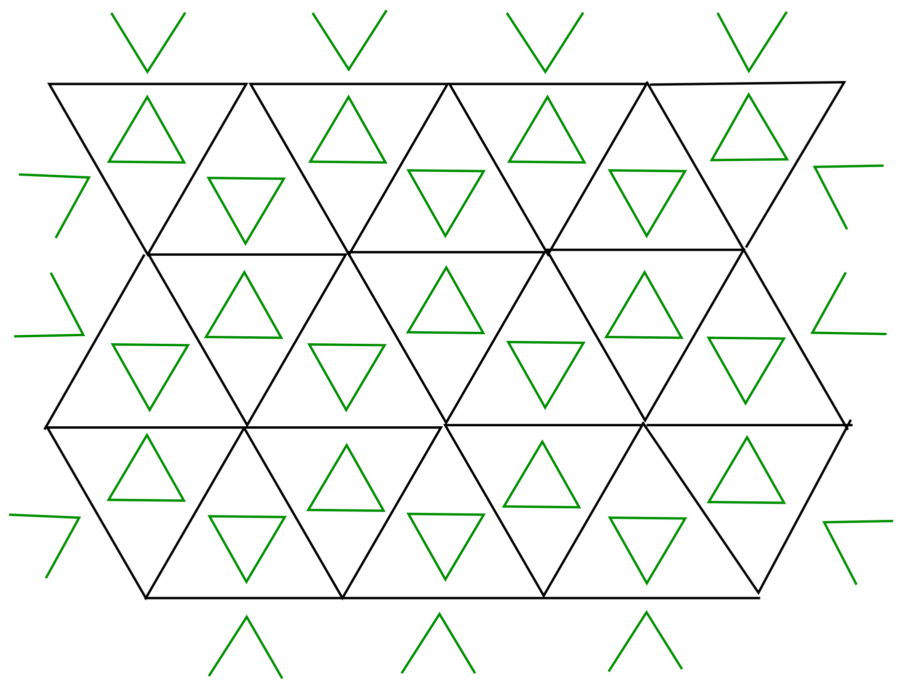
Good news #4:

Quantum self-duality means that on the square lattice, only four-spin interactions are required in the Hamiltonian!

In Levin and Wen's exactly solvable "string-net" models, 12-spin interactions are required.

Instead of writing the ground state $|\Psi\rangle$ in terms of nets, can also write them in terms of dual nets $|D\rangle$, in the $(\widehat{0}, \widehat{1})$ basis.

The dual nets live on the links of the dual of the net lattice, e.g. for loops on Kagomé when all vertices are in state $|\widehat{1}\rangle$:



The weight of each dual net $\left|D\right\rangle$ in the ground state is

$$\langle D|\Psi\rangle = \left(\frac{1}{\sqrt{d^2-1}}\right)^{L_D} \chi_{\widehat{D}}(d^2)$$

This is the same ground state $\left|\Psi\right\rangle$ in a new basis!

This quantum self-duality is highly non-obvious, and extremely useful.

A Hamiltonian H with $|\Psi\rangle$ a ground state can be found simply by demanding that H annihilate all states which are not nets and annihilate all states which are not dual nets.

For the square lattice:

$$H = \sum_{+}[P_1 P_0 P_0 P_0 + \text{rotations}] + \sum_{\square}[P_{\hat{1}} P_{\hat{0}} P_{\hat{0}} P_{\hat{0}} + \text{rotations}]$$

where P_i projects onto the states $|i\rangle$, and $P_{\hat{i}} = F P_i F$.

This is very much a non-abelian version of Kitaev's toric code.

Conclusions

- With the right inner product, we can crash the $d = \sqrt{2}$ barrier and find T-invariant lattice models with e.g. Fibonacci anyons.

- With the right inner product, loops and nets are equivalent.

- With the right inner product, the models exhibit quantum self-duality. The Hamiltonian on the square lattice needs involve only four-spin interactions.

- However, because $d > 1$, here the ground state should support non-abelian anyons!

- Pound your head on the wall enough, and sometimes the wall cracks before your head...