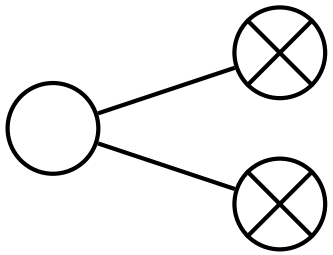
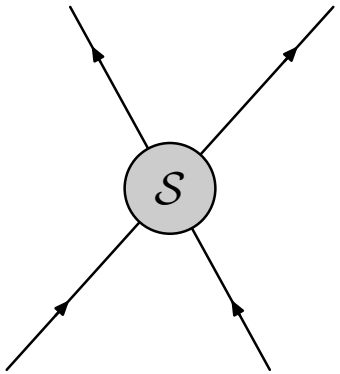
Integrability in AdS/CFT, the Hubbard Model and Quantum Algebra

Niklas Beisert

MPI für Gravitationsphysik
Albert-Einstein-Institut
Potsdam, Germany





ICFT 08
Edinburgh, April 12, 2008

Collaborations with P. Koroteev, F. Spill.
`hep-th/0511082`,`nlin/0610017`,`0704.0400`,`0708.1762`,`0802.0777`.

Strong/Weak Interpolation in the AdS/CFT Duality

AdS/CFT Duality

Duality between a string theory on AdS spacetime and CFT on boundary.

Main example:

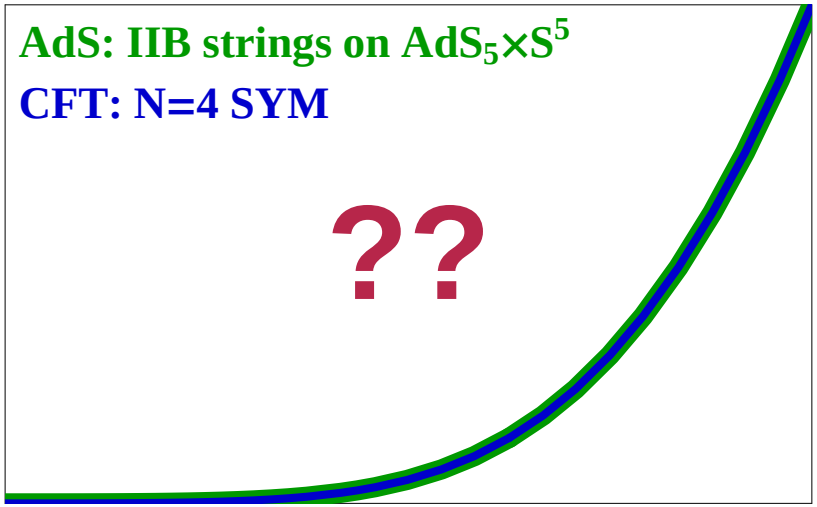
IIB **strings** on $AdS_5 \times S^5$:
- 2D string sigma model or
- 10D target space model.

$\mathrm{U}(N_\mathrm{c})$ $\mathcal{N} = 4$ susy **gauge theory**:
- 4D conformal QFT,
- technical similarities to QCD.

Two very different models are apparently equivalent. [hep-th/9711200 Maldacena]

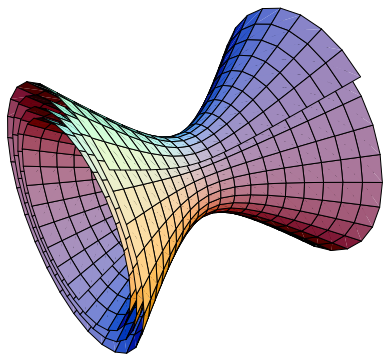
Parameters:

- N_c: string coupling constant $\sim 1/N_\mathrm{c}$,
  here: 't Hooft planar limit $N_\mathrm{c} \to \infty$,
- θ: theta angle, non-perturbative effects,
  here: irrelevant because of large N_c,
- λ: 't Hooft coupling,
  here: main parameter $\longrightarrow$



Spectrum of AdS/CFT

String Theory: $AdS_5 \times S^5$ background



States: Solutions X of classical equations of motion plus quantum corrections.

Energy: Charge E_X for translation along AdS-time.

Gauge Theory: Conformal $\mathcal{N} = 4$ SYM

States: Local operators. Local, gauge-inv. combinations of the fields, e.g.

$$\mathcal{O} = \mathrm{Tr}\, \Phi_1 \Phi_2 (\mathcal{D}_1 \mathcal{D}_2 \Phi_2)(\mathcal{D}_1 \mathcal{F}_{24}) + \ldots .$$

Energy: Scaling dimensions, e.g. two-point function in conformal theory

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = C |x - y|^{-2 D_{\mathcal{O}}(\lambda)}.$$

AdS/CFT: String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$?!

Strong/Weak Duality

Problem: Strong/weak duality.

Perturbative **strings** at $\lambda \to \infty$.

Perturbative **gauge theory** at $\lambda \approx 0$.

$$E(\lambda) = \sqrt{\lambda}\, E_0 + E_1 + E_2/\sqrt{\lambda} + \dots \qquad D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \dots$$

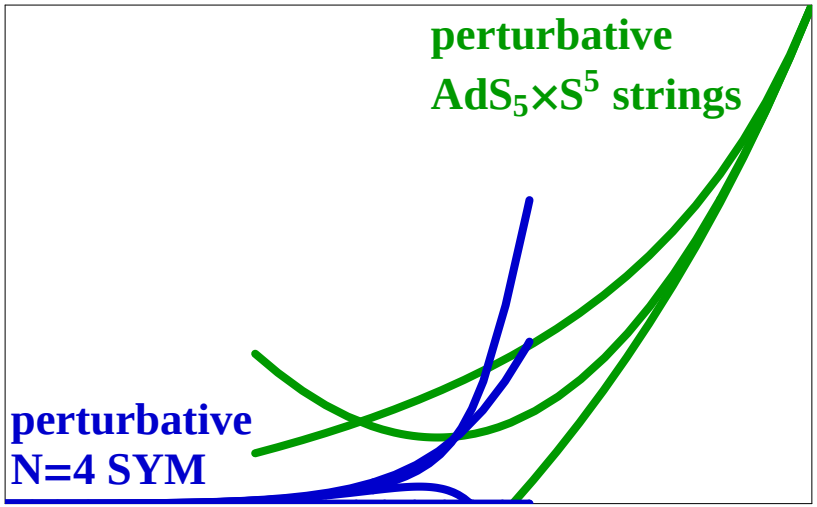
E_ℓ: ℓ loops (worldsheet model), practical limit: 1 or 2 loops.

D_ℓ: ℓ loops (Feynman diagram), practical limit: 3 or 4 loops.

Cannot compare:

- not analytically (term by term),
- not approximately (extrapolation),
- not numerically (lack of method).

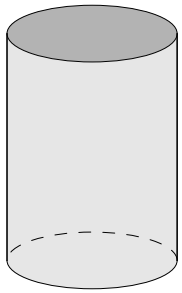
Need finite λ to compare!



Planar Limit

- Simplifications & surprises ['t Hooft Nucl. Phys. B72, 461] [Lipatov hep-th/9311037] [Faddeev Korchemsky] [Anastasiou, Bern Dixon, Kosower]
- AdS/CFT integrability [Lipatov ICTP 1997] [Mandal Suryanarayana Wadia] [Minahan Zarembo] [NB Kristjansen Staudacher] [Bena Polchinski Roiban] [NB, Staudacher hep-th/0307042] . . .

String Theory: $g_{\rm s} = 0$.

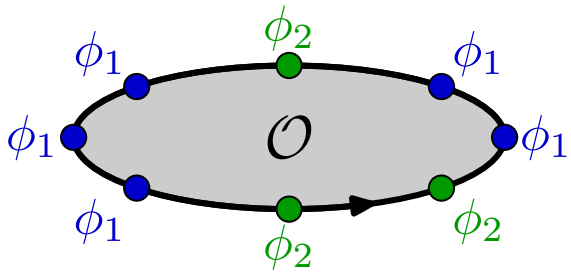


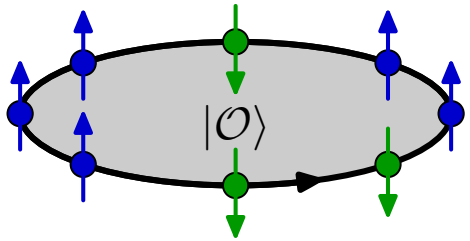
- Strictly cylindrical worldsheet.
- No string splitting or joining.

Gauge Theory: $N_{\rm c} = \infty$. Only single-trace operators relevant.

- Translate single-trace operators to spin chain states, e.g.

$$\mathcal{O} = \mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2 \qquad |\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$





- Energy spectrum: Eigenvalues of spin chain Hamiltonian.

Integrability in AdS/CFT

Planar AdS/CFT models apparently integrable.

Perturbative String Theory:

- integrable classical sigma model on supercoset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1)\times\mathrm{Sp}(2)}$, [Bena Polchinski Roiban]
- quantum corrections apparently integrable. [Arutyunov Frolov Staudacher] [Berkovits hep-th/0411170]

Perturbative Gauge Theory:

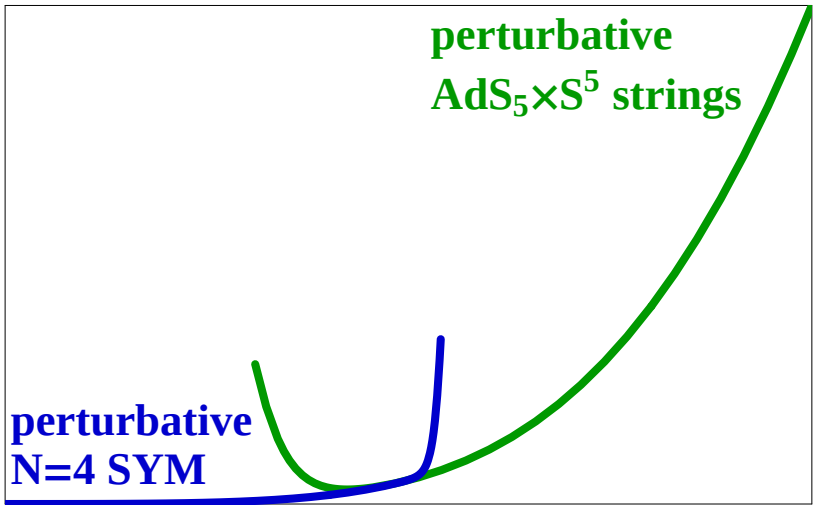
- one loop: integrable NN spin chain Hamiltonian, [Minahan Zarembo] [NB, Staudacher hep-th/0307042]
- higher loops: short-range corrections to Hamiltonian. [NB Kristjansen Staudacher]

Integrability enables precise computations:

Using constructions and educated guesses based on integrability proposed: [NB, Staudacher hep-th/0504190]

- all-order expansion at $\lambda \to \infty$, [NB Hernández López]
- all-order expansion at $\lambda \approx 0$. [NB, Eden Staudacher]

Small window of numerical overlap!



Strong/Weak Interpolation

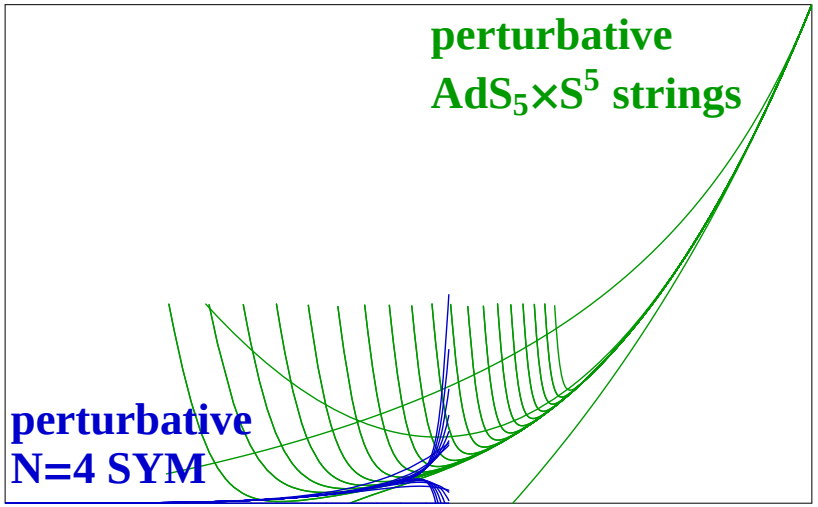
Convergence Properties:

Expansion at $\lambda \to \infty$

- Series asymptotic, no convergence,
- good approximation at low orders.

Expansion at $\lambda \approx 0$

- finite radius of convergence $|\lambda| < \pi^2$,
- defines holomorphic function.

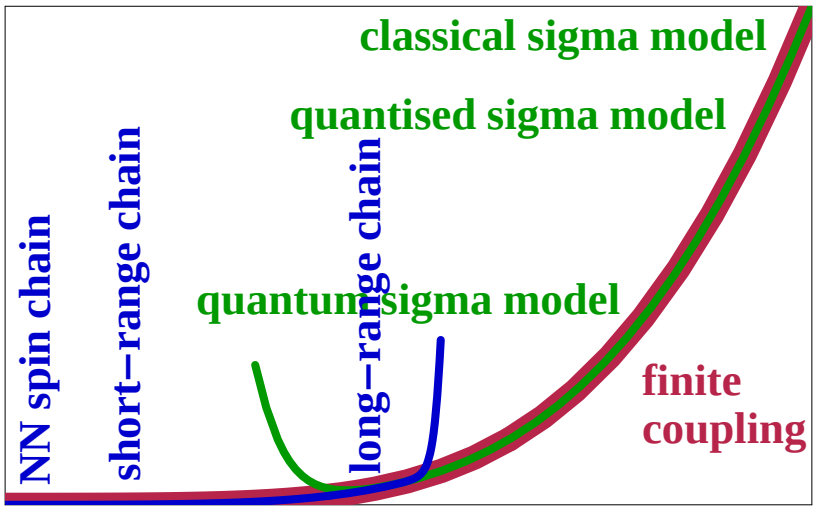


Finite Coupling:

- Integral representations exist,
- numerical evaluation convenient.

Finite coupling model:

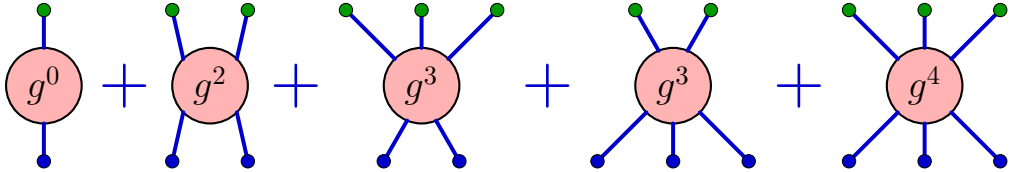
- quantum sigma model!
- long-range spin chain?!
- quantum algebra?



Bethe Ansatz

Short-Range Spin Chain

Action of perturbative spin chain Hamiltonian at $g \sim \sqrt{\lambda} \approx 0$

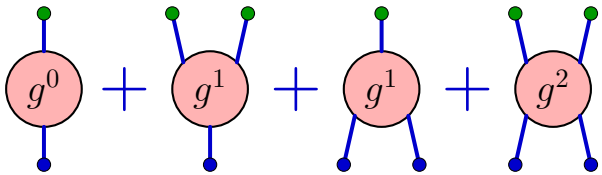
$\mathfrak{H}(g) =$ 

$+ \ldots$

- $\mathcal{O}(g^0)$: excitation number operator
- $\mathcal{O}(g^2)$: nearest-neighbour Hamiltonian (supersymmetric XXX$_{-1/2}$)
- $\mathcal{O}(g^4)$: next-nearest-neighbour corrections, range increases with order
- $\mathcal{O}(g^3)$: length fluctuates ...

At least: excitation number $\mathfrak{H}_0$ conserved, local action (on long chains)!

Representation of symmetry generators $\mathfrak{psu}(2,2|4)$

$\mathfrak{J}(g) =$ 

$+ \ldots .$

Not a coproduct! (or not clear how to interpret as such)

Asymptotic Bethe Ansatz

Spectrum? States of infinite spin chain with few "excitations": [Staudacher hep-th/0412188]

- Ferromagnetic vacuum: $|0\rangle = |\ldots 000\ldots\rangle$, all constituent particles '0'.
- One-magnon states with excitation $0 \to 1_A$ of momentum p

$$|A,p\rangle = \sum_a e^{ipa} |\ldots 0 \ldots \overset{\overset{a}{\downarrow}}{1_A} \ldots 0 \ldots\rangle, \qquad \mathcal{H}\,|A,p\rangle = E(p)|A,p\rangle.$$

$(8|\,8)$ admissible flavours 1_A of single excitations. [Berenstein Maldacena Nastase]

- Asymptotic two-magnon states, Hamiltonian eigenvalue $E(p) + E(q)$

$$|A,p;B,q\rangle \simeq \sum_{a \ll b} e^{ipa+iqb} |\ldots 0 \ldots \overset{\overset{a}{\downarrow}}{1_A} \ldots \overset{\overset{b}{\downarrow}}{1_B} \ldots 0 \ldots\rangle + \sum_{a \approx b} \ldots$$
$$+ S^{CD}_{AB}(p,q) \sum_{a \gg b} e^{ipa+iqb} |\ldots 0 \ldots \overset{\overset{b}{\downarrow}}{1_D} \ldots \overset{\overset{a}{\downarrow}}{1_C} \ldots 0 \ldots\rangle.$$

- Factorised scattering for three or more magnons (?!).

Residual Symmetry

QM particle model of 8 bosonic and 8 fermionic flavours on the circle.
Integrability: S-matrix as R-matrix of quasi-triangular Hopf algebra?!

Excitations transform as $(\mathbf{2}|\mathbf{2}) \times (\mathbf{2}|\mathbf{2})$ of $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$.

Consider just $(2|2)$ flavours and one copy of $\mathfrak{psu}(2|2)$. Generators:

- $\mathfrak{R}^a{}_b$: $\mathfrak{su}(2)$ subalgebra of internal symmetry $\mathfrak{su}(4)$.
- $\mathfrak{L}^\alpha{}_\beta$: $\mathfrak{su}(2)$ subalgebra of conformal symmetry $\mathfrak{su}(2,2)$.
- $\mathfrak{Q}^\alpha{}_b$: 4 (Poincaré) supercharges.
- $\mathfrak{S}^a{}_\beta$: 4 (conformal) supercharges.

$\mathfrak{psu}(2|2)$ has three-dimensional (exceptional!) central extension.

Need this central extension $\mathfrak{h} := \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ for consistency: [NB hep-th/0511082]

- $\mathfrak{C}$: Hamiltonian (up to integer shift),
- $\mathfrak{P}$: (classical) gauge variation,
- $\mathfrak{K}$: (quantum) gauge variation.

Lie Algebra $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ and Coproduct

Commutators define Lie algebra

- $\mathfrak{R}^a{}_b, \mathfrak{L}^\alpha{}_\beta$: canonical brackets of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ generators,
- $\mathfrak{C}, \mathfrak{P}, \mathfrak{K}$: central elements, • $\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^a{}_\beta$: supercharges

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^c{}_\delta\} = \delta^c_b \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\delta \mathfrak{R}^c{}_b + \delta^c_b \delta^\alpha_\delta \mathfrak{C},$$

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{Q}^\gamma{}_d\} = \varepsilon^{\alpha\gamma} \varepsilon_{bd} \mathfrak{P},$$

$$\{\mathfrak{S}^a{}_\beta, \mathfrak{S}^c{}_\delta\} = \varepsilon^{ac} \varepsilon_{\beta\delta} \mathfrak{K}.$$

Length fluctuations lead to non-trivial coproduct [Gomez Hernández] [Plefka Spill Torrielli]

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes 1 + \mathfrak{U}^{[A]} \otimes \mathfrak{J}^A$$

with $[\mathfrak{P}] = +2$, $[\mathfrak{Q}] = +1$, $[\mathfrak{R}] = [\mathfrak{L}] = [\mathfrak{C}] = 0$, $[\mathfrak{S}] = -1$, $[\mathfrak{K}] = -2$.
Abelian group-like generator $\mathfrak{U}$ measures magnon momentum $e^{ip/2}$.
Cocommutativity on centre: $\mathfrak{P} = g\alpha^{+1}(1 - \mathfrak{U}^{+2})$, $\mathfrak{K} = g\alpha^{-1}(1 - \mathfrak{U}^{-2})$.

Fundamental Representation

Have $(2|2)$ flavours of particles $\{|\phi^a\rangle, |\psi^\alpha\rangle\}$. Represent algebra! [NB hep-th/0511082]

Most general action compatible with $\mathfrak{su}(2) \times \mathfrak{su}(2)$

$$\mathfrak{Q}^\alpha{}_b|\phi^c\rangle = a\,\delta^c_b|\psi^\alpha\rangle, \qquad \mathfrak{S}^a{}_\beta|\phi^c\rangle = c\,\varepsilon^{ac}\varepsilon_{\beta\delta}|\psi^\delta\rangle,$$
$$\mathfrak{Q}^\alpha{}_b|\psi^\gamma\rangle = b\,\varepsilon^{\alpha\gamma}\varepsilon_{bd}|\phi^d\rangle, \qquad \mathfrak{S}^a{}_\beta|\psi^\gamma\rangle = d\,\delta^\gamma_\beta|\phi^a\rangle.$$

Imposing consistency of superalgebra

- fixes central charges $C = \frac{1}{2}(ad + bc)$, $P = ab$, $K = cd$,
- yields constraint $ad - bc = 1$ or $C^2 - PK = \frac{1}{4}$.

Cocommutativity constraints $P, K = g\alpha^{\pm 1}(1 - \mathfrak{U}^{\pm 2})$

- provide dispersion relation $C^2 = \frac{1}{4} + 4g^2 \sin^2(\frac{1}{2}p)$.
- Lattice-like (Brillouin zones) and almost relativistic: Deformed Poincaré.
- Elliptic curve with modulus $k = 4ig$ (rectangular complex torus).

Fundamental R-Matrix

Ansatz for fundamental R-matrix with $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry

$$\begin{aligned}
\mathcal{R}|\phi^a\phi^b\rangle &= A_{12}|\phi^{\{a}\phi^{b\}}\rangle - B_{12}|\phi^{[a}\phi^{b]}\rangle + \tfrac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\gamma\delta}|\psi^\gamma\psi^\delta\rangle,\\
\mathcal{R}|\psi^\alpha\psi^\beta\rangle &= -D_{12}|\psi^{\{\alpha}\psi^{\beta\}}\rangle + E_{12}|\psi^{[\alpha}\psi^{\beta]}\rangle - \tfrac{1}{2}F_{12}\varepsilon^{\alpha\beta}\varepsilon_{cd}|\phi^c\phi^d\rangle,\\
\mathcal{R}|\phi^a\psi^\beta\rangle &= G_{12}|\phi^a\psi^\beta\rangle + H_{12}|\psi^\beta\phi^a\rangle,\\
\mathcal{R}|\psi^\alpha\phi^b\rangle &= K_{12}|\phi^b\psi^\alpha\rangle + L_{12}|\psi^\alpha\phi^b\rangle.
\end{aligned}$$

Invariance $\mathcal{R}\circ\Delta(\mathfrak{J}^A) = \Delta_{\mathrm{op}}(\mathfrak{J}^A)\circ\mathcal{R}$ fixes $A,\ldots,L$ up to phase function.

YBE $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ fulfilled.

Questions to be addressed:

- What spin chain model does this R-matrix generate?
- What is the Hopf algebra and its universal R-matrix?
  Steps: $\star$ Quantum deformation? $\star$ Yangian? $\star$ Classical r-matrix?

Hubbard Model

One-Dimensional Hubbard Model

Electronic model: Fermionic oscillator $c_s, c_s^\dagger$, four states per site

$$|\circ\rangle = |0\rangle, \qquad |\uparrow\rangle = c_\uparrow^\dagger|0\rangle, \qquad |\downarrow\rangle = c_\downarrow^\dagger|0\rangle, \qquad |\updownarrow\rangle = c_\uparrow^\dagger c_\downarrow^\dagger|0\rangle.$$

Simple nearest-neighbour Hamiltonian with $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry

$$\mathcal{H}_{j,k}^{\mathrm{Hub}} = \sum_{\alpha=\uparrow,\downarrow} \left(c_{\alpha,j}^\dagger c_{\alpha,k} + c_{\alpha,k}^\dagger c_{\alpha,j} \right) + U c_{\uparrow,j}^\dagger c_{\downarrow,j}^\dagger c_{\downarrow,j} c_{\uparrow,j}.$$

Diagonalised by Lieb–Wu equations and $E = \sum_k \cos k_k$ [Lieb, Wu Phys. Rev. Lett. 20, 1445 (1968)]

$$1 = \exp(-ik_k K) \prod_{j=1}^{M} \frac{2\sin k_k - 2\Lambda_j + \frac{i}{2}U}{2\sin k_k - 2\Lambda_j - \frac{i}{2}U},$$

$$1 = \prod_{j=1}^{N} \frac{2\Lambda_k - 2\sin k_j + \frac{i}{2}U}{2\Lambda_k - 2\sin k_j - \frac{i}{2}U} \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{2\Lambda_k - 2\Lambda_j - iU}{2\Lambda_k - 2\Lambda_j + iU}.$$

Shastry's R-Matrix

One-dimensional Hubbard model has an interesting R-matrix [Shastry PRL 56,2453]

- Intricate form with around 10 coefficient functions.
- Not of difference form $\mathcal{R}(u_1, u_2) \neq \mathcal{R}(u_1 - u_2)$.
- Spectral parameters u_k defined on elliptic curve.
- Does not fit standard scheme $\mathrm{Y}(\mathfrak{g})$ for simple Lie (super)algebras $\mathfrak{g}$.
- Exceptional integrable structure?!

Consider the identification of bosonic/fermionic states

$$|\circ\rangle = |\phi^1\rangle, \qquad |\uparrow\rangle = |\psi^1\rangle, \qquad |\downarrow\rangle = |\psi^2\rangle, \qquad |\updownarrow\rangle = |\phi^2\rangle.$$

Integrable structures agree: [NB nlin.SI/0610017]

- Bethe equations can be made to coincide (choice of Bethe roots).
- Fundamental R-matrix equivalent to R-matrix of Hubbard chain.
- Hubbard chain integrability explained, simple construction for $\mathcal{R}$.
- Exceptional case of centrally extended $\mathfrak{psu}(2|2)$.

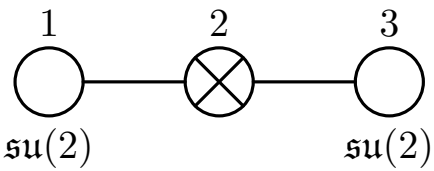
Quantum Deformation

Quantum Deformations $\mathrm{U}_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$

Can we quantum deform all of the above? [NB Koroteev]

- More convenient to formulate Hopf algebra & R-matrix at $q \neq 1$.
- Yangian double as contraction of quantum affine algebra at $q \to 1$.
- Quantum deformation of the Hubbard model?

Use **Chevalley–Serre basis**: $\mathfrak{E}_k, \mathfrak{H}_k, \mathfrak{F}_k$, $k = 1, 2, 3$



Drop two Serre relations for central extension (consistent with coalgebra!)

$$\big\{[\mathfrak{E}_1, \mathfrak{E}_2]_q, [\mathfrak{E}_3, \mathfrak{E}_2]_q\big\}_q = \mathfrak{P} \neq 0, \quad \big\{[\mathfrak{F}_1, \mathfrak{F}_2]_q, [\mathfrak{F}_3, \mathfrak{F}_2]_q\big\}_q = \mathfrak{K} \neq 0.$$

Construction of algebra and coalgebra standard, but deform by $\mathfrak{U}$

$$\Delta(\mathfrak{E}_2) = \mathfrak{E}_2 \otimes 1 + q^{-\mathfrak{H}_k}\mathfrak{U}^{+1} \otimes \mathfrak{E}_2, \quad \Delta(\mathfrak{F}_2) = \mathfrak{F}_2 \otimes q^{\mathfrak{H}_k} + \mathfrak{U}^{-1} \otimes \mathfrak{F}_2.$$

Fundamental Representation

Can construct a $(2|2)$-dimensional representation as before:

- Everything quantum-deformed, e.g. constraint $[C]_q^2 - PK = [\frac{1}{2}]_q^2$.
- Still rectangular elliptic curve with $k = 4ig\sqrt{1 - g^2(q - q^{-1})^2}$.

Invariant fundamental R-matrix can be constructed.

- Big mess (bi-elliptic functions).
- Satisfies YBE!
- $q = 1$ limit is previous fundamental R-matrix.

Three-parametric (g, q, u) NN integrable spin chain Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1}. \qquad \mathcal{H}_{12} = -i\, \mathcal{R}_{12}^{-1} \frac{d}{du_1} \mathcal{R}_{12} \bigg|_{u_{12}=u}.$$

Hamiltonian includes Alcaraz–Bariev deformation of Hubbard model. [Alcaraz Bariev]

Yangian

Yangian $\mathbf{Y}(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3)$

Can we find a Yangian generator $\widehat{\mathfrak{J}}^A$ to enhance $\mathfrak{J}^A$ of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$?
Coproduct of $\mathfrak{J}^A$ is deformed by $\mathfrak{U}$

$$\Delta(\mathfrak{J}^A) = \mathfrak{J}^A \otimes 1 + \mathfrak{U}^{[A]} \otimes \mathfrak{J}^A.$$

Educated guess for deformed coproduct of $\widehat{\mathfrak{J}}^A$

$$\Delta(\widehat{\mathfrak{J}}^A) = \widehat{\mathfrak{J}}^A \otimes 1 + \mathfrak{U}^{[A]} \otimes \widehat{\mathfrak{J}}^A - \tfrac{i}{2} f^A_{BC}\, \mathfrak{J}^B \mathfrak{U}^{[C]} \otimes \mathfrak{J}^C.$$

Define evaluation representation

$$\widehat{\mathfrak{J}}^A |X, p\rangle = \tfrac{1}{2} \cot(\tfrac{1}{2}p) \sqrt{1 + 16 g^2 \sin^2(\tfrac{1}{2}p)}\, \mathfrak{J}^A |X, p\rangle.$$

Fundamental R-matrix is invariant under Yangian! [NB 0704.0400]

- Hopf algebra probably Yangian double generated by $\mathfrak{J}^A_n$, $n \in \mathbb{Z}$.
- Presumably $q = 1$ contraction of quantum affine algebra.
- One more symmetry discovered. How to complete algebra? [Matsumoto Moriyama Torrielli] [NB Spill]

Classical Limit

Classical r-matrix & Lie Bialgebra

R-matrix has a classical limit $g \to \infty$, $p \simeq g^{-1}$ [Torrielli hep-th/0701281]

$$\mathcal{R}_{12} = 1 \otimes 1 + g^{-1} r_{12} + \mathcal{O}(g^{-2})$$

with classical r-matrix r. Coproduct becomes cobracket δ

$$\Delta(\mathfrak{J}) - \Delta_{\mathrm{op}}(\mathfrak{J}) = g^{-1}\delta(\mathfrak{J}) + \mathcal{O}(g^{-2}).$$

Treatment simplifies:

- Need only loop algebra $\mathfrak{g}$. No universal enveloping. No deformation.
- Classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$.
- Cobracket given by $\delta(\mathfrak{J}_n^A) = [\mathfrak{J}_n^A, r]$.
- Classical YBE: $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.

Proposal for deformed $\mathfrak{u}(2|2)$ loop algebra & r-matrix. [Moriyama Torrielli]

Deformation of the $\mathfrak{u}(2|2)$ Loop Algebra

Alternative proposal for deformed (β) $\mathfrak{u}(2|2)$ loop algebra [NB Spill]

$$\{(\mathfrak{Q}_m)^\alpha{}_b, (\mathfrak{S}_n)^c{}_\delta\} = \delta^c_b(\mathfrak{L}_{m+n})^\alpha{}_\delta + \delta^\alpha_\delta(\mathfrak{R}_{m+n})^c{}_b + \delta^c_b\delta^\alpha_\delta(\mathfrak{C}_{m+n}),$$

$$\{(\mathfrak{Q}_m)^\alpha{}_b, (\mathfrak{Q}_n)^\gamma{}_d\} = +2\alpha\beta\varepsilon^{\alpha\gamma}\varepsilon_{bd}\mathfrak{C}_{m+n-1},$$

$$\{(\mathfrak{S}_m)^a{}_\beta, (\mathfrak{S}_n)^c{}_\delta\} = -2\alpha^{-1}\beta\varepsilon^{ac}\varepsilon_{\beta\delta}\mathfrak{C}_{m+n-1},$$

$$[\mathfrak{B}_m, (\mathfrak{Q}_n)^\alpha{}_b] = +(\mathfrak{Q}_{m+n})^\alpha{}_b - 2\alpha\beta\varepsilon^{\alpha\gamma}\varepsilon_{bd}(\mathfrak{S}_{m+n-1})^d{}_\gamma,$$

$$[\mathfrak{B}_m, (\mathfrak{S}_n)^a{}_\beta] = -(\mathfrak{S}_{m+n})^a{}_\beta - 2\alpha^{-1}\beta\varepsilon^{ac}\varepsilon_{\beta\delta}(\mathfrak{Q}_{m+n-1})^\delta{}_c.$$

Properties:

- Satisfies Jacobi identities.
- Non-homogeneous loop level.
- Includes additional (automorphism) generators $\mathfrak{B}_m$.
- Central extensions $\mathfrak{P}, \mathfrak{K}$ (and thus $\mathfrak{U}$) replaced by loop charge $\mathfrak{C}_{-1}$.

Classical r-matrix

Classical r-matrix is almost standard $\mathfrak{u}(2|2)$ r-matrix

$$r = r_{\mathfrak{psu}(2|2)} - \sum_{m=-1}^{\infty} \mathfrak{B}_{-1-m} \otimes \mathfrak{C}_m - \sum_{m=+1}^{\infty} \mathfrak{C}_{-1-m} \otimes \mathfrak{B}_m$$

with the classical r-matrix $r_{\mathfrak{psu}(2|2)}$ for $\mathfrak{psu}(2|2)$

$$r_{\mathfrak{psu}(2|2)} = + \sum_{m=0}^{\infty} (\mathfrak{R}_{-1-m})^c{}_d \otimes (\mathfrak{R}_m)^d{}_c - \sum_{m=0}^{\infty} (\mathfrak{L}_{-1-m})^\gamma{}_\delta \otimes (\mathfrak{L}_m)^\delta{}_\gamma$$
$$+ \sum_{m=0}^{\infty} (\mathfrak{Q}_{-1-m})^\gamma{}_d \otimes (\mathfrak{S}_m)^d{}_\gamma - \sum_{m=0}^{\infty} (\mathfrak{S}_{-1-m})^c{}_\delta \otimes (\mathfrak{Q}_m)^\delta{}_c.$$

Satisfies CYBE $\Rightarrow$ quasi-triangular Lie bialgebra.

Double structure $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+$ with subalgebra decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$:
Standard $\mathfrak{J}_{n\geq 0} \in \mathfrak{g}_+$ (but $\mathfrak{B}_0 \in \mathfrak{g}_-$) and $\mathfrak{J}_{n<0} \in \mathfrak{g}_-$ (but $\mathfrak{C}_{-1} \in \mathfrak{g}_+$).

Restriction of Maximally Extended Algebra

Can start with maximally extended algebra $\mathfrak{h}_+ = \mathfrak{sl}(2) \ltimes \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$.

Has r-matrix

$$r_{\mathfrak{h}_+} = r_{\mathfrak{psu}(2|2)} - \sum_{m=0}^{\infty} (\mathfrak{B}_{-1-m})^{\mathfrak{c}}{}_{\mathfrak{d}} \otimes (\mathfrak{C}_m)^{\mathfrak{d}}{}_{\mathfrak{c}} - \sum_{m=0}^{\infty} (\mathfrak{C}_{-1-m})^{\mathfrak{c}}{}_{\mathfrak{d}} \otimes (\mathfrak{B}_m)^{\mathfrak{d}}{}_{\mathfrak{c}}.$$

Restrict automorphisms $(\mathfrak{B}_m)^{\mathfrak{a}}{}_{\mathfrak{b}}$ to subalgebra spanned by

$$\mathfrak{B}_n = (\mathfrak{B}_n)^1{}_1 - (\mathfrak{B}_n)^2{}_2 + 2\alpha^{-1}\beta(\mathfrak{B}_{n-1})^1{}_2 + 2\alpha\beta(\mathfrak{B}_{n-1})^2{}_1.$$

Factor out an ideal spanned by some $(\mathfrak{C}_m)^{\mathfrak{a}}{}_{\mathfrak{b}}$ such that

$$\mathfrak{C}_n = (\mathfrak{C}_n)^1{}_1 = -(\mathfrak{C}_n)^2{}_2 = (\mathfrak{C}_{n+1})^1{}_2/2\alpha\beta = (\mathfrak{C}_{n+1})^2{}_1/2\alpha^{-1}\beta.$$

Modified r-matrix belongs to $\mathfrak{g} \otimes \mathfrak{g}$ and satisfies CYBE!

$$r := r_{\mathfrak{h}_+} + (\mathfrak{C}_{-1})^1{}_1 \wedge \big((\mathfrak{B}_0)^1{}_1 - (\mathfrak{B}_0)^2{}_2\big) \in \mathfrak{g} \otimes \mathfrak{g}.$$

Conclusions

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⋆ **Extended $\mathfrak{psu}(2|2)$ Algebra**

- Symmetry for AdS/CFT scattering picture.
- Hopf algebra structure yields interesting fundamental R-matrix.
- Quantum deformation $\mathrm{U}_q(\mathfrak{psu}(2|2))$ appears to work.
- Integrable structure of Hubbard model and AB deformation explained.
- R-matrix appears to have Yangian symmetry.
- Classical bialgebra and r-matrix identified.
  Agrees with classical limit of R-matrix.

⋆ **Outlook**

- Find Yangian double (quantum affine algebra) and universal R-matrix.
- Understand integrable structure for long-range $\mathfrak{psu}(2,2|4)$ chain.