Low Energy Constants in QCD from Matrix Models

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collaboration with:

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Outline & Motivation

1. **QCD** $\rightarrow$ epsilon regime chiral Perturbation theory:

box $V = L^4$ $\qquad$ $\boxed{\Lambda_{\mathbf{QCD}} \gg p \sim 1/L \gg m_\pi \sim 1/L^2}$

LEC	F Pion decay constant	Σ chiral condensate
source term	μ chemical potential	m quark mass

2. Equivalent Matrix Models: easier

3. Lattice Data: fitting plots

- Results for:
  spectral density & individual $\not{D}$-eigenvalues
  currents $\langle S(x)S(0)\rangle$

QCD and Banks-Casher Relation

$$\mathcal{Z}_{\mathbf{QCD}} = \int [DA][d\Psi] \, \exp[-\int \bar{\Psi}(\not{D}[A]+M)\Psi - S_{YM}[A] + i\Theta \int F\tilde{F}]$$

$$\boxed{\mathcal{Z}_{\mathbf{QCD}} = \sum_{\nu=topo} \int [DA]_\nu \prod_f^{N_f} \det[\not{D}[A] + m_f] \, \mathrm{e}^{-S_{YM}[A]+i\Theta\nu}}$$

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spectral density $\rho_{\not{D}}(\lambda) \equiv \langle \sum_i' \delta(\lambda - \lambda_i) \rangle_{\mathbf{QCD}}$

Banks-Casher $\boxed{\rho_{\not{D}}(\lambda \approx 0) = \frac{1}{\pi} V \Sigma}$, $\Sigma = \langle \bar{\Psi}\Psi \rangle$

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Cartoon of QCD Phasediagram

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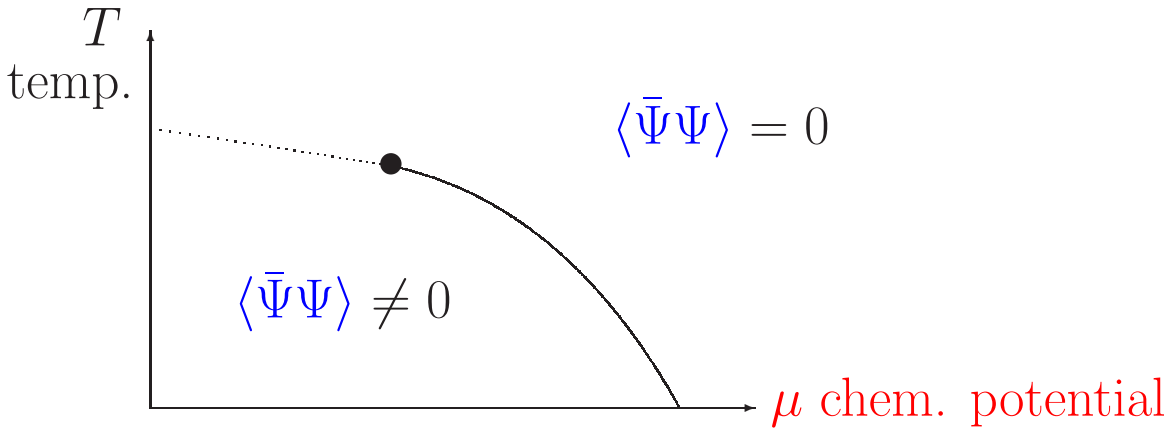

schematic phase diagram for **QCD** with 2 light flavors [Halasz et al. 98]

Matrix Models

$$\mathcal{Z}_{MM} \equiv \int dW \, \Pi_{f=1}^{N_f} \det \begin{pmatrix} m_f & iW \\ iW^\dagger & m_f \end{pmatrix} \mathrm{e}^{-N\sigma^2 \mathrm{Tr}\, WW^\dagger}$$

[Shuryak, Verbaarschot 93]

fixed topology: $W = N \times (N+\nu)$ random matrix

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- solvable for all $\rho_k(\lambda_1, \ldots, \lambda_k)$ and $P_k(\lambda)$

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- limit $N \rightarrow \infty$ universal & **equivalent to LO** $\varepsilon\chi PT$

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[Stephanov 96]

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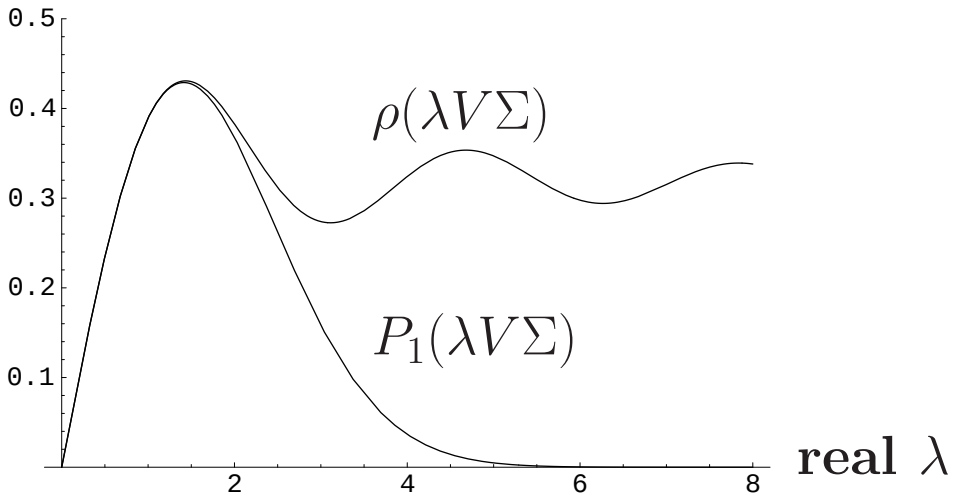
Spectral Density and Individual Eigenvalues



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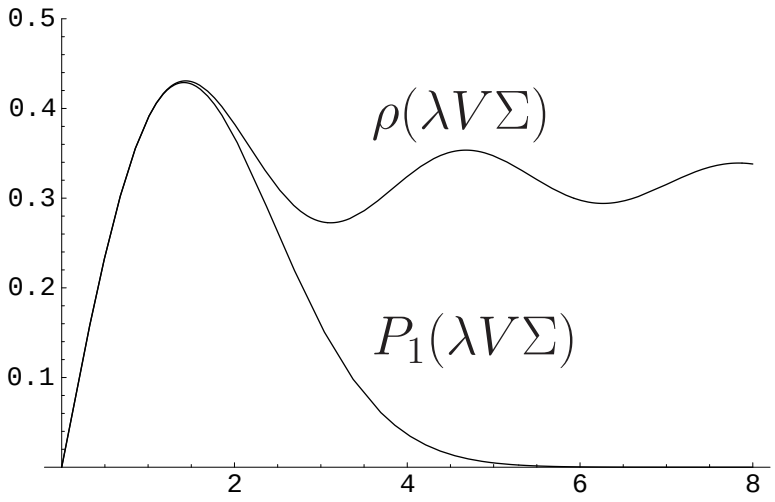
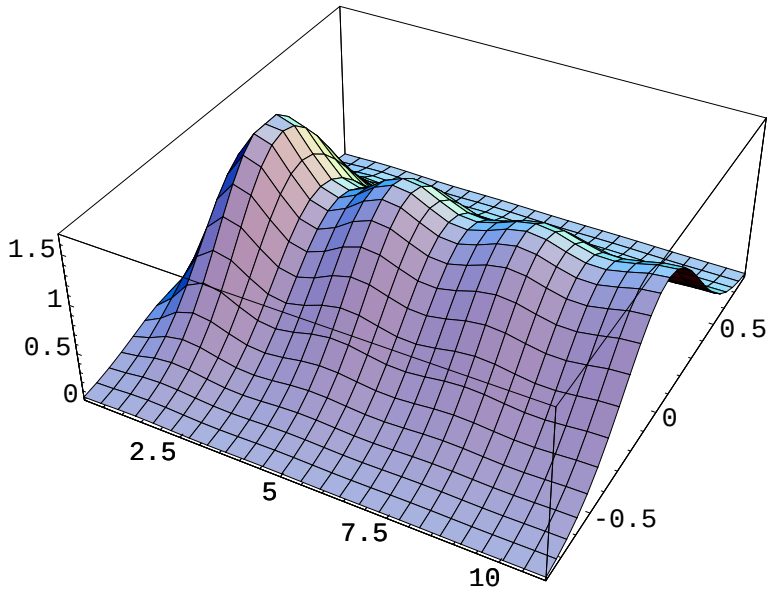


real λ



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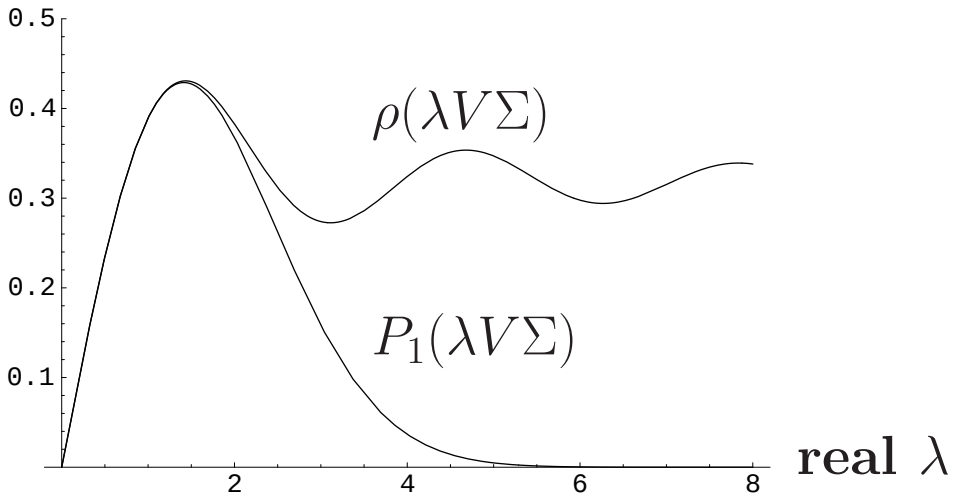

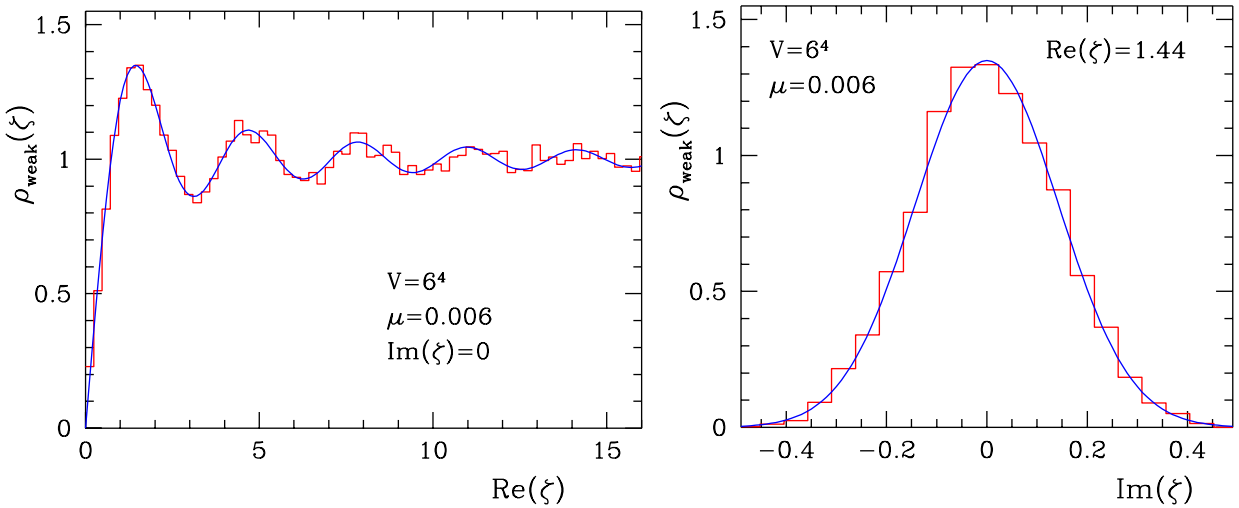

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[G.A., Wettig 03]

Chiral Perturbation Theory and epsilon-Regime

spontaneous χSB: $SU(N_f) \times SU(N_f) \longrightarrow SU(N_f)$

$$\boxed{\mathcal{L} = \frac{1}{4} F^2 \nabla_a U(x) \nabla_a U^\dagger(x) + \frac{1}{2} \Sigma M \left(\mathrm{e}^{i\frac{\Theta}{N_f}} U(x) + \mathrm{e}^{-i\frac{\Theta}{N_f}} U^\dagger(x) \right)}$$

- $\nabla_a = \partial_a + i[B, \ \cdot \ ]$ for vector current, e.g. $B \sim \mu\sigma_3$
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ε-Regime: $m_\pi \sim \dfrac{1}{L^2} \ll \dfrac{1}{L}$ [Gasser, Leutwyler 87]

- zero-momentum modes non-perturbativ:

$$U(x) = U_0 \ \mathrm{e}^{i\xi(x)}$$

$$\boxed{LO: \quad \mathcal{Z}_{\varepsilon\chi PT} = \int dU_0 \det[U_0]^\nu \mathrm{e}^{-\frac{1}{2}\Sigma V \mathrm{Tr}\left(M(U_0 + U_0^\dagger)\right)} \ \times \ \int [d\xi(x)] \mathrm{e}^{-\int \frac{1}{2}\partial_a \xi \partial_a \xi}}$$

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Objects of study: Currents & Densities

- **currents:** scalar source $S(x) = \bar{\Psi}(x)\mathbf{1}_{N_f}\Psi(x) \leftrightarrow$ mass

  [Hansen 90]

$$\langle S(x)S(0)\rangle_{\varepsilon\chi PT} \sim \frac{1}{\mathcal{Z}_{\varepsilon\chi PT}}\partial_m^2 \mathcal{Z}_{\varepsilon\chi PT} = \left\langle \left(\mathrm{Tr}[U_0 + U_0^\dagger]\right)^2 \right\rangle_{\varepsilon\chi PT}$$

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- **densities:** SUSY-$\varepsilon\chi PT$ [Damgaard et al 98, Basile, G.A. 07]

$$\Sigma_\nu(m) \equiv \partial_m \left\langle \frac{\det[\not{D} + m]}{\det[\not{D} + m']} \right\rangle_{\varepsilon\chi PT}^{m=m'} = \left\langle \mathrm{Tr}\, \frac{1}{\not{D} + m} \right\rangle_{\varepsilon\chi PT} = \int d\lambda\, \frac{\rho_{\not{D}}(\lambda) 2m}{\lambda^2 + m^2}$$

$$\boxed{\rho_{\not{D}}(m) = \Im\mathrm{m}[\Sigma_\nu(m)] \quad (\text{or } \partial_m^* \Sigma_\nu(m) \text{ when } \mu \neq 0)}$$

1st eigenvalue : $P_1(\lambda) = \partial_\lambda \sum_k \frac{(-)^k}{k!} \int_0^\lambda d\lambda_1 \ldots d\lambda_k\, \rho_k(\lambda_1, \ldots, \lambda_k)$

[G.A., Damgaard 04]

Results: Some Examples

scaling $\boxed{\hat{\lambda} \;=\; V\Sigma\lambda}$

- **density:** $\rho(\hat{\lambda}) \;=\; \int_0^1 dt J_\nu(t\hat{\lambda}) J_\nu(t\hat{\lambda})$

- k**-point:** $\rho_k(\{\hat{\lambda}\}) \;=\; \det_{i,j}[\;\; \hat{\lambda}_i \;\; \hat{\lambda}_j \;\;]/\text{Vandermonde}$

$$\Rightarrow P_1(\hat{\lambda}) = \partial_{\hat{\lambda}} \sum_k \int \rho_k \;=\; \tfrac{1}{2}\hat{\lambda}\;\exp[\tfrac{1}{4}\hat{\lambda}^2]$$

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- **current:**

$\langle S(x)S(0)\rangle_{\varepsilon\chi PT} \;=\; \partial^2_{\hat{m}} \log[\mathcal{Z}_{eff}] + \bar{\Delta}(x)(A + \hat{\mu}^2 B + \hat{\mu}^4 C)$

$\boxed{\Sigma_{eff} = \Sigma(1+\frac{c_1}{\sqrt{V}}) \quad F_{eff} = F(1+\frac{c_2}{\sqrt{V}})\;,}$ and $\Delta \sim \frac{1}{\sqrt{V}}$

NEW $\mathcal{Z}_{\varepsilon\chi PT} = \int dU_0 \det[U_0]^\nu \mathrm{e}^{-\mathrm{Tr}\left(\frac{1}{2}\Sigma V M(U_0+U_0^\dagger) \;+\; \sum_K a_K \left([B,U_0][B,U_0^\dagger]\right)^K\right)}$

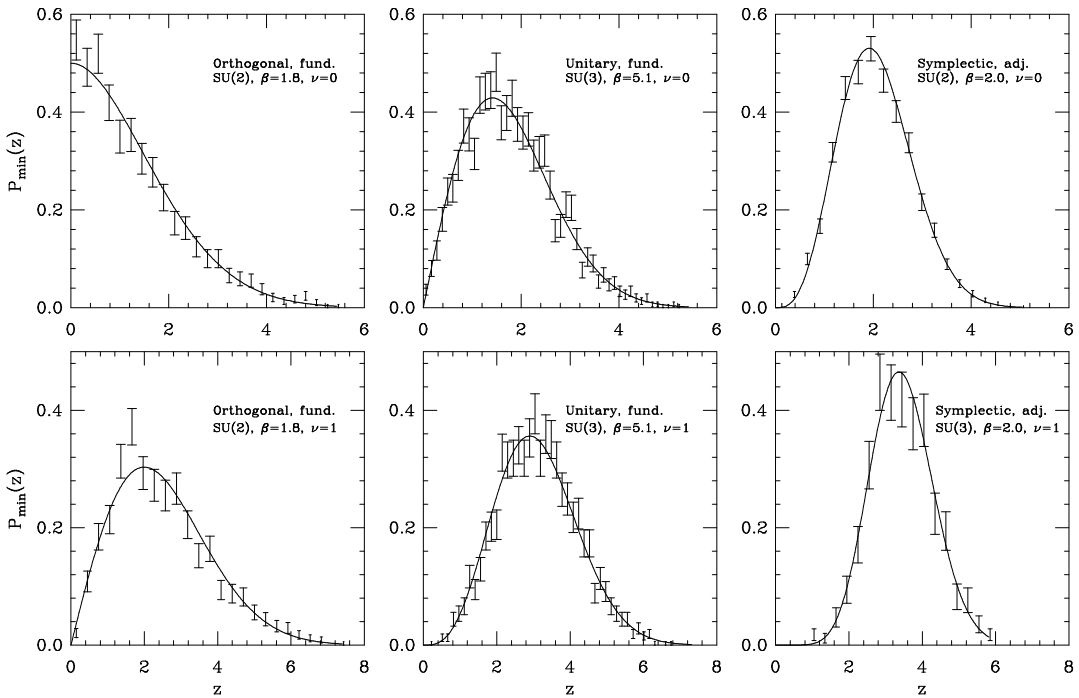
Real Spectra vs. Lattice

[R.G. Edwards et al 99]

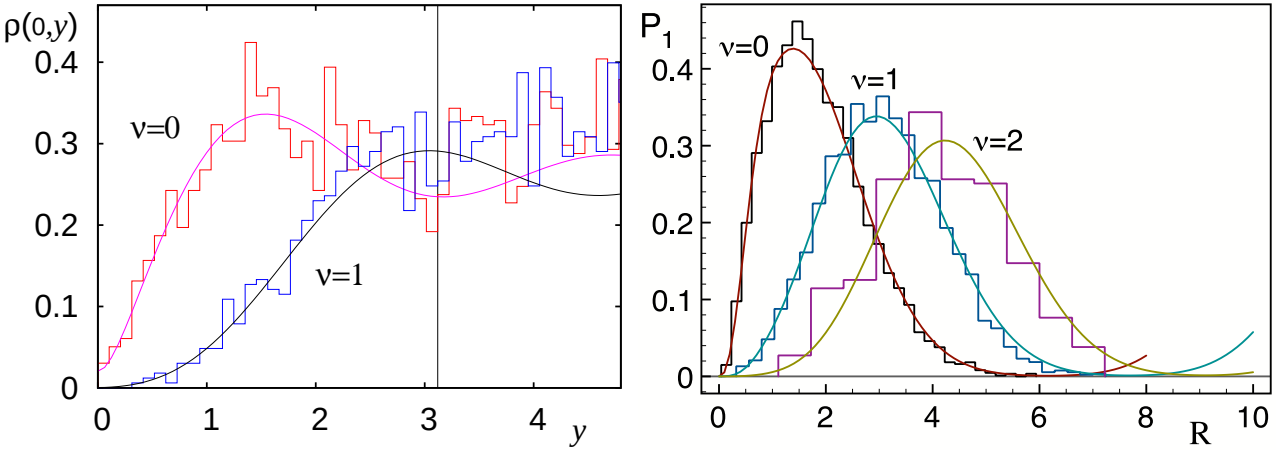
$SU(2)$ $SU(3)$ $SU(2)$ $adj.$

$\nu = 0$

$\nu = 1$

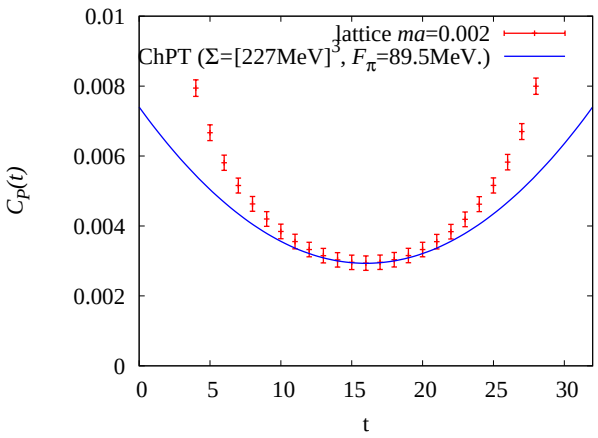


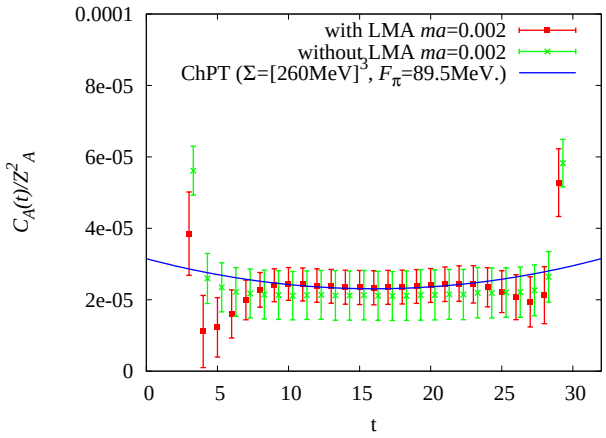
Complex Spectra vs. Lattice



density ρ (left) [Bloch, Wettig 06], P_k (right) [+ G.A., Shifrin 07]

Currents vs. Lattice





$N_f = 2$ Pseudo-scalar (left) and Axial-vector (right)

[JLQCD 07]

Some Open Problems

- complete all χSB classes with $\mu \neq 0$:
  real and complex spectrum of adjoint **QCD**

- simple form for $P_1(\lambda)$ on $\mathbb{C}$?

- Polyakov loop $\longleftrightarrow$ $\not{D}$ eigenvalues

- other applications of new group integral ?