Factorization of correlation functions and fermionic structure of the XXZ model

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joint works with: M. Jimbo, T.Miwa, F. Smirnov, Y. Takeyama

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- Our main object is the infinite spin-$\frac{1}{2}$ XXZ chain

$$\mathcal{H}_{\mathrm{XXZ}} = J \sum_{j=-\infty}^{\infty} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1) \right)$$

  - anti-ferromagnetic regime: $J > 0, \quad \Delta > -1$
  - critical regime: $-1 < \Delta \leq 1$
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Introduction

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- Main problem: obtain ground state correlation functions of local operators $\mathcal{O}$ at disorder field α in a compact form

$$\frac{\langle \mathrm{vac} | q^{2\alpha S(0)} \mathcal{O} | \mathrm{vac} \rangle}{\langle \mathrm{vac} | q^{2\alpha S(0)} | \mathrm{vac} \rangle}, \qquad S(k) = \frac{1}{2} \sum_{j=-\infty}^{k} \sigma_j^z$$

- Some methods:
  - Vertex-operator approach: bosonization, representation theory of infinite-dimensional algebras, multiple integrals Jimbo, Miwa, Miki, Nakayashiki (92)

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$$\mathcal{H}_{\mathrm{XXZ}} = \mathcal{H}_{\mathrm{Gauss}} + \sum_{j\geq 1} \lambda_j \, a^{d_j - 2} \int dx \, \mathcal{O}_j(x)$$

$\mathcal{O}_j$ - set of irrelevant operators with scaling dimensions d_j

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Hard problem: obtain lattice operators via

$$\mathcal{O} \sim C_{m_1} \, a^{d_{m_1}} \, \mathcal{O}_{m_1} + C_{m_2} \, a^{d_{m_2}} \, \mathcal{O}_{m_2} + \cdots$$ λ and C depend on normalization of scaling fields

Free fermions $\Delta = 0$: Jordan-Wigner-transformation

$$\psi_k^\pm = \sigma_k^\pm \, e^{\mp\pi i S(k-1)}$$

with the canonical anti-commutation relations

$$\left[\psi_k^\pm, \psi_l^\pm\right]_+ = 0, \quad \left[\psi_k^\pm, \psi_l^\mp\right]_+ = \delta_{k,l}$$

Take Fourier-transformation: $\quad \psi_k^\pm = \frac{1}{2\pi i} \int_{\zeta^2=1} \left(\frac{1-\zeta^2}{1+\zeta^2}\right)^{\pm k} \psi^\pm(\zeta) \frac{2d\zeta^2}{1-\zeta^4}$

and define 'Dirac-vacuum' $|\mathrm{vac}\rangle$: $\quad \psi^\mp(\zeta)|\mathrm{vac}\rangle = 0, \quad \arg\left(\frac{1-\zeta^2}{1+\zeta^2}\right) \gtreqqless 0$

Apply Wick-theorem and get Slater determinant formula

$$\frac{\langle \mathrm{vac}|q^{2\alpha S(0)} \psi_-^+(\zeta_1^+) \cdots \psi_-^+(\zeta_m^+) \psi_+^-(\zeta_1^-) \cdots \psi_+^-(\zeta_m^-)|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{2\alpha S(0)}|\mathrm{vac}\rangle} = \det_{i,j} \left|\sigma(\zeta_i^+, \zeta_j^-; \alpha)\right|$$

$$\psi_+^\pm(\zeta) := \sum_{k>0} \left(\frac{1+\zeta^2}{1-\zeta^2}\right)^{\pm k} \psi_k^\pm; \quad \psi_-^\pm(\zeta) := \sum_{k<0} \left(\frac{1+\zeta^2}{1-\zeta^2}\right)^{\pm k} \psi_k^\pm$$

note: $\psi_\pm^\pm(\psi_\pm^\mp)$ are singular (regular) at $\zeta^2 = 1$

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Main motivation:

- can we find such a basis for generic q?
- does Wick-theorem and Slater formula work for it?

Factorization and exponential formula

NEW INGREDIENTS:

- **factorization:** reducibility of correlation functions to special form with algebraic and transcendental parts

Example for XXX at $\alpha = 0$: $\langle \mathrm{vac}|S_1^z S_3^z|\mathrm{vac}\rangle = \frac{1}{4} - 4\ln 2 + \frac{9}{4}\zeta(3)$

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- **exponential formula:** $\frac{\langle \mathrm{vac}|q^{2\alpha S(0)}\mathcal{O}|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{2\alpha S(0)}|\mathrm{vac}\rangle} = \mathbf{tr}^\alpha\left(e^{\Omega}\left(q^{2\alpha S(0)}\mathcal{O}\right)\right)$

  JMSTB (04-07)

$$\mathbf{tr}^\alpha(X) = \cdots \mathrm{tr}_1^\alpha\, \mathrm{tr}_2^\alpha\, \mathrm{tr}_3^\alpha \cdots (X), \quad \mathrm{tr}^\alpha(x) = \mathrm{tr}\left(q^{-\frac{1}{2}\alpha\sigma^3} x\right)/\mathrm{tr}\left(q^{-\frac{1}{2}\alpha\sigma^3}\right)$$

$$\Omega = \mathrm{res}_{\zeta_1^2=1}\mathrm{res}_{\zeta_2^2=1}\left(\omega(\zeta_1/\zeta_2,\alpha)\,\mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)\frac{d\zeta_1^2}{\zeta_1^2}\frac{d\zeta_2^2}{\zeta_2^2}\right),$$

$\omega(\zeta,\alpha)$ is a scalar transcendental function and $\mathbf{b},\mathbf{c}$ are ‘new fermions’

Space of quasi-local operators

- $X = q^{2\alpha S(0)}\mathcal{O}$ is called quasi-local operator with tail α if there exist $k \le l$ such that X stabilizes as $q^{\alpha\sigma_j^z}$ for $j < k$ and as identity I_j for $j > l$.

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- Denote by $\mathcal{W}_\alpha$ a space of all quasi-local operators with tail α, and by $\mathcal{W}_{\alpha,s}$ the subspace of those with spin s. Also we introduce
$$\mathcal{W} = \bigoplus_{\alpha\in\mathbb{C}} \mathcal{W}_\alpha$$

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- Define an operator $\hat{\alpha} \quad : \hat{\alpha}\mathcal{W}_\alpha = \alpha\mathcal{W}_\alpha$

Fermionic annihilation operators b and c

- Block structure:

$$\mathbf{b}(\zeta):\ \mathcal{W}_{\alpha-1,s+1}\to\mathcal{W}_{\alpha,s}\,;\quad \mathbf{c}(\zeta):\ \mathcal{W}_{\alpha+1,s-1}\to\mathcal{W}_{\alpha,s}$$

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- ‘Fourier-mode’ expansions:

$$\mathbf{b}(\zeta) = \zeta^{-\hat{\alpha}-\mathbb{S}}\Big(\mathbf{b}_0 + \sum_{p=1}^{\infty} (\zeta^2-1)^{-p}\mathbf{b}_p\Big), \quad \mathbf{c}(\zeta) = \zeta^{\hat{\alpha}+\mathbb{S}}\Big(\mathbf{c}_0 + \sum_{p=1}^{\infty} (\zeta^2-1)^{-p}\mathbf{c}_p\Big)$$

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- Annihilation property:

$$\mathbf{b}_p\big(X\big)=0,\quad \mathbf{c}_p\big(X\big)=0\quad \text{for}\quad p>\mathrm{length}\big(X\big)$$

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and has the expansion in $\zeta^2 - 1$: $\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^p \mathbf{t}_p^* \quad$ where $\mathbf{t}_1^* = 2\tau$

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- $\mathbf{t}^*_p$ increase the length in a controllable way:

$$\text{length}\left(\mathbf{t}^*_p(X)\right) \leq \text{length}\,(X) + p$$

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We define operators $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ acting on $\mathcal{W}$ with block structure:

$$\mathbf{b}^*(\zeta): \ \mathcal{W}_{\alpha+1,s-1} \to \mathcal{W}_{\alpha,s}\,, \qquad \mathbf{c}^*(\zeta): \ \mathcal{W}_{\alpha-1,s+1} \to \mathcal{W}_{\alpha,s}$$

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Operators $\mathbf{b}^*$ and $\mathbf{c}^*$ create quasi-local operators:

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and satisfy canonical anti-commutation relations

$$\{\mathbf{b}(\zeta_1),\mathbf{c}^*(\zeta_2)\}=\{\mathbf{c}(\zeta_1),\mathbf{b}^*(\zeta_2)\}=0,\qquad \{\mathbf{b}_p,\mathbf{c}^*_q\}=\{\mathbf{c}_p,\mathbf{b}^*_q\}=0\,,$$

$$\{\mathbf{b}(\zeta_1),\mathbf{b}^*(\zeta_2)\}=-\psi(\zeta_2/\zeta_1,\hat\alpha+\mathbb{S}),\qquad \{\mathbf{b}_p,\mathbf{b}^*_q\}=\delta_{p,q}$$

$$\{\mathbf{c}(\zeta_1),\mathbf{c}^*(\zeta_2)\}=\psi(\zeta_1/\zeta_2,\hat\alpha+\mathbb{S}),\qquad \{\mathbf{c}_p,\mathbf{c}^*_q\}=\delta_{p,q},$$

$$\psi(\zeta,\alpha)=\zeta^{\alpha}/2\ (\zeta^2+1)/(\zeta^2-1)$$

‘Left’ and ‘right’ vacuum

Primary field $q^{2\alpha S(0)}$ is in the common kernel of the annihilation operators and plays the role of ‘right vacuum’.

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Important property:

$$\mathbf{tr}^{\alpha}\big(e^{\Omega_0}\mathbf{b}^*(\zeta)(q^{2(\alpha+1)S(0)}\mathcal{O}_1)\big)=0,\quad \mathbf{tr}^{\alpha}\big(e^{\Omega_0}\mathbf{c}^*(\zeta)(q^{2(\alpha-1)S(0)}\mathcal{O}_2)\big)=0$$

where $\mathcal{O}_1$, $\mathcal{O}_2$ have respectively spins -1 and 1

$$\Omega_0=-\mathrm{res}_{\zeta_1^2=1}\mathrm{res}_{\zeta_2^2=1}\left(\omega_0\left(\zeta_1/\zeta_2,\alpha\right)\mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)\frac{d\zeta_1^2}{\zeta_1^2}\frac{d\zeta_2^2}{\zeta_2^2}\right)$$

with $\quad \omega_0(\zeta,\alpha)=-\left(\frac{1-q^{\alpha}}{1+q^{\alpha}}\right)^2\Delta_{\zeta}\left(\psi(\zeta,\alpha)\right),\quad \Delta_{\zeta}(f(\zeta))=f(\zeta q)-f(\zeta q^{-1})$

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It allows to define ‘left vacuum’ via linear functional

$$\mathrm{v}^{\alpha}(\,\cdot\,)=\mathbf{tr}^{\alpha}\big(e^{\Omega_0}(\,\cdot\,)\big)$$

it vanishes on the image of creation operators.

Special basis and Wick theorem

- First, starting from primary field $q^{2\alpha S(0)}$ let us define inductively quasi-local operators :

$$X^{\varepsilon_1\cdots\varepsilon_k}(\zeta_1,\cdots,\zeta_k;\alpha)=\begin{cases}\mathbf{b}^*(\zeta_k)X^{\varepsilon_1\cdots\varepsilon_{k-1}}(\zeta_1,\cdots,\zeta_{k-1};\alpha) & (\varepsilon_k=+)\\ \mathbf{c}^*(\zeta_k)(-1)^{\mathbb{S}}X^{\varepsilon_1\cdots\varepsilon_{k-1}}(\zeta_1,\cdots,\zeta_{k-1};\alpha) & (\varepsilon_k=-)\\ \frac{1}{2}\mathbf{t}^*(\zeta_k)X^{\varepsilon_1\cdots\varepsilon_{k-1}}(\zeta_1,\cdots,\zeta_{k-1};\alpha) & (\varepsilon_k=0)\end{cases}$$

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- Then we rewrite: $\dfrac{\langle \mathrm{vac}|q^{2\alpha S(0)}\mathcal{O}|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{2\alpha S(0)}|\mathrm{vac}\rangle}=\mathrm{v}^{\alpha}\left(e^{\Omega-\Omega_0}\left(q^{2\alpha S(0)}\mathcal{O}\right)\right)$

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- Finally we apply Wick theorem and come to our **main result:**

$$\frac{\langle \mathrm{vac}|X^{\varepsilon_1\cdots\varepsilon_n}(\zeta_1,\cdots,\zeta_n,\alpha)|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{2\alpha S(0)}|\mathrm{vac}\rangle}=\det\Big((\omega-\omega_0)(\zeta_{j_p^+}/\zeta_{j_q^-},\alpha)\Big)_{p,q=1,\cdots,l}$$

where $j_1^+<\cdots<j_l^+$ are the indices with $\varepsilon_{j_p^+}=+$
and $j_1^-<\cdots<j_l^-$ are those with $\varepsilon_{j_p^-}=-$

- **Remark 1**

  Fermionic operators commute completely with the adjoint action of local integrals of motion:

$$[\mathbf{t}^*(\zeta_1), \mathbf{b}(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{c}(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{c}^*(\zeta_2)] = 0$$

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- **Remark 2**

  We do not prove, but only conjecture remaining commutation relations:

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- **Remark 2**

  We do not prove, but only conjecture remaining commutation relations:

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- **Remark 3**

  We can prove completeness of fermionic basis obtained via the generating function $X^{\varepsilon_1\cdots\varepsilon_k}(\zeta_1,\cdots,\zeta_k;\alpha)$

Some remarks

- **Remark 1**

  Fermionic operators commute completely with the adjoint action of local integrals of motion:

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- **Remark 4**

  We constructed fermionic operators $\mathbf{b},\mathbf{c}$ and their ‘conjugates’ $\mathbf{b}^*,\mathbf{c}^*$ but we failed to construct conjugate of $\mathbf{t}^*$

Conclusions

- If we had conjugates for $\mathbf{t}^*$ we would have a novel description of the space of quasi-local operators: it is simply the tensor product of Fock spaces of fermions and bosons. For the descendant operators created by the latter, the VEV's can be computed as in free theory.

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- Unlike the usual considerations we introduce operators not on the space of states but rather on the space of quasi-local operators. In this sense our philosophy seems to be rather close to CFT.

Conclusions

- In our recent joint work with Göhmann, Klümper and Suzuki we conjectured that the same algebraic construction works for the thermal averages. Only transcendental part becomes dependent on temperature. This suggests the universal character of our algebraic construction.