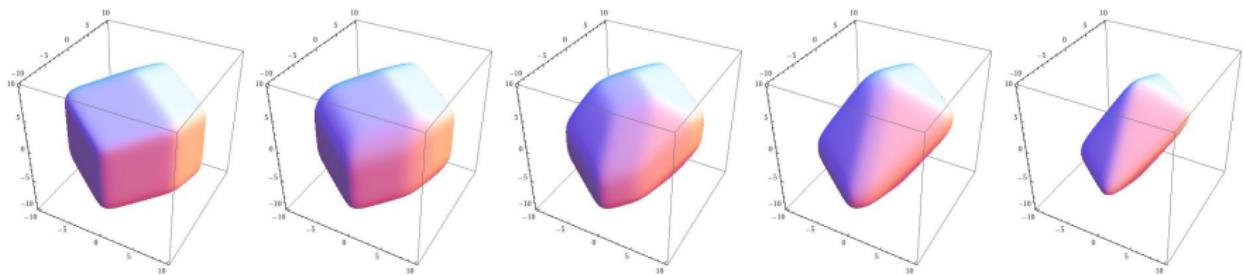


# Identical particles in $\kappa$ -deformed QFT

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ICFT08



C.Y., Robin Zegers, 0711.2206 [hep-th] 0803.2659 [hep-th]

$\mathcal{U}(P)$ , the universal enveloping algebra of the Poincaré algebra  $P$ , admits a certain one-parameter family of deformations

$$\mathcal{U}_\kappa(P) \quad \text{with} \quad \mathcal{U}_\infty(P) = \mathcal{U}(P).$$

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$$so(1, 4) \xrightarrow{\text{Inönü-Wigner contraction}} so(1, 3) \ltimes \mathbb{R}^{1,3} = P$$

$$\begin{aligned} M_{ab} &= J_{ab} \\ N_a &= J_{a0} \\ P_a &= J_{a4}/R, \quad R \rightarrow \infty \end{aligned}$$

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$$\begin{array}{ccc} \mathcal{U}(so(1,4)) & \xrightarrow{\text{Inönü-Wigner contraction}} & \mathcal{U}(P) \\ \mathcal{U}_q(so(1,4)) & \longrightarrow & \mathcal{U}_\kappa(P) \\ \\ R \rightarrow \infty, & & R \log q \rightarrow \frac{1}{\kappa} \end{array}$$

# The $\kappa$ -deformed Poincaré algebra in 3+1 dimensions

Rotations  $M_i$ , boosts  $N_i$ , translations  $P_0, P_i, \quad i = 1, 2, 3.$

$$[M_i, M_j] = \epsilon_{ijk} M_k \quad [M_i, N_j] = \epsilon_{ijk} N_k \quad [N_i, N_j] = -\epsilon_{ijk} M_k$$

$$[P_i, P_0] = 0 \quad [P_i, P_j] = 0 \quad [M_i, P_0] = 0 \quad [M_i, P_j] = \epsilon_{ijk} P_k$$

$$[N_i, P_0] = P_i \quad [N_i, P_j] = \delta_{ij}\kappa \sinh \frac{P_0}{\kappa}$$

$$[N_i, N_j] = -\epsilon_{ijk} \left( M_k \cosh \frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k \vec{P} \cdot \vec{M} \right)$$

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Coalgebra:

$$\Delta M_i = M_i \otimes 1 + 1 \otimes M_i$$

$$\Delta N_i = N_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes N_i + \frac{1}{2\kappa} \epsilon_{ijk} \left( P_j \otimes M_k e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} M_j \otimes P_k \right)$$

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta P_i = P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i$$

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[Majid,Ruegg 1994]

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# Casimir: $\kappa$ -deformed Dispersion Relation

$$P_0^2 - \vec{P} \cdot \vec{P} = m^2$$



$$\left(2\kappa \sinh \frac{P_0}{2\kappa}\right)^2 - \vec{P} \cdot \vec{P} = m^2,$$

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- Doubly Special Relativity

[Amelino-Camelia 2002]

- AdS/CFT magnons

[Beisert 2005]

- Lattice Phonons

[Celeghini, Giachetti, Sorace, Tarlini 1990]

# Non-Commutative Spacetime

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[see e.g. Szabo 2003]

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Twist deformation of **commutative** coordinate algebra?

- $\theta$ -deformation: Yes  $\mathcal{F} = \exp(i\theta^{\mu\nu} P_\mu \otimes P_\nu)$
- $\kappa$ -Minkowski spacetime: Yes [Bu et al 2006, Meljanac et al 2008]

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- $\kappa$ -Minkowski spacetime: Yes [Bu et al 2006, Meljanac et al 2008]
- but  $\kappa$ -Poincaré Hopf algebra  $\mathcal{U}_\kappa(P)$  is **not** a twist of  $\mathcal{U}(P)$

[Lukierski 2006]

and no quantum  $R$  matrix is known.

# $\kappa$ -deformed Quantum Field Theory...

Two approaches:

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  - Formulate classical field theory on  $\kappa$ -Minkowski spacetime
  - Quantize it
- QFT is the theory of quantum mechanical **particles** with  $\kappa$ -Poincaré symmetry.  
Fields, field equations etc. emerge later.

Single particle states = Irreps of  $\kappa$ -Poincaré.

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Many particles?

$$|p\rangle \otimes |q\rangle \mapsto \Delta A |p\rangle \otimes |q\rangle$$

$$\begin{aligned}\Delta P_i |p\rangle \otimes |q\rangle &= \left( P_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_i \right) |p\rangle \otimes |q\rangle \\ &= \left( p_i + e^{-\frac{P_0}{\kappa}} q_i \right) |p\rangle \otimes |q\rangle\end{aligned}$$

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$$\Delta P_i |q\rangle \otimes |p\rangle = \left( q_i + e^{-\frac{q_0}{\kappa}} p_i \right) |q\rangle \otimes |p\rangle$$

Order of particles in state matters. Familiar in 1+1 dimensional theories.  
Bad news in 3+1 or more dimensions.

# Particle exchange in undeformed QFT

Two particle states (bosons):

$$V \otimes V \Big/ \tau, \quad \tau : |p\rangle \otimes |q\rangle \longrightarrow |q\rangle \otimes |p\rangle$$

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$N$ -particle states:

$$V^{\otimes N} \Big/ S_N$$

where symmetric group  $S_N$  is generated by  $\tau_{ij}$ , all commuting with the action of  $\mathcal{U}(P)$ .

Fock space:

$$V \oplus \frac{V^{\otimes 2}}{S_2} \oplus \frac{V^{\otimes 3}}{S_3} \oplus \dots$$

Encoded in

$$a^\dagger(p)a^\dagger(q) = a^\dagger(q)a^\dagger(p)$$

# Particle exchange in $\kappa$ -deformed QFT

$$\tau : |r\rangle \otimes |s\rangle \rightarrow |f(r,s)\rangle \otimes |g(r,s)\rangle$$

Covariance:

with respect to  $\Delta P_\mu$ :

$$f_0 + g_0 = r_0 + s_0, \quad f_i + e^{-\frac{f_0}{\kappa}} g_i = r_i + e^{-\frac{r_0}{\kappa}} s_i$$

[Lukierski et al 2007]

with respect to **boosts**  $\Delta N_i$ :

$$D_i f_0 = -f_i$$

$$D_i f_j = -\delta_{ij} \left( \frac{\kappa}{2} \left( 1 - e^{-\frac{2f_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{f} \cdot \vec{f} \right) + \frac{1}{\kappa} f_i f_j$$

$$D_i g_0 = -g_i e^{-\frac{f_0}{\kappa}}$$

$$D_i g_j = -\delta_{ij} e^{-\frac{f_0}{\kappa}} \left( \frac{\kappa}{2} \left( 1 - e^{-\frac{2g_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{g} \cdot \vec{g} \right) + \frac{1}{\kappa} e^{-\frac{f_0}{\kappa}} g_i g_j + \frac{1}{\kappa} f_j g_i - \frac{1}{\kappa} \delta_{ij} \vec{f} \cdot \vec{g}.$$

where

$$\begin{aligned} D_i &:= -r_i \frac{\partial}{\partial r_0} - \left( \frac{\kappa}{2} \left( 1 - e^{-\frac{2r_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{r} \cdot \vec{r} \right) \frac{\partial}{\partial r_i} + \frac{1}{\kappa} r_i r_j \frac{\partial}{\partial r_j} \\ &\quad - e^{-\frac{r_0}{\kappa}} s_i \frac{\partial}{\partial s_0} - e^{-\frac{r_0}{\kappa}} \left( \frac{\kappa}{2} \left( 1 - e^{-\frac{2s_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{s} \cdot \vec{s} \right) \frac{\partial}{\partial s_i} + \frac{1}{\kappa} e^{-\frac{r_0}{\kappa}} s_i s_j \frac{\partial}{\partial s_j} \\ &\quad + \frac{1}{\kappa} r_j s_i \frac{\partial}{\partial s_j} - \frac{1}{\kappa} \vec{r} \cdot \vec{s} \frac{\partial}{\partial s_i} \end{aligned}$$

is the realization of the boost operator  $N_i$  on such states.

$$f_0(r, s) = s_0 + \dots$$

$$f_i(r, s) = s_i + \dots$$

$$g_0(r, s) = r_0 + \dots$$

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$$f_0(r, s) = s_0 + \frac{1}{\kappa} \vec{r} \cdot \vec{s} + \dots$$

$$f_i(r, s) = s_i + \frac{1}{\kappa} r_i s_0 + \dots$$

$$g_0(r, s) = r_0 - \frac{1}{\kappa} \vec{r} \cdot \vec{s} + \dots$$

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$$f_0(r, s) = s_0 + \frac{1}{\kappa} \vec{r} \cdot \vec{s} + \frac{1}{2\kappa^2} (s_0 \vec{r} \cdot \vec{r} - r_0 \vec{s} \cdot \vec{s} + r_0 \vec{r} \cdot \vec{s} + s_0 \vec{r} \cdot \vec{s}) + \dots$$

$$f_i(r, s) = s_i + \frac{1}{\kappa} r_i s_0 + \frac{1}{2\kappa^2} (r_i \vec{r} \cdot \vec{s} - s_i \vec{r} \cdot \vec{s} + r_i r_0 s_0 - s_i r_0 s_0 - r_i s_0 s_0) + \dots$$

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f_0(r, s) &= s_0 + \frac{1}{\kappa} \vec{r} \cdot \vec{s} + \frac{1}{2\kappa^2} (s_0 \vec{r} \cdot \vec{r} - r_0 \vec{s} \cdot \vec{s} + r_0 \vec{r} \cdot \vec{s} + s_0 \vec{r} \cdot \vec{s}) \\
&\quad + \frac{1}{4\kappa^3} (\vec{r} \cdot \vec{s} \vec{r} \cdot \vec{r} + \vec{r} \cdot \vec{s} \vec{s} \cdot \vec{s} + 2r_0 s_0 \vec{r} \cdot \vec{r} - 2r_0 s_0 \vec{r} \cdot \vec{s} - 2r_0 s_0 \vec{s} \cdot \vec{s}) + \dots
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&\quad + \frac{1}{4\kappa^3} \left( r_i (2r_0 \vec{r} \cdot \vec{s} - r_0 \vec{s} \cdot \vec{s} - 2r_0 s_0 s_0 + s_0 \vec{r} \cdot \vec{r} - 2s_0 \vec{r} \cdot \vec{s} + \frac{2}{3}s_0 s_0 s_0) \right. \\
&\quad \left. + s_i (-2r_0 \vec{r} \cdot \vec{s} + r_0 \vec{s} \cdot \vec{s} - s_0 \vec{r} \cdot \vec{r}) \right) + \dots
\end{aligned}$$

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&\quad - \frac{1}{4\kappa^3} (\vec{r} \cdot \vec{s} \vec{r} \cdot \vec{r} + \vec{r} \cdot \vec{s} \vec{s} \cdot \vec{s} + 2r_0 s_0 \vec{r} \cdot \vec{r} - 2r_0 s_0 \vec{r} \cdot \vec{s} - 2r_0 s_0 \vec{s} \cdot \vec{s}) + \dots
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&\quad + \frac{1}{4\kappa^3} \left( s_i (2s_0 \vec{r} \cdot \vec{s} + s_0 \vec{r} \cdot \vec{r} + 2s_0 r_0 r_0 - r_0 \vec{s} \cdot \vec{s} - 2r_0 \vec{r} \cdot \vec{s} - \frac{2}{3}r_0 r_0 r_0) \right. \\
&\quad \left. + r_i (2s_0 \vec{r} \cdot \vec{s} + s_0 \vec{r} \cdot \vec{r} - r_0 \vec{s} \cdot \vec{s}) \right) + \dots
\end{aligned}$$

(To fourth order in  $\frac{1}{\kappa}$ ) there exists a unique non-trivial intertwiner

$$\tau : V_m \otimes V_m \longrightarrow V_m \otimes V_m.$$

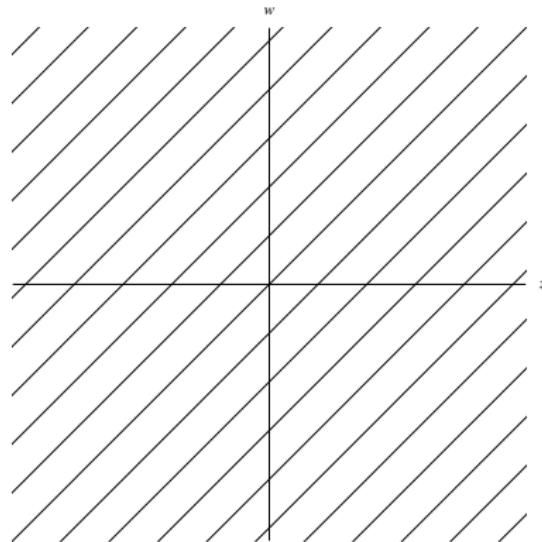
It has the property that

$$\tau^2 = 1.$$

[C.Y., R Zegers, 0711.2206]

# Exact results in 1+1 dimensions

Undeformed case,  $\kappa = \infty$ :



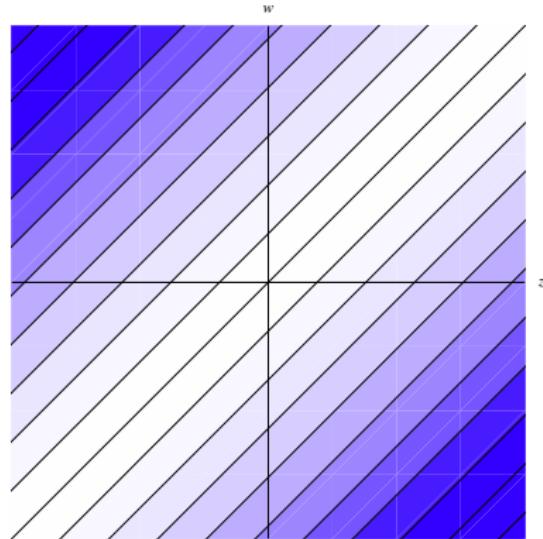
$$\vec{p} = (m \cosh z, m \sinh z)$$

$$\vec{q} = (m \cosh w, m \sinh w)$$

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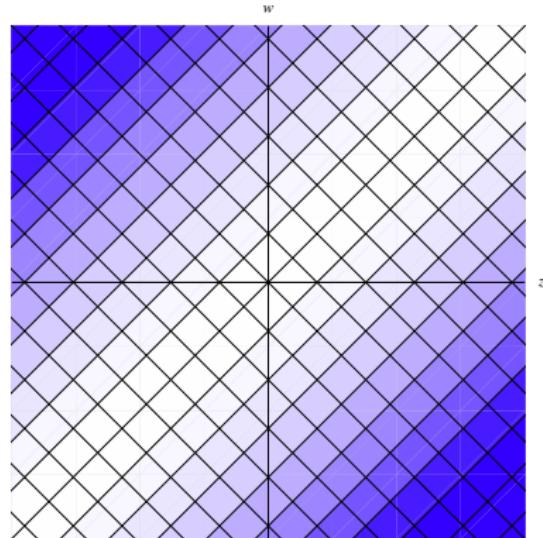
$$\vec{q} = (m \cosh w, m \sinh w)$$

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$$C_{12} = \left( 2m \cosh \frac{1}{2}(z - w) \right)^2$$

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$$\vec{p} = (m \cosh z, m \sinh z)$$

$$\vec{q} = (m \cosh w, m \sinh w)$$

$$\tau : \begin{cases} \phi = \frac{1}{2}(z + w) \rightarrow \phi \\ \mu = \frac{1}{2}(z - w) \rightarrow -\mu \end{cases}$$

$$C_{12} = (2m \cosh \mu)^2$$

General deformed case [C.Y., R. Zegers, 0803.2659], set:

$$\vec{p} = (-km \operatorname{Am}(z, k), -im \operatorname{Dn}(z, k))$$

$$\vec{q} = (-km \operatorname{Am}(w, k), -im \operatorname{Dn}(w, k))$$

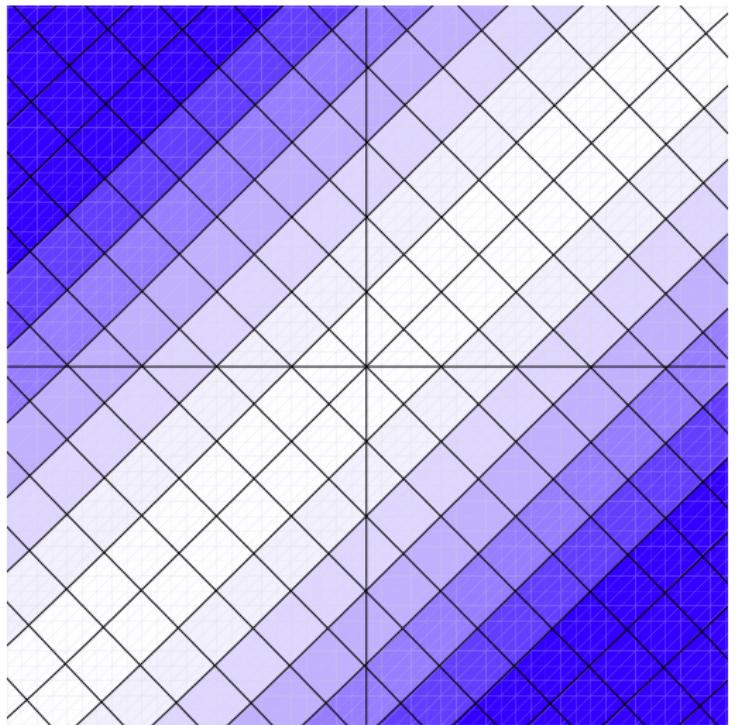
where  $k = 2i\kappa/m$ .

$$\begin{aligned}\mu &= -\frac{1}{2} \log \frac{1}{k} (\operatorname{Dn}z - k \operatorname{Cn}z) (\operatorname{Dn}z - ik \operatorname{Sn}z) \\ &\quad -\frac{1}{2} \log \frac{1}{k} (\operatorname{Dn}w - k \operatorname{Cn}w) (\operatorname{Dn}w + ik \operatorname{Sn}w) + \frac{1}{2} \log \left(1 - \frac{1}{k^2}\right).\end{aligned}$$

$$\begin{aligned}\phi &= -\frac{1}{2} \log \frac{1}{k} (\operatorname{Dn}z - k \operatorname{Cn}z) (\operatorname{Dn}z + ik \operatorname{Sn}z) \\ &\quad +\frac{1}{2} \log \frac{1}{k} (\operatorname{Dn}w - k \operatorname{Cn}w) (\operatorname{Dn}w - ik \operatorname{Sn}w) + \frac{i\pi}{2}.\end{aligned}$$

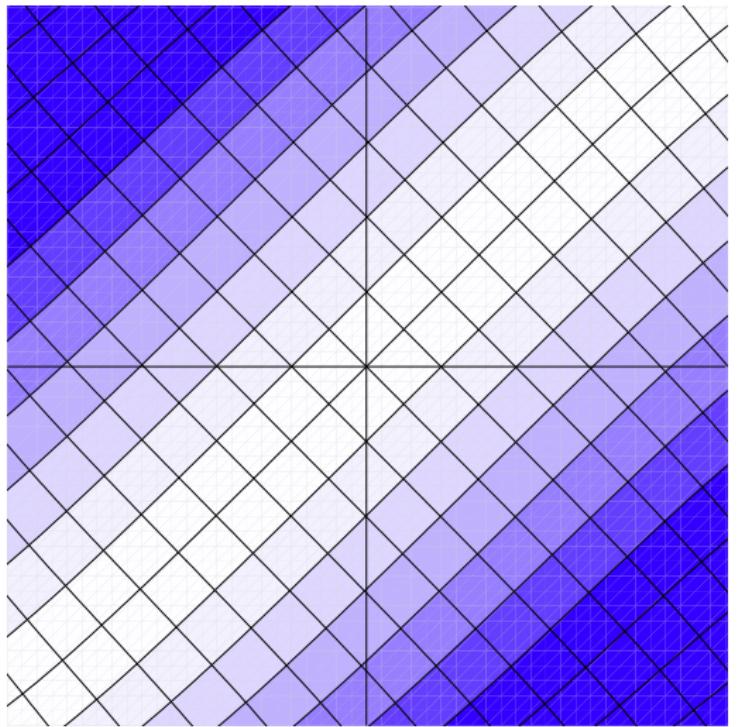
$$\tau : \begin{cases} \phi \rightarrow \phi \\ \mu \rightarrow -\mu \end{cases}, \quad C_{12} = \left(1 - \frac{1}{k^2}\right) (2m \cosh \mu)^2$$

*w*



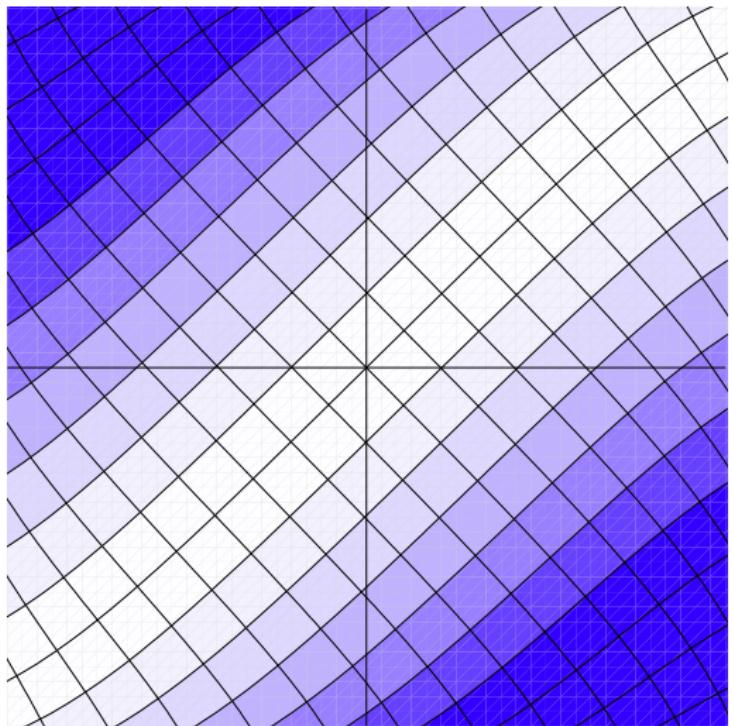
$$\kappa = \frac{\exp(7)}{2}$$

*w*



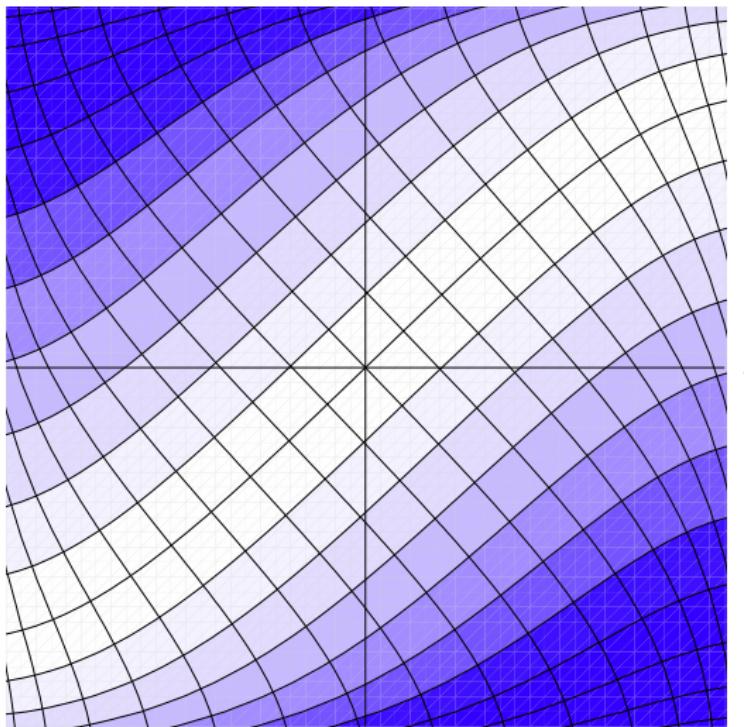
$$\kappa = \frac{\exp(6)}{2}$$

*w*



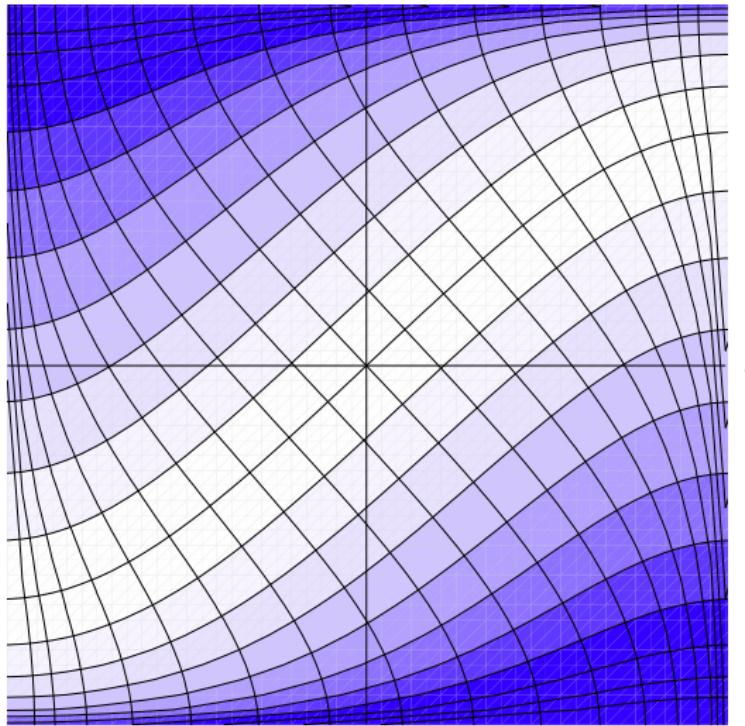
$$\kappa = \frac{\exp(5)}{2}$$

*w*

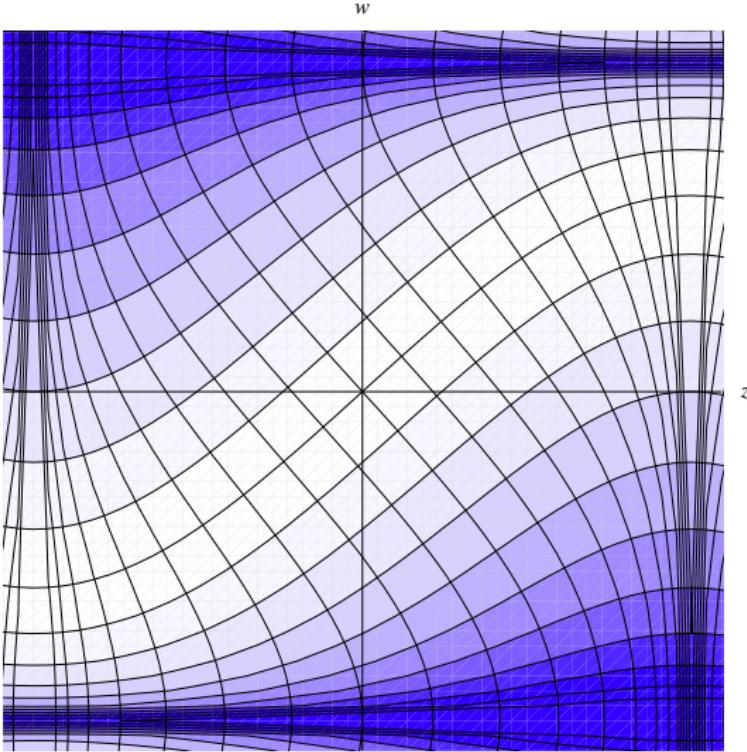


$$\kappa = \frac{\exp(4)}{2}$$

*w*



$$\kappa = \frac{\exp(3.5)}{2}$$



$$\kappa = \frac{\exp(3)}{2}$$

# 3 particles

General state:  $|r_\mu(z_1)\rangle \otimes |s_\mu(z_2)\rangle \otimes |t_\mu(z_3)\rangle \in V_m^{\otimes 3}$

3-particle boost operator:

$$N = e^{-\frac{r_0}{2\kappa} - \frac{s_0}{2\kappa}} \frac{\partial}{\partial z_3} + e^{-\frac{r_0}{2\kappa} + \frac{t_0}{2\kappa}} \frac{\partial}{\partial z_2} + e^{\frac{s_0}{2\kappa} + \frac{t_0}{2\kappa}} \frac{\partial}{\partial z_1}$$

All solutions to  $N\mu(z_1, z_2, z_3) = 0$  are functions of  $\mu_{12}, \mu_{23}$ :

$$\begin{aligned} \mu_{i,i+1} &= -\frac{1}{2} \log \frac{1}{k} (D_n z_i - k C_n z_i) (D_n z_i - ik S_n z_i) \\ &\quad -\frac{1}{2} \log \frac{1}{k} (D_n z_{i+1} - k C_n z_{i+1}) (D_n z_{i+1} + ik S_n z_{i+1}) + \frac{1}{2} \log \left(1 - \frac{1}{k^2}\right) \end{aligned}$$

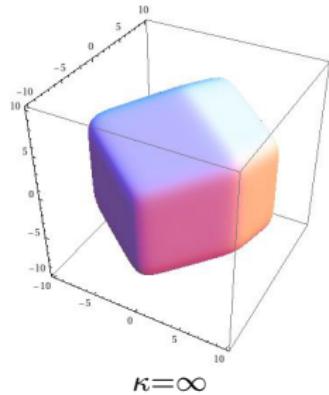
These  $\mu_{12}, \mu_{23} \therefore$  label the irreducible components of  $V^{\otimes 3}$ . Total Casimir is

$$\begin{aligned} \frac{C_{123}}{m^2} &= \left(1 - \frac{1}{k^2}\right)^2 (e^{2\mu_{12}+2\mu_{23}} + e^{-2\mu_{12}-2\mu_{23}}) - \frac{1}{k^2} \left(1 - \frac{1}{k^2}\right) (e^{2\mu_{12}-2\mu_{23}} + e^{-2\mu_{12}+2\mu_{23}}) \\ &\quad + \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{2}{k^2}\right) (e^{2\mu_{12}} + e^{-2\mu_{12}} + e^{2\mu_{23}} + e^{-2\mu_{23}}) + 3 \left(1 - \frac{1}{k^2}\right)^2 + \frac{1}{k^4}. \end{aligned}$$

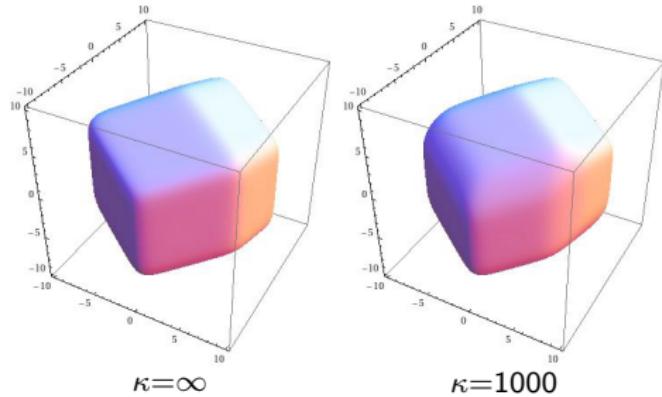
# 4 particles

$$\begin{aligned}
 \frac{C_{1234}}{m^2} = & + \left(1 - \frac{1}{k^2}\right)^3 (e^{2\mu_{12}+2\mu_{23}+2\mu_{34}} + e^{-2\mu_{12}-2\mu_{23}-2\mu_{34}}) \\
 & - \frac{1}{k^2} \left(1 - \frac{1}{k^2}\right)^2 (e^{-2\mu_{12}+2\mu_{23}+2\mu_{34}} + e^{2\mu_{12}+2\mu_{23}-2\mu_{34}} \\
 & \quad + e^{-2\mu_{12}-2\mu_{23}+2\mu_{34}} + e^{2\mu_{12}-2\mu_{23}-2\mu_{34}}) \\
 & + \frac{1}{k^4} \left(1 - \frac{1}{k^2}\right) (e^{2\mu_{12}-2\mu_{23}+2\mu_{34}} + e^{-2\mu_{12}+2\mu_{23}-2\mu_{34}}) \\
 & + \left(1 - \frac{1}{k^2}\right)^2 \left(1 - \frac{2}{k^2}\right) (e^{2\mu_{12}+2\mu_{23}} + e^{-2\mu_{12}-2\mu_{23}} + e^{2\mu_{23}+2\mu_{34}} + e^{-2\mu_{23}-2\mu_{34}}) \\
 & - \frac{1}{k^2} \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{2}{k^2}\right) (e^{2\mu_{12}-2\mu_{23}} + e^{-2\mu_{12}+2\mu_{23}} + e^{2\mu_{23}-2\mu_{34}} + e^{-2\mu_{23}+2\mu_{34}}) \\
 & - \frac{2}{k^2} \left(1 - \frac{1}{k^2}\right)^2 (e^{2\mu_{12}+2\mu_{34}} + e^{-2\mu_{12}+2\mu_{34}} + e^{2\mu_{12}-2\mu_{34}} + e^{-2\mu_{12}-2\mu_{34}}) \\
 & + \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{2}{k^2}\right)^2 (e^{2\mu_{12}} + e^{-2\mu_{12}} + e^{2\mu_{23}} + e^{-2\mu_{23}} + e^{2\mu_{34}} + e^{-2\mu_{34}}) \\
 & + 4 \left(1 - \frac{1}{k^2}\right)^3 + \frac{4}{k^4} \left(1 - \frac{1}{k^2}\right)
 \end{aligned} \tag{1}$$

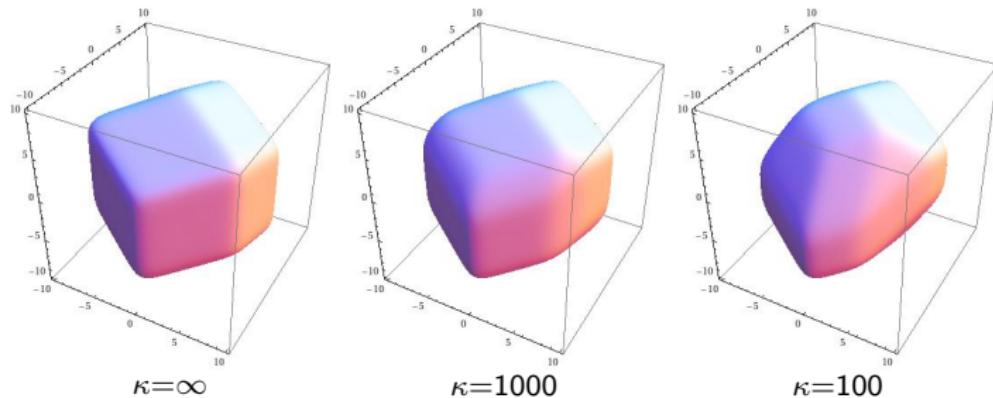
# Contour of $C_{1234}(\mu_{12}, \mu_{23}, \mu_{34})$ .



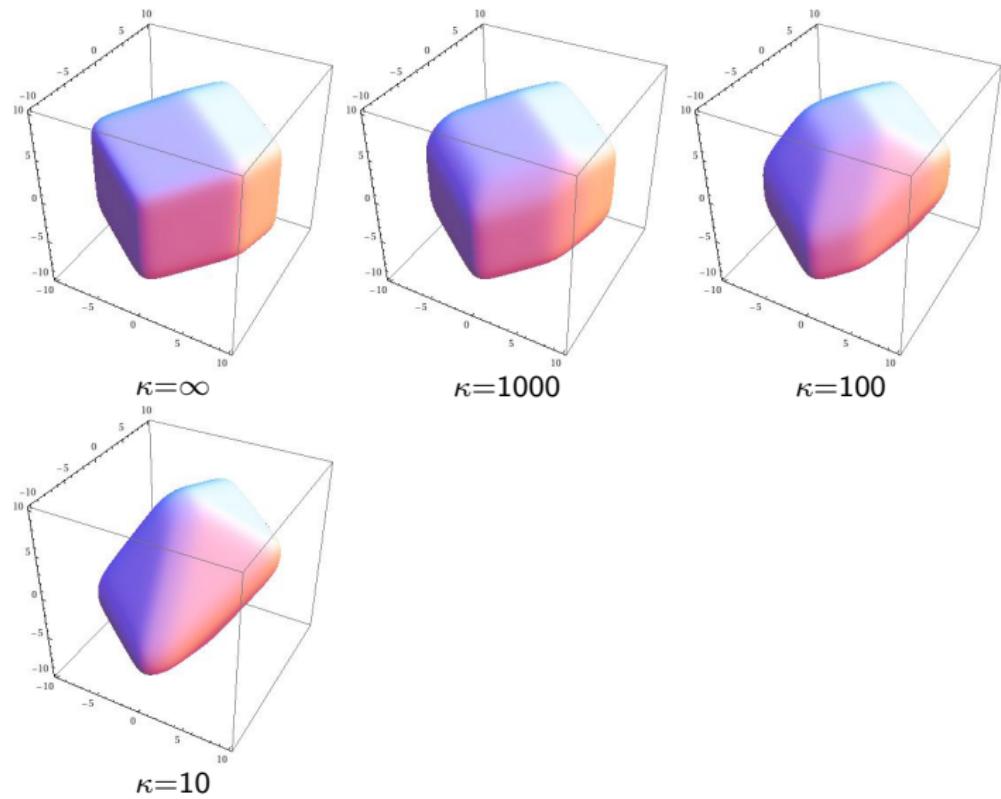
# Contour of $C_{1234}(\mu_{12}, \mu_{23}, \mu_{34})$ .



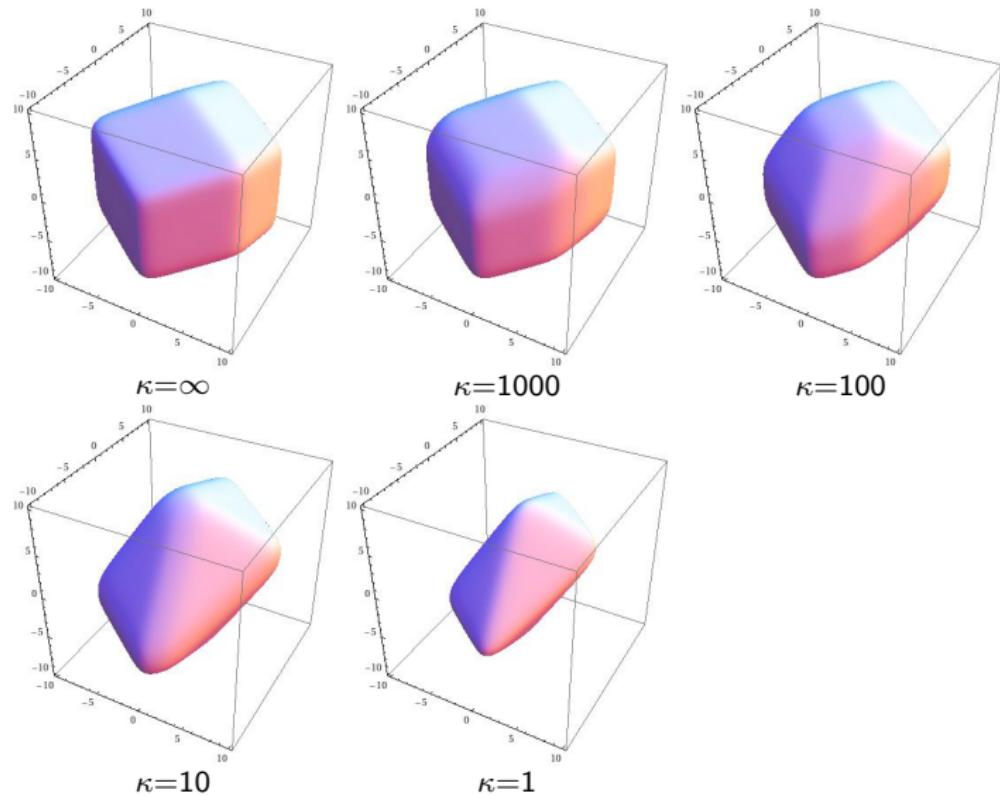
# Contour of $C_{1234}(\mu_{12}, \mu_{23}, \mu_{34})$ .



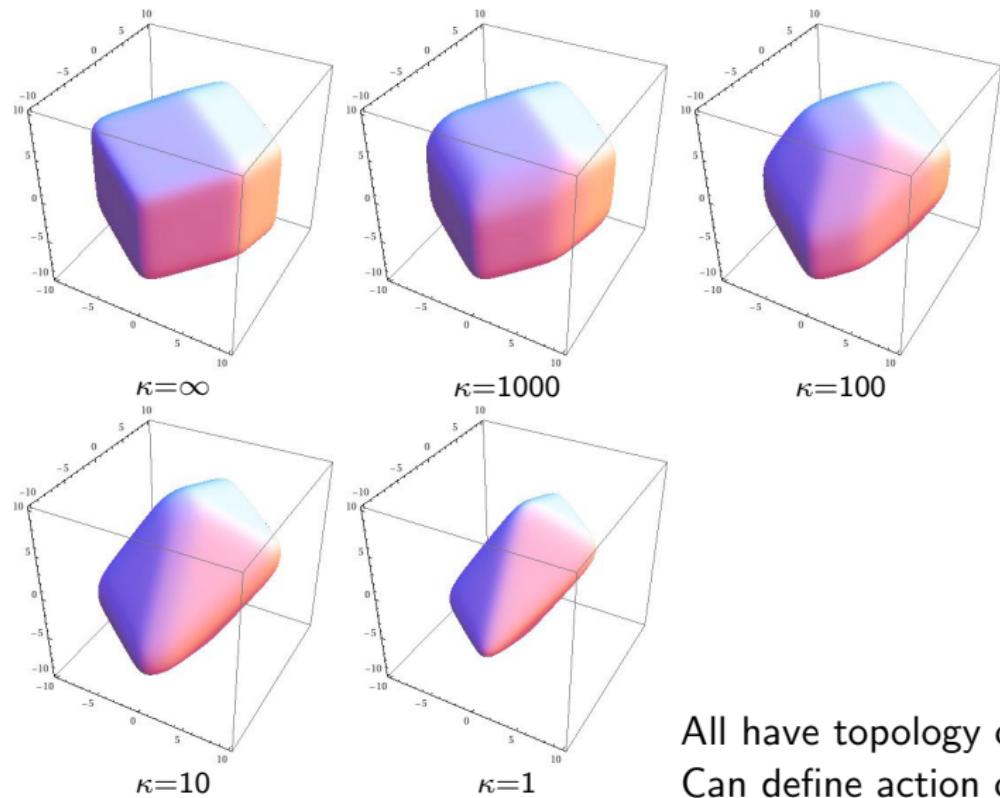
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All have topology of  $\mathbb{S}^3$   
Can define action of  $S_4 \cong A_3$

# Overview

- To understand many particle quantum theory with  $\kappa$ -Poincaré symmetry, at least in 3+1 or higher dimensions, need a notion of identical particles.
- Amounts to finding realization of the “particle exchange group”  $S_N$  on tensor product states  $V_m^{\otimes N}$  which
  - is  $\kappa$ -covariant, i.e. commutes with action of  $\mathcal{U}_\kappa(P)$
  - collapses to exchange of tensor factors as  $\kappa \rightarrow \infty$ , so that usual notion of particle exchange is recovered.

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Conclusions:

- Such a notion of particle exchange exists
- Unique for  $N = 2$ .
- For  $N > 2$  need some additional input to pick out the natural choice.

Correct labelling of states in  $\kappa$ -deformed theories? Purely algebraic approach? Algebra of creation/annihilation operators in closed form? CCRs of fields....