

# Fusion hierarchies and defect lines

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## Outline:

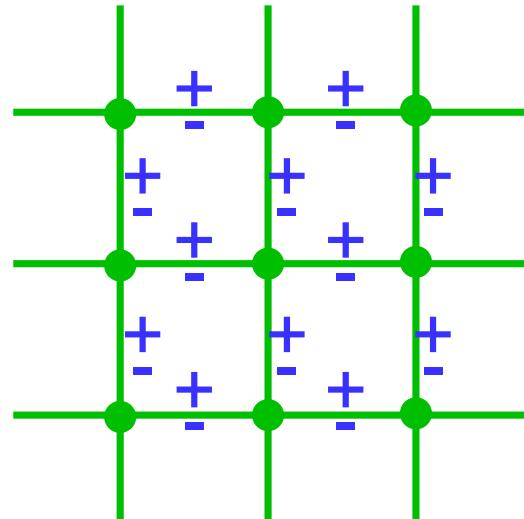
- integrable lattice models
- a free field construction in CFT
- perturbed defects in CFT

## Integrable lattice models : 6-vertex

**References:** Bazhanov, Reshetikhin '89  
Klümper, Pearce '92

**here:** Kuniba, Nakanishi, Suzuki '93  
Kuniba, Sakai, Suzuki '98

**review:** Dorey, Dunning, Tateo '07



$$\begin{array}{c} + \\ | \\ v \\ | \\ + \end{array} = \frac{\sin \frac{\theta(v+2)}{2}}{\sin \theta}$$

$$\begin{array}{c} + \\ | \\ v \\ | \\ - \end{array} = \frac{\sin \frac{\theta v}{2}}{\sin \theta}$$

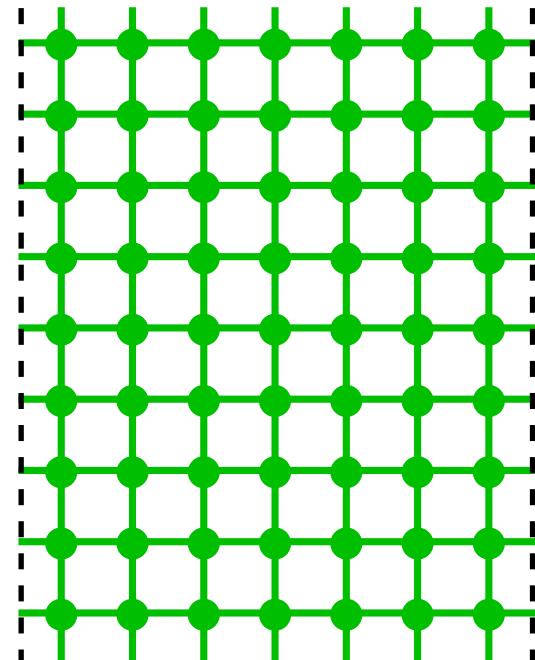
$$\begin{array}{c} + \\ | \\ v \\ | \\ - \end{array} = 1$$

## ... 6-vertex

Transfer matrix. Fix  $u \in \mathbb{C}$

$$T(v)_{\epsilon_1 \dots \epsilon_N}^{\sigma_1 \dots \sigma_N} = \sum_{\delta_k = \pm} \text{Diagram}$$

The diagram shows a horizontal chain of vertices connected by green horizontal edges. Each vertex has two vertical edges extending upwards and downwards. The top edge is labeled  $\sigma_1, \sigma_2, \dots, \sigma_N$  from left to right. The bottom edge is labeled  $\delta_1, \delta_2, \dots, \delta_N$  from left to right. The vertical edges at each vertex are labeled  $u+iv, u-iv, u-iv$  from left to right. Below the vertices, the labels  $\epsilon_1, \epsilon_2, \dots, \epsilon_N$  are placed under the bottom edges. The bottom edge is labeled  $\delta_1$  at both ends.



Partition function

$$Z_{M \times N}(\theta, u, v) = \text{tr}(T(v)^M)$$

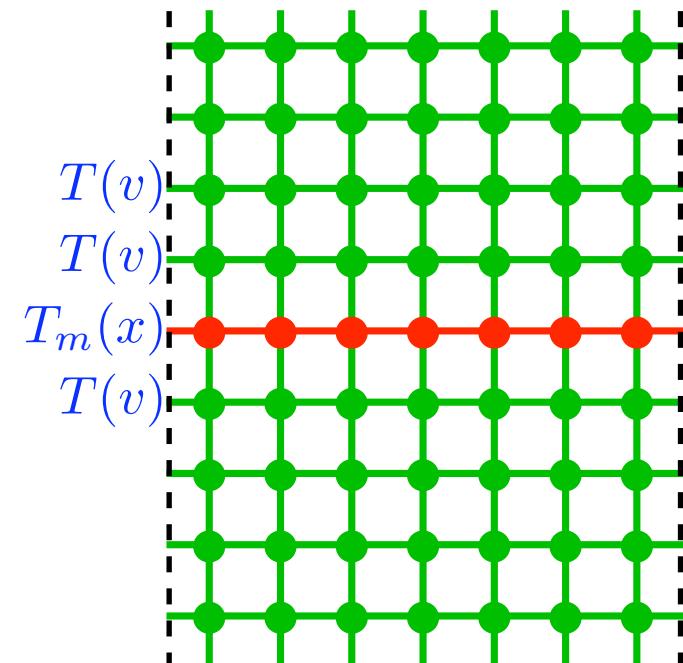
Transfer matrices commute  $[T(v), T(v')] = 0$

## ... 6-vertex

### Fused transfer matrix

$$T_m(v)_{\epsilon_1 \dots \epsilon_N}^{\sigma_1 \dots \sigma_N} = \sum_{j_k=1}^m \text{Diagram}$$

The diagram shows a horizontal red line with vertices labeled  $j_1, j_2, \dots, j_N$  from left to right. Above the line, vertical green lines labeled  $\sigma_1, \sigma_2, \dots, \sigma_N$  connect to the vertices  $j_1, j_2, \dots, j_N$  respectively. Below the line, vertical blue lines labeled  $\epsilon_1, \epsilon_2, \dots, \epsilon_N$  connect to the vertices  $j_1, j_2, \dots, j_N$  respectively. The labels  $u+iv$  and  $u-iv$  are placed between the red line and the green lines.



## ... 6-vertex

Properties of fused transfer mat.

$$T_2(v) = T(v)$$

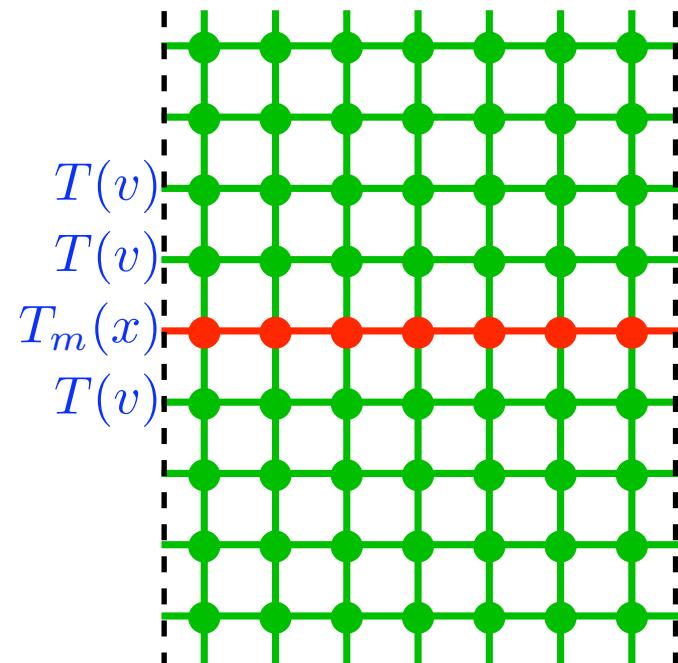
$$T_m(v + 2\pi i/\theta) = T_m(v)$$

$$[T_m(v), T_n(v')] = 0$$

Fusion hierarchy

$$\begin{aligned} & T_m(v+i)T_m(v-i) \\ &= T_{m-1}(v)T_{m+1}(v) + T_1(v+in)T_1(v-in) \end{aligned}$$

$$T_1(v) = s(\theta, u, v)^N \cdot id$$



## ... 6-vertex

Remember:

- $T_m(x)$  “defect lines” in lattice model
- linear operators on space of states
- commute
- fusion hierarchy

$$\begin{aligned} & T_m(v+i)T_m(v-i) \\ &= (\text{scalar fn}) \cdot id + T_{m-1}(v)T_{m+1}(v) \end{aligned}$$

## Free field construction

References: Bazhanov, Lukyanov, Zamolodchikov '94 '96 '98

Free boson with bg charge :  $c = 13 - 6(\beta^2 + \beta^{-2})$

$\mathcal{F}_p$  : Fock space, momentum  $p$   $\beta = 1 \Rightarrow c = 1$

$$\hat{\mathcal{F}}_p = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{p+n\beta}$$

$V_j$  :  $2j+1$  dim. irrep. of  $U_q(sl(2))$ ,  $j=0, \frac{1}{2}, 1, \dots$

$$q = e^{i\pi\beta^2}$$

Next: an operator on  $\hat{\mathcal{F}}_p \otimes V_j$

## ... free field construction

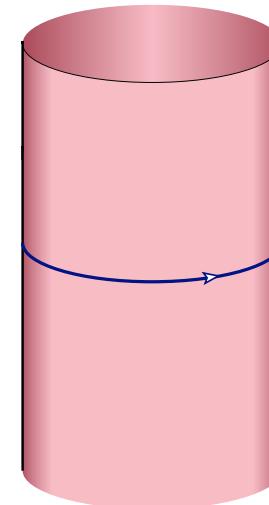
Vertex operator  $h = \beta^2 \times \text{Matrix}$

$$K(x) = :e^{-2\varphi(x)}: \otimes q^{\frac{H}{2}} E + :e^{2\varphi(x)}: \otimes q^{-\frac{H}{2}} F$$

An operator on  $\hat{\mathcal{F}}_p \otimes V_j$  :

$$e^{2\pi i PH} P \exp \left( \lambda \int_0^{2\pi} K(x) dx \right)$$

$$\mathbb{T}_j(\lambda) = \text{tr}_{V_j} \left( \cdots \right) : \hat{\mathcal{F}}_p \rightarrow \hat{\mathcal{F}}_p$$



## ... free field construction

Have  $\mathbb{T}_j(\lambda) = \mathbb{T}_j(-\lambda)$   $\rightarrow$  set  $\tilde{T}_s(\lambda^2) = \mathbb{T}_{\frac{1}{2}(s-1)}(\lambda)$

Properties ( $q = e^{i\pi\beta^2}$ )

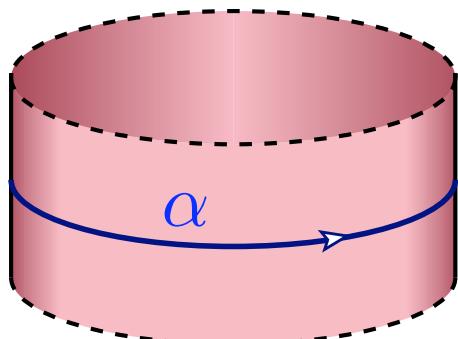
- $[L_0, \tilde{T}_s(\mu)] = 0$
- $\tilde{T}_1(\mu) = id$
- $[\tilde{T}_m(\mu), \tilde{T}_n(\mu')] = 0$
- $\tilde{T}_m(q\lambda) \tilde{T}_m(q^{-1}\lambda) = id + \tilde{T}_{m-1}(\lambda) \tilde{T}_{m+1}(\lambda)$

## Perturbed defects in CFT

minimal model  $\mathsf{M}_{p,p'} : c = 13 - 6(t^2 + t^{-2}) \quad t = \frac{p}{p'}$   
 $\mathcal{I}$ : irreps , Kac-labels  $(r,s)$  mod  $\mathbb{Z}_2$

$$\mathcal{H} = \bigoplus_{i \in \mathcal{I}} R_i \otimes \bar{R}_i$$

## Defect operators



$$D_\alpha : \mathcal{H} \rightarrow \mathcal{H}$$

## ... perturbed defects in CFT

- conformally invariant defect

$$[L_m - \bar{L}_m, D_\alpha] = 0$$

- topological defects

Petkova,  
Zuber '00

$$[L_m, D_\alpha] = 0 = [\bar{L}_m, D_\alpha]$$

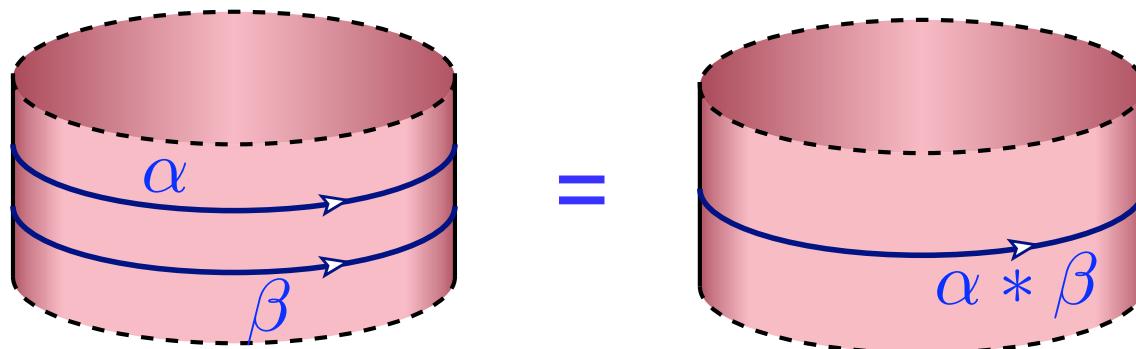
top. def. labelled by  $\alpha \in \mathcal{I}$

$$D_\alpha = \sum_{k \in \mathcal{I}} \frac{S_{\alpha k}}{S_{1k}} id_{R_k \otimes \bar{R}_k}$$

$$D_\alpha D_\beta = \sum_{\gamma \in \mathcal{I}} N_{\alpha\beta}^\gamma D_\gamma$$

... perturbed defects in CFT

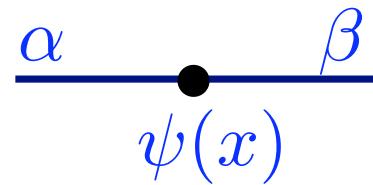
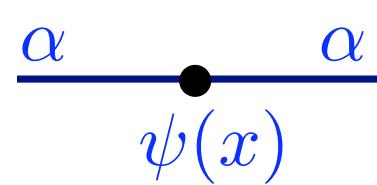
Fusion of topological defects



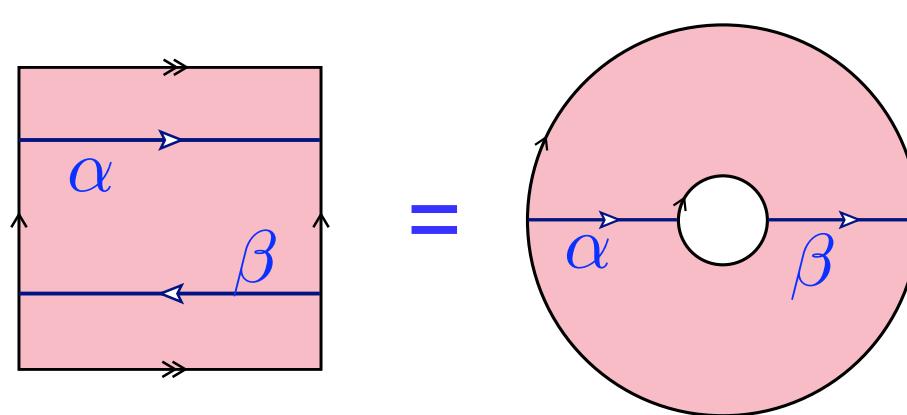
$$D_\alpha D_\beta = D_{\alpha * \beta} = \sum_{\gamma \in \mathcal{I}} N_{\alpha\beta}^\gamma D_\gamma$$

## ... perturbed defects in CFT

### Defect fields



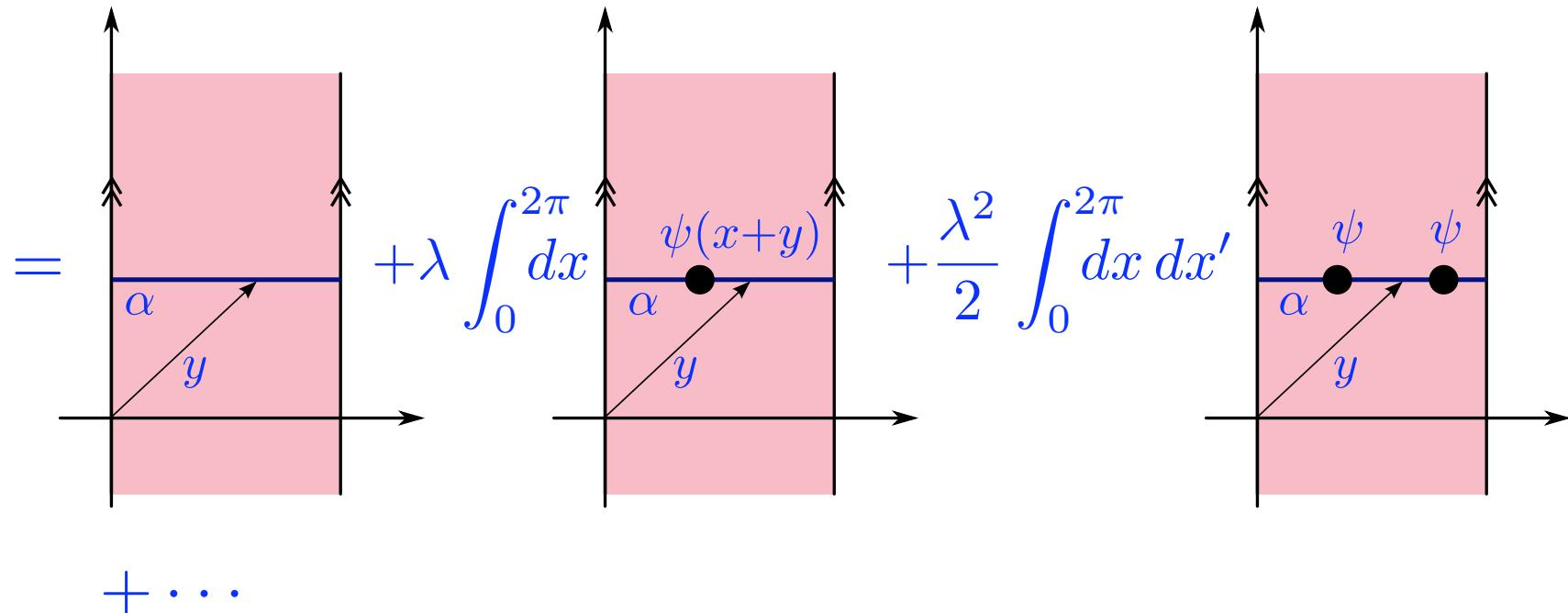
### Field content from torus with two defect lines



## ... perturbed defects in CFT

Perturb defect operator by  $\psi(x) \equiv \psi^{\alpha \rightarrow \alpha}(x)$

$$D_\alpha(\lambda, y) = \exp\left(\lambda \int_0^{2\pi} \psi(x+y) dx\right)$$



## ... perturbed defects in CFT

$$D_\alpha(\lambda, y) = \exp\left(\lambda \int_0^{2\pi} \psi(x+y) dx\right)$$

Suppose  $\psi(x)$  has weight  $(h, 0)$ .

Then  $\frac{\partial}{\partial \bar{y}} D_\alpha(\lambda, y) = 0$  as  $\frac{\partial}{\partial \bar{y}} \psi(x + y) = 0$ .

Also  $\frac{\partial}{\partial a} D_\alpha(\lambda, a + ib) = 0$ . Thus independent of  $y$ .

Write  $D_\alpha(\lambda)$ .

$$[L_0 + \bar{L}_0, D_\alpha(\lambda)] = 0 \quad [\bar{L}_m, D_\alpha(\lambda)] = 0$$

## ... perturbed defects in CFT

Fusion without field insertion

$$\begin{array}{c} \alpha \\ \beta \end{array} = \sum_{\gamma \in \mathcal{I}} \text{Diagram} \quad \text{Diagram: } \begin{array}{c} \alpha \\ \beta \end{array} \text{ and } \begin{array}{c} \alpha \\ \beta \end{array} \text{ connected by a horizontal line labeled } \gamma \text{ between them.}$$

Fusion with field insertion  $\rightarrow$  get defect changing field

$$\begin{array}{c} \alpha \\ \beta \end{array} \psi(x) = \sum_{\gamma, \delta \in \mathcal{I}} \text{Diagram} \quad \text{Diagram: } \begin{array}{c} \alpha \\ \beta \end{array} \text{ and } \begin{array}{c} \alpha \\ \beta \end{array} \text{ connected by a horizontal line labeled } \gamma \text{ between them, with a circle labeled } \psi \text{ between them, and a horizontal line labeled } \delta \text{ between them.}$$

Fusion of perturbed defects

$$D_\alpha(\lambda) D_\beta(\mu) = D_{\alpha * \beta} \left( \sum_{\gamma, \delta} c_{\gamma, \delta}(\lambda, \mu) \psi^{\gamma \rightarrow \delta} \right)$$

## ... perturbed defects in CFT

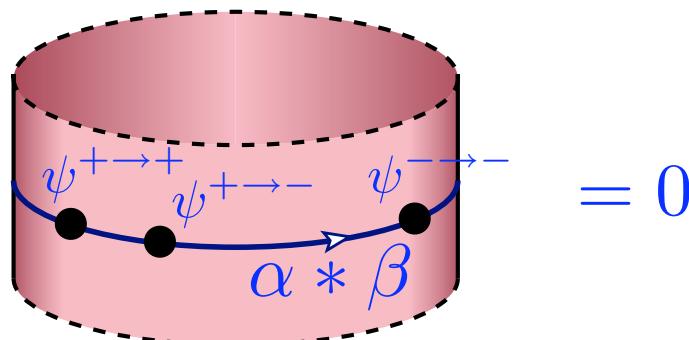
$$D_\alpha(\lambda) D_\beta(\mu) = D_{\alpha * \beta} \left( \sum_{\gamma, \delta} c_{\gamma, \delta}(\lambda, \mu) \psi^{\gamma \rightarrow \delta} \right)$$

e.g. Min. Mod.:  $\Psi$  of weight  $(h_{l,3}, 0)$ ,  $\alpha = (l, 2)$ ,  $\beta = (l, s)$

$$\alpha * \beta = (l, s-l) + (l, s+l)$$

Use 3-d TFT approach (see Fröhlich, Fuchs, Schweigert, IR '06 )

$$c_{s\pm 1, s\mp 1} = (\text{const}) \left( \lambda - \mu e^{\pm i\pi st} \right)$$



## ... perturbed defects in CFT

Functional relation     $q = e^{i\pi p/p'}$

IR '07

$$\begin{aligned} & D_{(1,2)}(\lambda) D_{(1,s)}(q^s \lambda) \\ &= D_{(1,s-1)}(q^{s+1} \lambda) + D_{(1,s+1)}(q^{s-1} \lambda) \end{aligned}$$

This implies, with  $\hat{T}_s(\lambda) = D_{(1,s)}(\lambda)$ ,

$$\hat{T}_s(q\lambda) \hat{T}_s(q^{-1}\lambda) = id + \hat{T}_{s-1}(\lambda) \hat{T}_{s+1}(\lambda)$$

Furthermore

$$[\hat{T}_s(\lambda), \hat{T}_s(\mu)] = 0 \quad \hat{T}_1(\lambda) = id \quad \hat{T}_{p'}(\lambda) = D_{(1,p')}$$

## ... perturbed defects in CFT

Chiral perturbation:  $D_{(1,s)} + \lambda\psi$  of weight (h,0)

$$D_{(1,s)}(\lambda)|_{R_i \otimes \bar{R}_i} = ( * ) \otimes id_{\bar{R}_i}$$

Anti-chiral perturbation:  $D_{(1,s)} + \mu\bar{\psi}$  of weight (0,h)

$$\bar{D}_{(1,s)}(\mu)|_{R_i \otimes \bar{R}_i} = id_{R_i} \otimes ( * )$$

Puzzle:

Two sets of fused transfer matrices in lattice model ?

## ... perturbed defects in CFT

What about  $D_{(1,s)} + \lambda\psi + \mu\bar{\psi}$  ?

$$[L_0 + \bar{L}_0, \mathbb{D}_{(1,s)}(\lambda, \mu)] \neq 0$$

Expect:

$D_{(1,s)}(\lambda)$  and  $\bar{D}_{(1,s)}(\mu)$  flow to topological defect.

$\mathbb{D}_{(1,s)}(\lambda, \mu)$  flows to conformal defect.

## Example: Ising model

topological defects

Petkova, Zuber '00

$$id \quad D_\epsilon \quad D_\sigma$$

conformal defects

$$\mathbb{D}_D(\varphi) \quad \varphi \in [0, \pi] \qquad \text{Affleck, Oshikawa, '96}$$

$$\mathbb{D}_N(\varphi) \quad \varphi \in [0, \frac{\pi}{2}]$$

# Conformal defects in the Ising model

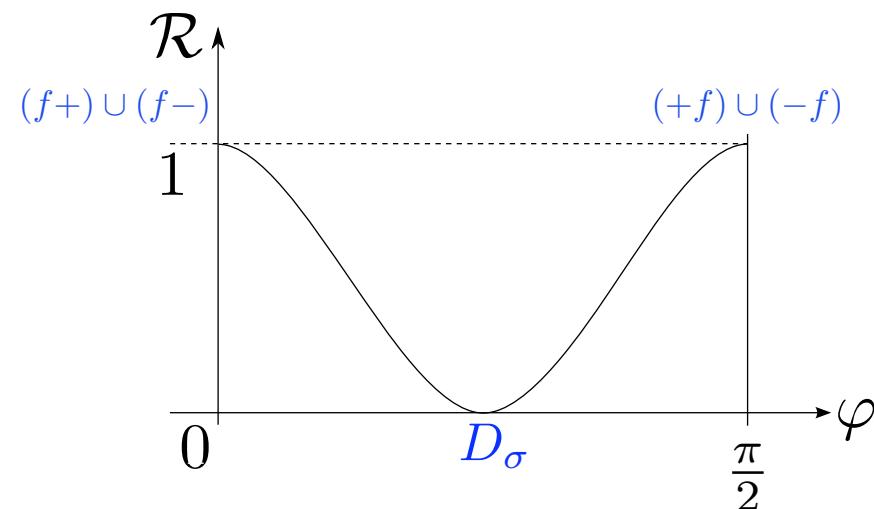
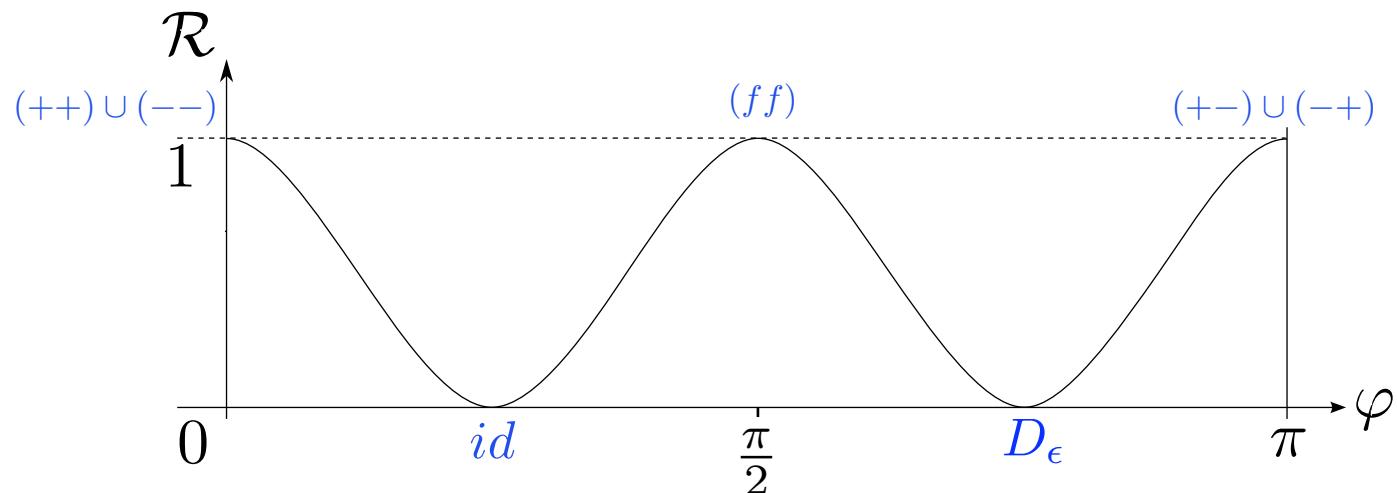
Quella, Watts, IR '06

Dirichlet  
Line

$$\mathbb{D}_D(\varphi)$$

Neumann  
Line

$$\mathbb{D}_N(\varphi)$$



## ... example: Ising model

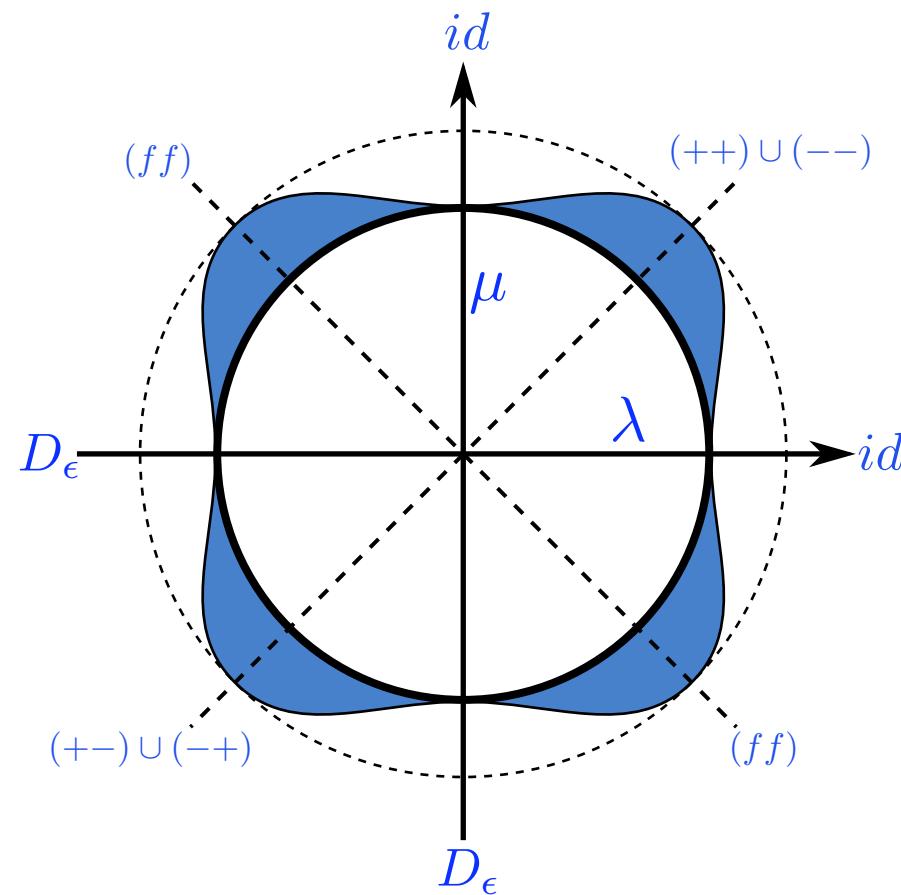
work in progress with  
Gérard Watts and Márton Kormos

perturb  $D_\sigma + \lambda\psi + \mu\bar{\psi}$  , weights  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$

→ flows to

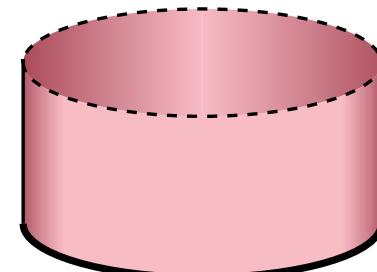
$$\mathbb{D}_D(\varphi)$$

$$\varphi = -\frac{\pi}{4} + \text{atan} \frac{\lambda}{\mu}$$



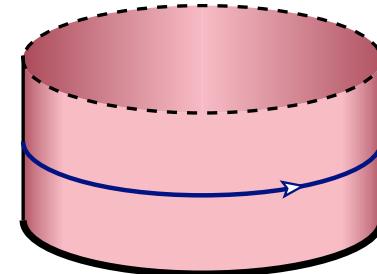
## Application: boundary flows

1) conformal b.c. a  
+ bnd field  $\psi$ , weight h



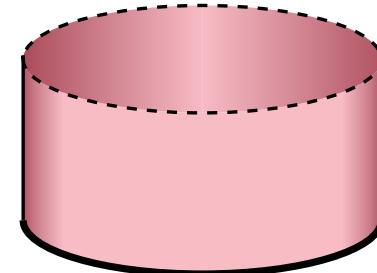
$$|a\rangle + \lambda\psi$$

2) decompose into  
• defect  $\alpha$  +  $\psi'$ , weight (h,0)  
• conformal b.c. b



$$D_\alpha + \lambda\psi'$$
  
 $|b\rangle$

3) use fusion hierarchy to analyse flow  $D_\alpha \xrightarrow{+\psi'} D_*$



$$D_*|b\rangle$$

4) move defect back to bnd ,  
get endpoint of bnd flow

## Summary

- topological defect
  - + perturbation by chiral defect field

$$[L_0 + \bar{L}_0, \hat{T}_s(\lambda)] = 0 \quad [\hat{T}_s(\lambda), \hat{T}_s(\mu)] = 0$$

- fusion of perturbed defects gives functional relation

$$\hat{T}_s(q\lambda) \hat{T}_s(q^{-1}\lambda) = id + \hat{T}_{s-1}(\lambda) \hat{T}_{s+1}(\lambda)$$

TODO:

- Q-operators in CFT/defect picture
- other examples than  $sl(2)$

...