The graph expansion of an ordered groupoid

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Abstract

We generalise the Margolis-Meakin graph expansion of a group to a construction for ordered groupoids, and show that the graph expansion of an ordered groupoid enjoys structural properties analogous to those for graph expansions of groups. We also use the Cayley graph of an ordered groupoid to prove a version of McAlister's P–theorem for incompressible ordered groupoids.

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1 Introduction

The Margolis-Meakin *graph expansion* [11] constructs, from the Cayley graph of a group G and a given set of generators X, an inverse semigroup $\mathcal{M}(G,X)$ with numerous interesting properties. It is shown in [11] that $\mathcal{M}(G,X)$ is an X-generated, E–unitary inverse semigroup with maximum group image G, and that the subgroups of $\mathcal{M}(G,X)$ are precisely the finite subgroups of G. It follows that $\mathcal{M}(G,X)$ is combinatorial if and only if G is torsion-free. The structure of the expansion also captures subtle information on the presentation of G on the given generating set X, since there is an inverse semigroup presentation

$$\mathcal{M}(G,X) = \mathrm{Inv}[X : w^2 = w \text{ if } w =_G 1]. \tag{1.1}$$

We note that presentations of this form are *contractive* in the sense of Stephen [20]. Moreover, considered as a functor from the category of X-

generated groups to the category of X-generated E-unitary inverse semigroups, the graph expansion is left adjoint to the maximum group image functor, and so preserves coproducts.

Various generalisations of the graph expansion have been developed. Gould [5] constructs an expansion from the Cayley graph of a monoid S, and proves that if S is right cancellative then its expansion is a proper left ample monoid with maximum right cancellative image S. Noonan Heale [16] constructs an expansion from the Cayley graph of a semigroup S, and proves that if S is E–dense then so is its expansion and they have the same universal group. Lawson, Margolis and Steinberg [10] construct expansions for inverse semigroups, treating both the graph expansion and the prefix expansion of Birget and Rhodes [1]. Their version of the graph expansion is an expansion of inverse semigroups (cut down to generators) in the sense of [1], and it is first defined by an inverse semigroup presentation analogous to (1.1). The structure of the expansion of an inverse semigroup S is then described in terms of subgraphs of the Cayley graph of S in [10, Theorem 5.4].

The approach of [10] then naturally generalises to the setting of ordered groupoids. Groups are ordered groupoids, with one identity and the trivial partial order, and inverse semigroups are ordered groupoids in which the identities form a meet semilattice. Ordered groupoids with this property are called inductive, and the Ehresmann-Schein-Nambooripad Theorem (see [7, Theorem 4.1.8]) gives an isomorphism between the categories of inverse semigroups and inductive groupoids. The construction of the graph expansion of an ordered groupoid was outlined in [3] and its basic properties as an ordered groupoid were set out. However, no attempt was made in [3] at any deeper structural investigation, based upon the properties enjoyed by the graph expansion of a group. The aim of the present paper is to carry out such an investigation.

The main structural property of interest is incompressibility, as defined in [4], which generalises the E–unitary property for inverse semigroups. An inverse semigroup S is E–unitary if and only if the map $\sigma : S \to \widehat{S}$ from S to its maximum group image $\widehat{S}$ is injective when restricted to each $\mathcal{R}$–class (see [7, Proposition 3.2.14]) or, in the language of inductive groupoids, if and only if σ is star injective. For an ordered groupoid G, the map σ is replaced by a groupoid map $\lambda : G \to G_{\updownarrow}$, where $G_{\updownarrow}$ is a quotient of the universal groupoid (in the sense of Higgins [6]) constructed from G by identifying identities that are related in the partial order, and is called the *level groupoid* of G. If G is inductive then all the identities are identified, and $G_{\updownarrow}$ is a group. For an ordered groupoid G we say that G is *incompressible* if λ is star injective.

We revisit the brief discussion of the graph expansion of an ordered groupoid in [4], adopting a more rigorous definition of what it means for an ordered groupoid G to be generated by a set X. Our main result, Theorem 3.1, shows that the graph expansion $\mathcal{M}(G, X)$ of an X–generated ordered groupoid G is again X–generated, has level groupoid isomorphic to the level groupoid $G_{\updownarrow}$ of G, and is incompressible if and only if G is. We also show (in Theorem 2.9) that the subgroups of $\mathcal{M}(G, X)$ are precisely the finite subgroups of G. These results mirror the key properties of the Margolis-Meakin graph expansion of a group.

Any study of the properties of E–unitary inverse semigroups must touch upon its central structural result, McAlister's celebrated P–theorem, first proved in [12]. The theorem has been regularly revisited and a range of proofs have been put forward, some of which are described in the survey [9]. A short and lucid proof has been given by Steinberg [18], using the Cayley graph of the maximum group image $\widehat{S}$ of an E–unitary inverse semigroup S to give an explicit construction of the structural ingredients required in the P–theorem. As noted in [9], the P–theorem can be rephrased as a statement about ordered groupoids, and it is then seen to be a consequence of the Maximum Enlargement Theorem of Ehresmann, see [7, 17]. A proof of the P–theorem for ordered groupoids was given in [4], using a direct approach modelled on Munn's paper [14], and further discussion of connections with the Maximum Enlargement Theorem is given in [13]. Here we present a proof of the P–theorem for ordered groupoids as stated in [4], using the methods of [18]. As in Steinberg's approach, we are able to give an explicit construction in terms of the Cayley graph of the level groupoid $G_{\updownarrow}$.

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2 Ordered groupoids

A *groupoid* G is a small category in which every arrow is invertible. We think of a groupoid as an algebraic structure (as in [6, 7]) in which the elements are the arrows, and composition of arrows is a partially defined associative binary operation. The set of identities in G is denoted by $E(G)$, and an element $g \in G$ has domain $\mathbf{d}(g) = gg^{-1}$ and range $\mathbf{r}(g) = g^{-1}g$. The structure-preserving maps between groupoids are functors.

For an identity $e \in E(G)$, the *star* of e in G is the set $\text{star}_G(e) = \{g \in G : \mathbf{d}(g) = e\}$. A functor $\phi : G \to H$ is said to be *star injective* if, for each $e \in E(G)$, the restriction $\phi : \text{star}_G(e) \to \text{star}_H(e\phi)$ is injective. The *local group* at e is the subgroup $\{g \in G : \mathbf{d}(g) = e = \mathbf{r}(g)\}$ of $\text{star}_G(e)$: clearly the local groups are the maximal subgroups of G.
An *ordered groupoid* $(G, \leqslant)$ is a groupoid G with a partial order $\leqslant$ satisfying the following axioms:

OG1 for all $g, h \in G$, if $g \leqslant h$ then $g^{-1} \leqslant h^{-1}$,

OG2 if $g_1 \leqslant h_1$, $g_2 \leqslant h_2$ and if the compositions g_1g_2 and h_1h_2 are defined, then $g_1g_2 \leqslant h_1h_2$,

OG3 if $g \in G$ and f is an identity of G with $f \leqslant \mathbf{d}(g)$, there exists a unique element $(f|g)$, called the *restriction* of g to f, such that $\mathbf{d}(f|g) = f$ and $(f|g) \leqslant g$.

As a consequence of OG1 and OG3, if $g \in G$ and f is an identity of G with $f \leqslant \mathbf{r}(g)$, there exists a unique element $(g|f)$, called the *corestriction* of g to f, such that $\mathbf{r}((g|f)) = f$ and $(g|f) \leqslant g$: we may define $(g|f)$ as $(f|g^{-1})^{-1}$.
Let G be an ordered groupoid and let $a, b \in G$. Suppose that the identities $\mathbf{r}(a)$ and $\mathbf{d}(b)$ have a greatest lower bound – which we write as $\mathbf{r}(a)\mathbf{d}(b)$ – in $E(G)$. Then we may define the *pseudoproduct* of a and b in G as

$$a * b = (a|\mathbf{r}(a)\mathbf{d}(b))(\mathbf{r}(a)\mathbf{d}(b)|b),$$

where the right-hand side is a composition that is defined in the groupoid G. As Lawson shows in Lemma 4.1.6 of [7], this is a partially defined associative operation on G, in the sense that if $a * (b * c)$ and $(a * b) * c$ both exist, then they must be equal. Of course, it is easy to concoct examples for which only one of $a * (b * c)$ and $(a * b) * c$ exists. We record a useful observation on the pseudoproduct:

Lemma 2.1. *If, for some bracketing of the terms, the pseudoproduct $a_1 * a_2 * \cdots * a_n$ exists in G, then it is equal in G to some composition of arrows $b_1b_2 \cdots b_n$ with $b_j \leqslant a_j$ for $j = 1, \dots, n$.*

An ordered groupoid $(G, \leqslant)$ is *inductive* if $E(G)$ is a meet semilattice. In an inductive groupoid G, the pseudoproduct is everywhere-defined and $(G, *)$ is then an inverse semigroup (see Proposition 4.1.7 of [7]). This correspondence is one half of the Ehresmann-Schein-Nambooripad theorem, establishing a isomorphism between the categories of inductive groupoids and inverse semigroups (see Theorem 4.1.8 of [7]). The inverse correspondence is easily described: an inverse semigroup S gives rise to a groupoid $\mathbb{G}(S)$ by restricting the multiplication $(s,t) \mapsto st$ of S to be defined only if $s^{-1}s = tt^{-1}$. The set of identities of $\mathbb{G}(S)$ is equal to the set of idempotents $E(S)$ of S and is ordered by the natural partial order on S. Since $E(S)$ is a semilattice, $\mathbb{G}(S)$ is inductive.

We recall that an inverse semigroup S is said to be *E-unitary* if whenever $e \in E(S)$ and $s \in S$ with $e \leqslant s$ in the natural partial order on S, then $s \in E(S)$. The minimal group congruence σ on an inverse semigroup S may be defined as follows:

$$a \sigma b \iff \text{there exists } c \in S \text{ such that } c \leqslant a \text{ and } c \leqslant b. \tag{2.1}$$

Now S is E-unitary if and only if the quotient map $\sigma_\sharp : S \to S/\sigma$ satisfies $1\sigma_\sharp^{-1} = E(S)$ (see [11, Lemma 1.1]). Under the Ehresmann-Schein-Nambooripad isomorphism, this translates to the fact that S is E-unitary if and only if $\sigma_\sharp : \mathbb{G}(S) \to S/\sigma$ is star-injective.

2.1 Generators and the Cayley graph

Let G be an ordered groupoid, and let X be a set with a given function $\gamma : X \to G$. We say that (X, γ) *generates* G if, for each arrow $g \in G$, there exists a sequence $(a_1, a_2, \cdots a_m)$ with $m \geqslant 1$ such that:

- for each j, $(1 \leqslant j \leqslant m)$ either $a_j \in X\gamma$ or $a_j^{-1} \in X\gamma$,
- if $m \geqslant 2$, the pseudoproducts

  $$a_1 * a_2, (a_1 * a_2) * a_3, \ldots, (\ldots((a_1 * a_2) * a_3) * \cdots * a_{m-1}) * a_m$$

  all exist,
- $g = (\ldots((a_1 * a_2) * a_3) * \cdots * a_{m-1}) * a_m$.

In short, (X, γ) generates G if every arrow in G is expressible as a left-normed pseudoproduct of elements of $X\gamma$ and their inverses. When the mapping γ is understood, we shall just say that X generates G, and we shall occasionally suppress mention of γ without comment.

Lemma 2.2. *Let $h = a_1 * a_2 * \cdots * a_m$ be a left-normed pseudoproduct in G.*

(a) If $g \in G$ and $\mathbf{r}(g) \leqslant \mathbf{d}(h)$ then $g * a_1 * \cdots * a_m$ is also a left-normed pseudoproduct, and is equal to $g * h$ in G.

(b) $a_1 * a_2 * \cdots * a_m * a_m^{-1} * a_{m-1}^{-1} * \cdots * a_1^{-1}$ *is a left-normed pseudoproduct, and is equal to* $\mathbf{d}(h)$ *in* G.

Proof. We prove part (a) by induction on m: the case $m = 1$ is trivial. If $m > 1$, let $h' = a_1 * \cdots * a_{m-1}$. This expression is a left-normed pseudoproduct, and $h = h' * a_m$. It follows that $\mathbf{d}(h) \leqslant \mathbf{d}(h')$ and so $\mathbf{r}(g) \leqslant \mathbf{d}(h')$: by induction, the pseudoproduct $g * h'$ exists and is equal to the left-normed pseudoproduct $g * a_1 * \cdots * a_{m-1}$. By Lemma 2.1, for each j with $1 \leqslant j \leqslant m-1$, there exist b_j, b'_j and b''_j with $b''_j \leqslant b'_j \leqslant b_j \leqslant a_j$, and $b'_m \leqslant b_m$, such that

$$\begin{aligned} h' &= b_1 b_2 \cdots b_{m-1}, \\ h &= h' * a_m = b'_1 b'_2 \cdots b'_m, \end{aligned}$$

and

$$g * h' = g b''_1 b''_2 \cdots b''_{m-1}$$

are compositions of arrows defined in G. Now $\mathbf{r}(g * h') = \mathbf{r}(b''_{m-1}) \leqslant \mathbf{r}(b'_{m-1}) = \mathbf{d}(b'_m) \leqslant \mathbf{d}(a_m)$. Hence the left-normed pseudoproduct $(g * h') * a_m = g * a_1 * \cdots * a_{m-1} * a_m$ exists, and is equal to the composition $g b''_1 b''_2 \cdots b''_{m-1} b''_m$ for some arrow b''_m with $b''_m \leqslant b'_m \leqslant a_m$. Now $b''_1 b''_2 \cdots b''_{m-1} b''_m \leqslant h$ in G and so is equal to the restriction $(\mathbf{r}(g)|h)$. But we have $g(\mathbf{r}(g)|h) = g * h$. This proves part (a).
For part (b), we again proceed by induction, and again the case $m = 1$ is trivial. If $m > 1$ we write $h = b'_1 b'_2 \cdots b'_m$. Now $\mathbf{r}(h) = \mathbf{r}(b'_m) \leqslant \mathbf{r}(a_m) = \mathbf{d}(a_m^{-1})$ and so $h * a_m^{-1}$ exists, and we have

$$\begin{aligned} h * a_m^{-1} &= b'_1 b'_2 \cdots b'_m (b'_m)^{-1} = b'_1 b'_2 \cdots b'_{m-1} \\ &\leqslant b'_1 b'_2 \cdots b'_{m-1} = a_1 * \cdots * a_{m-1} = h'. \end{aligned}$$

By induction, the left-normed pseudoproduct $h' * a_{m-1}^{-1} * \cdots * a_1^{-1}$ exists and is equal to $\mathbf{d}(h')$. Therefore the left-normed pseudoproduct $h * a_m^{-1} * a_{m-1}^{-1} * \cdots * a_1^{-1}$ exists, and is equal in G to the composition

$$b'_1 b'_2 \cdots b'_m (b'_m)^{-1} \cdots (b'_1)^{-1} = \mathbf{d}(h).$$

□

The *Cayley graph* $\Gamma(G, X, \gamma)$ of an ordered groupoid G, with respect to the generating set (X, γ), is now defined as follows. The vertex set is G, and the edge set consists of all pairs $(g, x) \in G \times X$ with $\mathbf{r}(g) \leqslant \mathbf{d}(x\gamma)$. The edge (g, x) has initial vertex g and final vertex $g * x\gamma = g(\mathbf{r}(g)|x\gamma)$. Since $\mathbf{d}(g) = \mathbf{d}(g(\mathbf{r}(g)|x\gamma))$, each component of $\Gamma = \Gamma(G, X, \gamma)$ contains a unique identity of G, and the component containing $e \in E(G)$ is denoted by Γ_e. If $(g_1, x_1)^{\varepsilon_1}, \ldots, (g_m, x_m)^{\varepsilon_m}, (\varepsilon_j = \pm 1)$ are the successive edges of a path p in $\Gamma(G, X, \gamma)$ starting at $a \in G$ and ending at $b \in G$ then we define the *label* of p to be the sequence $(x_1^{\varepsilon_1}, \ldots, x_m^{\varepsilon_m})$. The left-normed pseudoproduct $x_1^{\varepsilon_1} * \cdots * x_m^{\varepsilon_m}$ may or may not exist: however, $\mathbf{r}(a) * x_1^{\varepsilon_1} * \cdots * x_m^{\varepsilon_m}$ is always a left-normed pseudoproduct equal to $a^{-1}b$.

We note that the Cayley graph $\Gamma(G, X, \gamma)$ is an *ordered graph* in the sense of Steinberg [19], and if S is an inverse semigroup generated by a subset $X \subseteq S$ then $\mathbb{G}(S)$ is generated by X as an ordered groupoid, and the components of $\Gamma(G, X)$ are then the Schützenberger graphs of [20].

Example 2.3. Let FIS_1 be the free inverse semigroup on a single generator x. Using one possible coordinatisation of the elements of FIS_1 (see [15, Theorem IX.1.1]) we can describe its structure as an inductive groupoid. The idempotents of are in one-to-one correspondence with pairs of non-negative integers (p, q), not both zero, with the pair (p, q) corresponding to $x^{-p}x^{p+q}x^{-q}$. The ordering of the idempotents is described in terms of pairs by $(p, q) \leqslant (t, u)$ if and only if $p \geqslant t$ and $q \geqslant u$, and the greatest lower bound of (p, q) and (r, s) is $(\max(p, r), \max(q, s))$. The rest of the inductive groupoid structure is also easy to describe: there is a unique arrow from (p, q) to (r, s) if and only if $p + q = r + s$.

The component of the Cayley graph $\Gamma(FIS_1, \{x\})$ containing the idempotent (p, q) contains $p + q + 1$ vertices, arranged in a chain, starting at the vertex representing the element $x^q x^{-q-p}$ and finishing at the vertex representing $x^{-p}x^{p+q}$.

The free inverse monoid FIM_1 on a single generator is not generated as an inductive groupoid by a single element, since a generator is required to represent the identity $1 \in FIM_1$.

Example 2.4. The *bicyclic monoid* P_1 is the inverse monoid with presentation $P_1 = \mathrm{Inv}[p : pp^{-1} = 1]$. It is E–unitary, and has many other interesting properties (see [7, section 3.4], for example): as an incompressible ordered groupoid its structure is easy to describe. The set of identities $E(P_1) = \{p^{-n}p^n : n \geqslant 0\}$ may be identified with the natural numbers (including 0) with the reverse of the natural partial order, and there is a unique arrow between any two identities. The component Γ_n of its Cayley graph that contains $p^{-n}p^n$ has vertex set $\{p^{-n}p^m : m \geqslant 0\}$, and there is a directed edge from $p^{-n}p^m$ to $p^{-n}p^{m+1}$. We shall return to this example, and also consider the *polycyclic monoids* P_n for $n > 1$, in section 4.1.1 below.

We now show how left-normed pseudoproducts are related to paths in the Cayley graph.

Lemma 2.5. *Let G be an ordered groupoid generated by (X,γ). Suppose that $g \in G$ and that for some $x_1, x_2, \ldots, x_m \in X$, g is equal to the left-normed pseudoproduct $(x_1\gamma)^{\varepsilon_1} * (x_2\gamma)^{\varepsilon_2} * \cdots * (x_m\gamma)^{\varepsilon_m}$. Then there exists a path $\mathbf{x}$ in $\Gamma(G,X,\gamma)$ from $\mathbf{d}(g)$ to g with label $(x_1^{\varepsilon_1}, \ldots, x_m^{\varepsilon_m})$. Conversely, any path in $\Gamma(G,X,\gamma)$ from $\mathbf{d}(g)$ to g yields a left-normed pseudoproduct $\mathbf{d}(g) * y_1^{\varepsilon_1} * \cdots * y_m^{\varepsilon_m}$ (with $y_j \in X\gamma$) that is equal to g.*

Proof. We proceed by induction on m. If $m = 1$ then $g = (x\gamma)^{\varepsilon}$ for some $x \in X$ and $\varepsilon = \pm 1$, and $\mathbf{x}$ consists of a single edge in the Cayley graph $\Gamma(G,X,\gamma)$. Now if $m > 1$ let $g' = (x_1\gamma)^{\varepsilon_1} * \cdots * (x_{m-1}\gamma)^{\varepsilon_{m-1}}$. By our inductive assumption, there exists a path $\mathbf{x}'$ from $\mathbf{d}(g')$ to g' with label $(x_1^{\varepsilon_1}, \ldots, x_{m-1}^{\varepsilon_{m-1}})$. Now $g = g' * (x_m\gamma)^{\varepsilon_m}$, and it follows that $\mathbf{d}(g) \leqslant \mathbf{d}(g')$. We define $g_0 = \mathbf{d}(g)$, $g_1 = (\mathbf{d}(g)|x_1^{\varepsilon_1})$, and then iteratively define $g_k = (\mathbf{r}(g_{k-1})|x_k^{\varepsilon_k})$ for each k with $1 < k < m$. Since $\mathbf{r}(g_k) \leqslant \mathbf{d}(x_k^{\varepsilon_k})$ the edges $(g_k, x_k)^{\varepsilon_k}$ exist in $\Gamma(G,X,\gamma)$ and form a path from $\mathbf{d}(g)$ to $g_{m-1} = g_{m-2} * x_{m-1}^{\varepsilon_{m-1}}$ with label $(x_1^{\varepsilon_1}, \ldots, x_{m-1}^{\varepsilon_{m-1}}, x_m^{\varepsilon_m})$.
The converse is straightforward: each edge $(g,x)^{\varepsilon}$ of the path extends the left-normed pseudoproduct by one factor. □

A *pattern* in $\Gamma(G,X,\gamma)$ is a finite connected subgraph P that contains an identity vertex. A *marked* pattern is a pair (P,g) where P is a pattern, g is a vertex of P, and P contains some path $\mathbf{x}$ from $\mathbf{d}(g)$ to g derived from a representation of g as a left-normed pseudoproduct as in Lemma 2.5. Note that $(\mathbf{x}, g)$ is itself a marked pattern. If (P,g) is a marked pattern, then P contains the unique identity $\mathbf{d}(g)$, and for all vertices h of P we have $\mathbf{d}(h) = \mathbf{d}(g)$. If $e \in E(G)$ and $e \leqslant \mathbf{d}(g)$, then $(e|P)$ is the pattern obtained from P by taking the restriction of each vertex to e. An edge (h,x) of P then gives rise to an edge $((e|h),x)$ of $(e|P)$ from $(e|h)$ to $(e|h)(\mathbf{r}((e|h))|x)$. The *graph expansion* $\mathcal{M}(G,X,\gamma)$ is the set of all marked patterns in the Cayley graph $\Gamma(G,X,\gamma)$. It is an ordered groupoid, with the following structure as described in [3, Theorem 5.1].

Theorem 2.6. *The graph expansion $\mathcal{M}(G,X,\gamma)$ is an ordered groupoid, and the details of its structure are as follows:*

(a) the marked pattern (P,g) has domain $\mathbf{d}(P,g) = (P, \mathbf{d}(g))$ and range $\mathbf{r}(P,g) = (g^{-1}P, \mathbf{r}(g))$;

(b) the composition is $(P, g)(Q, h) = (P, gh)$, defined when $g^{-1}P = Q$ and $\mathbf{r}(g) = \mathbf{d}(h)$;

(c) the inverse of (P, g) is $(g^{-1}P, g^{-1})$;

(d) $(P, g) \leqslant (Q, h)$ if and only if $g \leqslant h$ and $P \supseteq (\mathbf{d}(g)|Q)$;

(e) if (Q, f) is an identity and $(Q, f) \leqslant \mathbf{d}(P, g)$ then the restriction $((Q, f)|(P, g))$ is defined to be $(Q, (f|g))$;

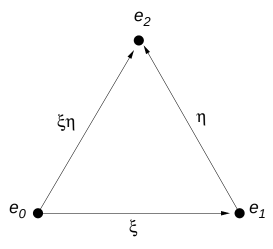
(f) if (P, g) and (Q, h) are marked patterns and the pseudoproduct $g * h$ exists in G, then $(P, g) * (Q, h)$ exists in $\mathcal{M}(G, X)$ and

$$(P, g) * (Q, h) = ((\mathbf{r}(g)\mathbf{d}(h)|g^{-1}P) \cup (\mathbf{r}(g)\mathbf{d}(h)|Q), g * h).$$

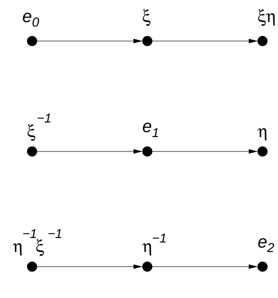
For point (b) we observe that the marked pattern P contains some path $\mathbf{x}$ from $\mathbf{d}(g)$ to g, and Q contains a path $\mathbf{y}$ from $\mathbf{d}(h) = \mathbf{r}(g)$ to h. Since $g^{-1}P = Q$, the left translate $g\mathbf{y}$ is a path in P from g to gh and so P also contains the concatenation $\mathbf{x}(g\mathbf{y})$ which is a path of the required form from $\mathbf{d}(g)$ to gh. Part (f) generalises what was said in [3, Theorem 5.1], where it is shown that if G is inductive, then so is the expansion $\mathcal{M}(G, X, \gamma)$. This links the ordered groupoid approach to the graph expansion with the expansion of inverse semigroups of [10].
We recall one further observation from [3, Corollary 5.3].

Corollary 2.7. *The projection $\theta : \mathcal{M}(G, X) \to G$ mapping $(P, g) \mapsto g$ is star-injective.*

Example 2.8. Recall that the simplicial groupoid Δ^n has $n + 1$ identities $e_i, 0 \leqslant i \leqslant n$ and a unique arrow from e_i to e_j for each pair (i, j). Then Δ^n is the free groupoid generated by a chain of n edges. We consider Δ^2 with the trivial ordering, generated by the arrows ξ and η, shown with its Cayley graph below.



Δ^2



$\Gamma(\Delta^2, \{\xi, \eta\})$

Let Γ_i be the component of $\Gamma(\Delta^2, \{\xi, \eta\})$ containing e_i. The graph expansion $\mathcal{M}(\Delta^2, \{\xi, \eta\})$ is the disjoint union $\mathbb{I}_\xi \sqcup \mathbb{I}_\eta \sqcup \Delta$, where $\mathbb{I}_\xi \cong \mathbb{I}$ is generated by the marked pattern p_ξ consisting of the directed edge (e_0, ξ) from e_0 to ξ marked at ξ, $\mathbb{I}_\eta$ is similarly generated by the marked pattern p_η consisting of the directed edge (e_1, η) from e_1 to η marked at η, and $\Delta \cong \Delta^2$ is generated by the marked patterns (Γ_0, ξ) and (Γ_1, η).

In the above example, the graph expansion has only trivial subgroups. This is an instance of our first new structural result for the graph expansion, which describes all the possible subgroups. The proof of the following theorem is simply a generalisation of the corresponding argument for [11, Corollary 3.3].

Theorem 2.9. *Each subgroup of $\mathcal{M}(G, X)$ is isomorphic to a finite subgroup of G, and every finite subgroup of G arises in this way.*

Proof. By Theorem 2.6 (a), the local subgroups of $\mathcal{M}(G, X)$ are finite, and by Corollary 2.7, the projection $\theta : \mathcal{M}(G, X) \to G$ embeds any subgroup of $\mathcal{M}(G, X)$ into G.
Conversely, let $U = \{u_0, u_1, \dots, u_m\}$ be a finite subgroup of G, with identity element u_0. For each u_j , $(1 \leqslant j \leqslant m)$ choose a pseudoproduct representation of u_j and let $\mathbf{x}_j$ be the corresponding path in $\Gamma(G, X)$ as given in Lemma 2.5. If g is a vertex of a path $\mathbf{x}_j$ then $\mathbf{d}(g) = u_0$, and so for each $u_i \in U$ the translate $u_i\mathbf{x}_j$ exists, and is a path in $\Gamma(G, X)$ from u_i to u_iu_j, with $u_0\mathbf{x}_j = \mathbf{x}_j$ for all j. Let P be the union of all the U-translates of all the paths $\mathbf{x}_j$:

$$P = \bigcup_{\substack{1 \leqslant j \leqslant m \\ 0 \leqslant i \leqslant m}} u_i\mathbf{x}_j \,.$$

Now P is a pattern in Γ, and for any $u \in U$, (P, u) is a marked pattern and $uP = P$. It then follows from parts (a),(b) and (c) of Theorem 2.6 that

$$V = \{(P, u_0), (P, u_1), \dots, (P, u_m)\}$$

is a subgroup of $\mathcal{M}(G, X)$, and it is clear that $\theta : \mathcal{M}(G, X) \to G$ restricts to an isomorphism $V \to U$. □

2.2 Levelling ordered groupoids

A functor ξ from an ordered groupoid G to a groupoid H is said to be *levelling* if, whenever $g \leqslant h$ in G, then $g\xi = h\xi$. In the theory of ordered groupoids, levelling functors play the role of maps from inverse semigroups to groups, and we now recall from [4] the analogue of the maximum group image. Since this notion is crucial for the rest of the paper, we give the construction in some detail.

We shall first need Higgins' construction [6] of *universal* groupoids. Let G be a groupoid, and $\sigma : E(G) \to V$ some function. To construct the universal groupoid $U_\sigma(G)$, we first define a graph G^σ as follows. Its vertex set is V and its edges are the non-identity arrows of G with incidence maps $\mathbf{d}^\sigma : a \mapsto (\mathbf{d}(a))\sigma$ and $\mathbf{r}^\sigma : a \mapsto (\mathbf{r}(a))\sigma$. Let $p = a_1 a_2 \cdots a_n$ be a path of length $n > 0$ in G^σ. An *elementary reduction* of p is either the deletion of some a_j where $a_j \in E(G)$, or the replacement of $a_j a_{j+1}$ by a when $a_j a_{j+1} = a$ in G. Modulo the equivalence relation $\simeq$ generated by elementary reduction, the path category $\mathcal{P}(G^\sigma)$ becomes a groupoid $U_\sigma(G) = \mathcal{P}(G^\sigma)/\simeq$. Now let $\updownarrow$ be the equivalence relation on G generated by the partial order $\leqslant$, and consider the restriction of $\updownarrow$ to the set $E(G)$ of identities. Let $\lambda : E(G) \to E(G)/\updownarrow$ be the quotient map, and construct the universal groupoid $U_\lambda(G)$. Now suppose that $a, b \in G$ have pseudoproduct $a * b$. Then $\mathbf{d}(a*b) \updownarrow \mathbf{d}(a)$ and $\mathbf{r}(a*b) \updownarrow \mathbf{r}(b)$, and therefore $b^{-1}a^{-1}(a*b)$ is in the local group at $(\mathbf{r}(b))\lambda$ in $U_\lambda(G)$. Let N be the normal subgroupoid of $U_\lambda(G)$ generated by all such elements, and set $G_\updownarrow = U_\lambda(G)/N$. The groupoid $G_\updownarrow$ is called the *level* groupoid of G. We also use λ to denote the quotient map $G \to G_\updownarrow$. The key universal property of the level groupoid is taken from [4].

Lemma 2.10 ([4])**.** *Let G be an ordered groupoid. Then $\lambda : G \to G_\updownarrow$ is a levelling functor, and for any levelling functor $\xi : G \to H$ to an ordered groupoid H, there exists a unique functor $\xi_\updownarrow : G_\updownarrow \to H$ such that $\xi = \lambda\xi_\updownarrow$.*

The definition of the minimum group congruence (2.1) on an inverse semigroup obviously generalises, without change, to give a relation σ on any ordered groupoid. (This relation is called β in [4]). Moreover σ is obviously reflexive and symmetric, but is transitive if and only if each principal order ideal in G is a directed set. We shall call ordered groupoids with this property *σ–transitive.* It is easy to see that inductive groupoids are σ–transitive: more generally, any *principally inductive* ordered groupoid (see [8]) is σ–transitive.

Lemma 2.11. *In a σ–transitive ordered groupoid G, the following are equivalent for $a, b \in G$:*

$$a\lambda = b\lambda\,, \quad a \updownarrow b\,, \quad a\sigma b\,.$$

Proof. Let G be a σ–transitive ordered groupoid and $a, b \in G$. If $a \leqslant b$ then certainly $a\sigma b$: hence σ is an equivalence relation containing $\leqslant$ and so contains the equivalence relation generated by $\leqslant$: that is $\updownarrow \subseteq \sigma$. But it is obvious that $\sigma \subseteq \updownarrow$. Therefore, we have $\updownarrow = \sigma$.

There is an obvious well-defined groupoid structure on the quotient set $G/\sigma = G/\updownarrow$; this is easy to check directly, and also follows from [8, Lemma 4]. Then G/σ and $G_{\updownarrow}$ are groupoids with the same set of identities. If $a\sigma = b\sigma$ then $a \updownarrow b$ and so $a\lambda = b\lambda$. But clearly $G \to G/\sigma$ is a levelling functor, and so by Lemma 2.10 there is a well-defined functor $G_{\updownarrow} \to G/\sigma$ that maps $a\lambda$ to $a\sigma$. It follows that if $a\lambda = b\lambda$ then $a\sigma = b\sigma$. □

2.2.1 Examples

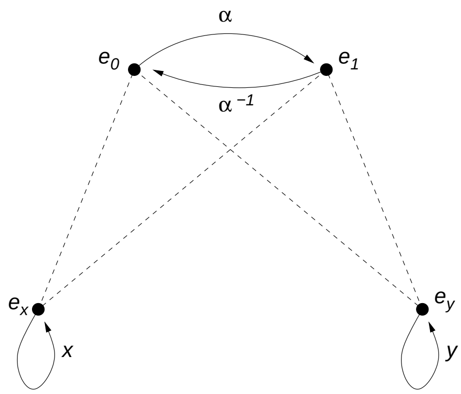
1. Our first example is a σ–transitive ordered groupoid that is principally inductive but not inductive. It is a trivial ordered groupoid, and so we just have a poset

$$X = \{a, b, f_0, f_1, f_2, \dots\}$$

with the ordering $f_0 \leqslant f_1 \leqslant f_2 \leqslant \cdots \leqslant a$ and $f_0 \leqslant f_1 \leqslant f_2 \leqslant \cdots \leqslant b$. So every f_j is a lower bound for a and b but there is no greatest lower bound. But for any $x, y \in X$ we have $x\lambda = y\lambda$, $x \updownarrow y$ and $x\sigma y$.

2. Let G be the ordered groupoid that consists of a copy of $\mathbb{I} = \{e_0, \alpha, \alpha^{-1}, e_1\}$ and two infinite cyclic groups, generated by x and y, with identities e_x, e_y. The ordering is given by

$$e_0 \geqslant e_x, e_0 \geqslant e_y, e_1 \geqslant e_x, e_1 \geqslant e_y, \alpha \geqslant x, \alpha \geqslant y, \alpha^{-1} \geqslant x^{-1}, \alpha^{-1} \geqslant y^{-1}.$$



Note that $x, y \leqslant \alpha$ but have no lower bound. So this example is not σ–transitive, and we show that the Lemma fails here. We observe that

$$x\lambda = (e_x * \alpha)\lambda = (e_x\lambda)(\alpha\lambda) = (e_y\lambda)(\alpha\lambda) = (e_y * \alpha)\lambda = y\lambda.$$

Therefore $x^2\lambda = y^2\lambda$, but in the partial order on G no elements are comparable with either x^2 or y^2 and so x^2 is not $\updownarrow$–related to y^2.

3 Incompressibility and expansion

The graph expansion $\mathcal{M}(G, X, \gamma)$ of a group G with generating set X is an X–generated E–unitary inverse semigroup, with maximum group image G. In this section we shall prove analogues of these facts for the expansion of an ordered groupoid.

Theorem 3.1. *Let G be an ordered groupoid generated by X. Then the graph expansion $\mathcal{M}(G, X)$ is an ordered groupoid generated by X, whose level groupoid $\mathcal{M}(G, X)_{\updownarrow}$ is isomorphic to $G_{\updownarrow}$. Moreover, $\mathcal{M}(G, X)$ is incompressible if and only if G is incompressible.*

Proof. We define a function $\delta : X \to \mathcal{M}(G, X)$ as follows: $x\delta$ is the marked pattern consisting of the single edge $(\mathbf{d}(x\gamma), x\gamma)$, marked at $x\gamma$. Now let (P, g) be a marked pattern in $\mathcal{M}(G, X)$ containing a path $\mathbf{x}$ corresponding to some representation of g as a left-normed pseudoproduct $g = (x_1\gamma)^{\varepsilon_1} * (x_2\gamma)^{\varepsilon_2} * \cdots * (x_m\gamma)^{\varepsilon_m}$. Then $(\mathbf{x}, g)$ is a marked pattern, and in $\mathcal{M}(G, X)$ we have $(\mathbf{x}, g) = (x_1\delta)^{\varepsilon_1} * (x_2\delta)^{\varepsilon_2} * \cdots * (x_m\delta)^{\varepsilon_m}$.

We now consider adding edges to marked patterns. Let (P, g) be a marked pattern, let h be a vertex of P, and suppose that for some $x \in X$ we have $\mathbf{r}(h) \leqslant \mathbf{d}(x)$. There exists a path in P from $\mathbf{d}(g)$ to h, and so, by Lemma 2.5 a left-normed pseudoproduct $\mathbf{d}(g) * (z_1\gamma)^{\varepsilon_1} * \cdots * (z_m\gamma)^{\varepsilon_m}$ equal to h. It follows that the left-normed pseudoproduct

$$(P, \mathbf{d}(g)) * (z_1\delta)^{\varepsilon_1} * \cdots * (z_m\delta)^{\varepsilon_m}$$

is equal to (P, h), and then $(P, h) * (x\delta) = (P', h * x\delta)$, where P' is obtained by adjoining the edge labelled (h, x) starting at h to P. Then by Lemma 2.2, $(P', \mathbf{d}(g))$ is a left-normed pseudoproduct of elements of $(X\delta) \cup (X\delta)^{-1}$, and hence so is $(P', g) = (P', \mathbf{d}(g)) * (\mathbf{x}, g)$. Similarly, we can add the inverse of an edge to a marked pattern P using left-normed pseudoproducts. Since the marked pattern (P, g) may be obtained by adding finitely many edges to $(\mathbf{x}, g)$, we see that $\mathcal{M}(G, X)$ is indeed generated by (X, δ).

Suppose that $g \in G$ has two pseudoproduct representations as

$$g = a_1 * a_2 * \cdots * a_m = b_1 * b_2 * \cdots * b_n$$

with $a_i, b_j \in (X\gamma) \cup (X\gamma)^{-1}$, yielding marked patterns $(\mathbf{a}, g)$ and $(\mathbf{b}, g)$ in the Cayley graph $\mathcal{M}(G, X)$. Then we also have a marked pattern $(\mathbf{a} \cup \mathbf{b}, g)$ with $(\mathbf{a}, g) \geqslant (\mathbf{a} \cup \mathbf{b}, g) \leqslant (\mathbf{b}, g)$, and so $(\mathbf{a}, g) \updownarrow (\mathbf{b}, g)$. Moreover, for any marked pattern (P, g), the subgraph P must contain some path $\mathbf{c}$ from $\mathbf{d}(g)$ to g so that $(P, g) \leqslant (\mathbf{c}, g)$. It follows that if $(P, g), (Q, g) \in \mathcal{M}(G, X)$

are two patterns marked at the same element $g \in G$ then $(P,g) \updownarrow (Q,g)$ and so $(P,g)\lambda = (Q,g)\lambda$. Therefore there exists a well-defined map $\alpha : G \to \mathcal{M}(G,X)_{\updownarrow}$ carrying $g \in G$ to $(P,g)\lambda$, where (P,g) is any pattern with marked vertex g.

Now suppose that $g,h \in G$ with $g \leqslant h$. Consider a marked pattern (Q,h). Then $((\mathbf{d}(g)|Q),g) \leqslant (Q,h)$ in $\mathcal{M}(G,X)$ and so $((\mathbf{d}(g)|Q),g)\lambda = (Q,h)\lambda$. It follows that $g\alpha = h\alpha$, so that α is levelling, and so induces a functor $\beta : G_{\updownarrow} \to \mathcal{M}(G,X)_{\updownarrow}$ carrying $g\lambda \mapsto (P,g)\lambda$.

Now the composition $\mathcal{M}(G,X) \xrightarrow{\theta} G \xrightarrow{\lambda} G_{\updownarrow}$ is levelling and so induces a functor $\gamma : \mathcal{M}(G,X)_{\updownarrow} \to G_{\updownarrow}$ that carries $(P,g)\lambda \mapsto g\lambda$. Clearly β and γ are inverse functors, and $G_{\updownarrow}$ and $\mathcal{M}(G,X)_{\updownarrow}$ are isomorphic.

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(G,X) & \xrightarrow{\lambda} & \mathcal{M}(G,X)_{\updownarrow} \\ \theta\big\downarrow & & \big\downarrow\gamma \\ G & \xrightarrow[\lambda]{} & G_{\updownarrow} \end{array}$$

with $(P,g)\lambda\gamma = (P,g)\theta\lambda = g\lambda$. Now γ is an isomorphism, and θ is star-injective by Corollary 2.7, and it then follows easily that $\lambda : G \to G_{\updownarrow}$ is star-injective if and only if $\lambda : \mathcal{M}(G,X) \to \mathcal{M}(G,X)_{\updownarrow}$ is star-injective. □

Since all groups are trivially E–unitary, no further hypothesis on a group G is required in [11] to ensure that the Margolis-Meakin graph expansion $\mathcal{M}(G,X,\gamma)$ is an E–unitary inverse semigroup.

If we take the definition of an E–unitary inverse semigroup S in terms of the natural partial order on S, and interpret it as a statement about an ordered groupoid, we obtain the notion of a filtered ordered groupoid, introduced in [4]. An ordered groupoid G is *filtered* if, whenever $g \in G$ and $g \geqslant e$ for some identity $e \in G$, then g is also an identity. It is easy to see that incompressible ordered groupoids are filtered, and it is shown in [4] that the two notions coincide for inductive groupoids. The graph expansion preserves the filtered property.

Proposition 3.2. *The expansion $\mathcal{M}(G,X)$ is filtered if and only if G is filtered.*

Proof. Suppose that G is filtered and that $(P,e) \leqslant (Q,g)$ where $e \in E(G)$. Then $e \leqslant g$ and so $g \in E(G)$, and it follows that (Q,g) is an identity in $\mathcal{M}(G,X)$.

Conversely, if $\mathcal{M}(G, X)$ is filtered and $e \leqslant g$ in G, let $(\mathbf{x}, g)$ be a marked pattern corresponding to some left-normed pseudoproduct representation of g, as in Lemma 2.5. Since $e \leqslant \mathbf{d}(g)$, we have $((e|\mathbf{x}), e) \leqslant (\mathbf{x}, g)$. Since $\mathcal{M}(G, X)$ is filtered, $(\mathbf{x}, g)$ must be an identity, and so $g \in E(G)$. □

We can get another viewpoint on the case of inductive groupoids by reading between the lines in [10]. Here the graph expansion is set up for inverse semigroups, and so for inductive groupoids. In the language of inverse semigroups, it is proved that if an inverse semigroup S is given by a presentation $S = \mathrm{Inv}[X : R]$, then its graph expansion $\mathcal{M}(S, X)$ is given by the presentation $\mathcal{M}(S, X) = \mathrm{Inv}[X : \mathcal{E}]$ where

$$\mathcal{E} = \{u^2 = u : u \text{ represents an idempotent in } S\}.$$

To get the maximum group image, we just replace these inverse semigroup presentations by group presentations, so that $S_{\updownarrow} = \langle X : R \rangle$ and

$$\mathcal{M}(S, X)_{\updownarrow} = \langle X : u = 1 \text{ if } u \text{ represents an idempotent in } S \rangle.$$

Now if u represents an idempotent in S, it must map to $1 \in S_{\updownarrow}$, and it follows that the identity map on X induces a group homomorphism $\gamma : \mathcal{M}(S, X)_{\updownarrow} \to S_{\updownarrow}$. On the other hand, if $p = q$ is a relation in R, so that p and q represent the same element of S, then pq^{-1} represents an idempotent in S (the same idempotent as represented by pp^{-1}), and so $pq^{-1} = 1$ is a relation in the presentation of $\mathcal{M}(S, X)_{\updownarrow}$. So we also have an inverse homomorphism $\beta : S_{\updownarrow} \to \mathcal{M}(S, X)_{\updownarrow}$.

4 A P–theorem for ordered groupoids

McAlister's P–theorem describes the structure of an E–unitary inverse semigroup in terms of an action of its maximum group image on a poset. The impact of the P–theorem on inverse semigroup theory, and some of the approaches to its proof, are reviewed in [9]. Steinberg [18] proves the P–theorem by constructing, from any E–unitary inverse semigroup S, a poset of subgraphs of the Cayley graph of the maximum group image G of S on which G acts. Steinberg's proof is elegant and concise, and we now show that it may be adapted to prove a more general P–theorem for incompressible ordered groupoids as given in [4]. The formulation of this more general theorem uses the action groupoid of a groupoid acting on a poset, a special case of Steinberg's semidirect product of ordered groupoids from [17].

Let L be a groupoid and J a poset. A (left) action of L on J consists of the following:

- a function $\mu : J \to E(L)$,
- a function $\{(g, x) : \mathbf{r}(g) = x\mu\} \to J$, denoted $(g, x) \mapsto g \cdot x$,

such that

(A1) $(g \cdot x)\mu = \mathbf{d}(g)$,

(A2) if $g, h \in L$ and the composition gh exists in L then $(g \cdot (h \cdot x)) = (gh) \cdot x$,

(A3) $x\mu \cdot x = x$.

The semidirect product $J \rtimes L$ is then an ordered groupoid, defined as follows. The set of arrows is the pullback $\{(x, g) : x\mu = \mathbf{d}(g)\}$, and an arow (x, g) has domain $\mathbf{d}(x, g) = (x, \mathbf{d}(g))$ and range $\mathbf{r}(x, g) = (g^{-1} \cdot x, \mathbf{r}(g))$. The composition $(x, g)(y, h)$ is defined if and only if $\mathbf{r}(g) = \mathbf{d}(h)$ and $x = g \cdot y$, and is then equal to (x, gh). The ordering on arrows is given by $(x, g) \leqslant (y, h)$ if and only if $x \leqslant y$ and $g = h$, and restriction is defined by $((x, e)|(y, g)) = (x, g)$ when $x \leqslant y$ and $e = \mathbf{d}(g)$. Moreover, $J \rtimes L$ is incompressible: the projection $(x, g) \mapsto g$ is a star-injective levelling functor, and Lemma 2.10 then implies that $\lambda : J \rtimes L \to (J \rtimes L)_{\updownarrow}$ is star-injective.
Now suppose that L acts on J and that $K \subseteq J$ is an order ideal in J. Define

$$P(J, K, L) = \{(y, g) \in J \rtimes L : y \in K, g \in L, g^{-1} \cdot y \in K\}.$$

It is easy to check that $P(J, K, L)$ is an ordered subgroupoid of the semidirect product $J \rtimes L$, and so is incompressible. We can now state our version of the P–theorem for ordered groupoids:

Theorem 4.1. *An ordered groupoid G is incompressible if and only if it is isomorphic to $P(J, K, L)$ for some poset J, order ideal K, and groupoid L acting on J.*

In fact, the proof of the theorem will show that we may take $L = G_{\updownarrow}$.

Proof. We shall construct J, K such that $L = G_{\updownarrow}$ acts on J, and show that we have an isomorphism $G \to P(J, K, G_{\updownarrow})$. Suppose that (X, γ) is a generating set for G, so that every arrow in G is a (left-normed) pseudoproduct of arrows in $X\gamma$ (and their inverses). Now for any pseudoproduct $a_1 * \cdots * a_m$ in G, the composition $(a_1\lambda) \cdots (a_m\lambda)$ is defined in $G_{\updownarrow}$, and so the level groupoid

$G_{\updownarrow}$ is generated by $(X, \gamma\lambda)$. Hereon, we shall suppress mention of γ and $\gamma\lambda$ when dealing with the generating set X.

Suppose that G is incompressible. Since $\lambda : G \to G_{\updownarrow}$ is star-injective, it induces an embedding of each component of the Cayley graph of G into the Cayley graph of $G_{\updownarrow}$. For each identity $e \in G$, let Γ_e be the embedded copy in $\Gamma(G_{\updownarrow}, X)$ of the component of $\Gamma(G, X)$ containing e. We let K be the set of all such embedded components: $K = \{\Gamma_e : e \in E(G)\}$.

Now $G_{\updownarrow}$ acts on K: the fibring function $\mu : K \to E(G_{\updownarrow})$ maps $\Gamma_e \mapsto e\lambda$, and any $g \in G_{\updownarrow}$ with $\mathbf{r}(g) = e\lambda$ acts on Γ_e by left multiplication on the vertex sets, so that $g \cdot \Gamma_e$ is some connected subgraph of $\Gamma(G_{\updownarrow}, X)$. We let J be the set of all such $G_{\updownarrow}$–translates of elements of K, partially ordered by reverse-inclusion, and we claim that K is an order ideal of J: that is, we claim that if $g \cdot \Gamma_e$ contains Γ_f for some $f \in E(G)$, then $g \cdot \Gamma_e = \Gamma_k$ for some $k \in E(G)$. Suppose that $\mathbf{d}(g) = d$. Then by assumption we have

$$\Gamma_f \subseteq g \cdot \Gamma_e \subseteq \Gamma_d(G_{\updownarrow}, X),$$

where $\Gamma_d(G_{\updownarrow}, X)$ is the connected component of $\Gamma(G_{\updownarrow}, X)$ containing d. Since $\Gamma_d(G_{\updownarrow}, X)$ contains a unique identity of $G_{\updownarrow}$, it follows that $f\lambda = d$ and $d \in g \cdot \Gamma_e$. Now there exists a path in $g \cdot \Gamma_e$ starting at g and ending at d, and we suppose that $(g_i, x_i)^{\varepsilon_i}, (1 \leqslant i \leqslant r)$ are the successive edges of this path. Then $x_1^{\varepsilon_1}, \ldots, x_r^{\varepsilon_r}$ also occur as edge labels on a path in $\Gamma(G, X)$, and so the (left-normed) pseudoproduct $u = e * x_1^{\varepsilon_1} * \cdots * x_r^{\varepsilon_r}$ exists in G and satisfies $\mathbf{d}(u) = e$ and $u\lambda = g^{-1}$. We claim that, as subgraphs of $\Gamma(G_{\updownarrow}, X)$, we have $g \cdot \Gamma_e = \Gamma_{u^{-1}u}$.

It is enough to check that their vertex sets are the same. A typical vertex of $g \cdot \Gamma_e$ is $g(h\lambda)$ where $h \in G$ and $\mathbf{d}(h) = e$. But then $g(h\lambda) = (u^{-1}h)\lambda$ and $\mathbf{d}(u^{-1}h) = \mathbf{r}(u) = u^{-1}u$. Hence $g(h\lambda)$ is also a vertex of $\Gamma_{u^{-1}u}$. Conversely, a typical vertex of $\Gamma_{u^{-1}u}$ has the form $k\lambda$ where $\mathbf{d}(k) = u^{-1}u$. Therefore $\mathbf{d}(k\lambda) = gg^{-1} = d$, and so $g^{-1}(k\lambda)$ is defined in $G_{\updownarrow}$. So we may write $k\lambda = g(g^{-1}(k\lambda))$ with $g^{-1}(k\lambda)$ a vertex of Γ_e.

We have now assembled the necessary ingredients to construct the ordered groupoid $P(J, K, G_{\updownarrow})$. Let $\alpha : G \to P(J, K, G_{\updownarrow})$ be defined by $\alpha : g \mapsto (\Gamma_{\mathbf{d}(g)}, g\lambda)$. We claim that α is an isomorphism of ordered groupoids. It is clear that α is a functor, since if g, h are composable arrows in G then $\Gamma_{\mathbf{d}(g)} = g \cdot \Gamma_{\mathbf{d}(h)}$, so that $(\Gamma_{\mathbf{d}(g)}, g\lambda)(\Gamma_{\mathbf{d}(h)}, h\lambda)$ is defined , and equal to $(\Gamma_{\mathbf{d}(g)}, (gh)\lambda)$. Moreover, if $g \leqslant h$ in G then $(\mathbf{d}(g)|\Gamma_{\mathbf{d}(h)}) \subseteq \Gamma_{\mathbf{d}(h)}$, so that $(\Gamma_{\mathbf{d}(g)}, g\lambda) \leqslant (\Gamma_{\mathbf{d}(h)}, h\lambda)$ and so α is an ordered functor. Now if $(\Gamma_{\mathbf{d}(g)}, a) \in P$ then $a^{-1} \cdot \Gamma_{\mathbf{d}(g)} = \Gamma_{\mathbf{d}(h)}$ for some h: in particular, $a^{-1} \cdot \Gamma_{\mathbf{d}(g)}$ contains an identity, so that $a \in \Gamma_{\mathbf{d}(g)}$ and therefore $\mathbf{d}(a) = \mathbf{d}(g)$. It follows that α is surjective.

Now consider the subgraph $\Gamma_{\mathbf{d}(g)}$ together with its distinguished vertex $g\lambda$. Now $\Gamma_{\mathbf{d}(g)}$ contains only one vertex labelled by an identity of $G_{\updownarrow}$, namely $d = \mathbf{d}(g)\lambda$. Now g is equal to some left-normed pseudoproduct $x_1^{\varepsilon_1} * \cdots * x_m^{\varepsilon_m}$, and by Lemma 2.5 there corresponds a path in $\Gamma(G, X)$ from $\mathbf{d}(g)$ to g, and so in $\Gamma_{\mathbf{d}(g)}$ a path from d to $g\lambda$ whose label is the sequence $(x_1^{\varepsilon_1}, \ldots, x_m^{\varepsilon_m})$. Amongst all the paths in $\Gamma_{\mathbf{d}(g)}$ from d to $g\lambda$, consider those whose labels possess a left-normed pseudoproduct in G. If $h = b_1 * \cdots * b_n$ is the left-normed pseudoproduct of the label of such a path, there will exist a path in $\Gamma(G, X)$ from $\mathbf{d}(g)$ to g with the same label. Therefore $g = \mathbf{d}(g) * b_1 * \cdots * b_n$, and so $g \leqslant b_1 * \cdots * b_n = h$, and g is the minimum element of G obtained as the evaluation of a left-normed pseudoproduct of the label of a path from d to $g\lambda$ in $\Gamma_{\mathbf{d}(g)}$. Hence the pair $(\Gamma_{\mathbf{d}(g)}, g\lambda)$ determines $g \in G$, and therefore α is injective. □

4.1 Examples

4.1.1 The polycyclic monoids

Recall the discussion of the bicyclic monoid P_1 in Example 2.4, and the description of the components Γ_n of its Cayley graph as one-way infinite directed chains. The maximum group image $(P_1)_{\updownarrow}$ is the group of integers $\mathbb{Z}$, and the component Γ_n embeds into the Cayley graph $\Gamma(\mathbb{Z}, \{1\})$ as the subtree spanned by the vertex set $\{k : k \geqslant -n\}$.

Let Λ_r be the subtree in $\Gamma(\mathbb{Z}, \{1\})$ spanned by the vertex set $\{k : k \geqslant r\}$. In the notation of the proof of Theorem 4.1, the set J consists of all such subtrees Λ_r , and K consists of all Λ_r with $r \leqslant 0$. Each $a \in \mathbb{Z}$ acts on J by translation: $a \cdot \Lambda_r = \Lambda_{a+r}$. Now if $\Lambda_{-n} \subseteq \Lambda_{a+r}$ then $a + r \leqslant -n \leqslant 0$ and $\Lambda_{a+r} \in Y$.

For $n > 1$, the bicyclic monoid generalises to the *polycyclic monoid* P_n (see [7, section 9.3]), which is the inverse monoid with zero presented by

$$P_n = \mathrm{Inv}[p_1, \ldots, p_n : p_i p_i^{-1} = 1\,, p_i p_j^{-1} = 0\,(i \neq j)].$$

Let Σ denote the generating set $\{p_1, \ldots, p_n\}$. A natural representation of P_n is then obtained by identifying its elements as the set of pairs of strings $\Sigma^* \times \Sigma^*$, together with 0 (again see [7, section 9.3]), with the multiplication of pairs defined by

$$(a, b)(c, d) = \begin{cases} (a, sd) & \text{if } b = sc \\ (sa, d) & \text{if } c = sb \\ 0 & \text{otherwise.} \end{cases}$$

The ordered groupoid structure is easy to describe. The set of identities may be identified with $\Sigma^* \cup \{0\}$, with strings being given the reverse suffix ordering: $u \leqslant v$ if and only if $u = sv$. There is a unique arrow between any two non-zero identities, and $(a, b) \leqslant (c, d)$ if and only if $a = sc$ and $b = sd$ for some $s \in \Sigma^*$. Let G_n be the ordered groupoid $P_n \setminus \{0\}$. Then G_n is incompressible, and its level groupoid $(G_n)_\updownarrow$ is the free group on Σ, with $\lambda : G_n \to F(\Sigma)$ by $(a, b) \mapsto a^{-1}b$. Now given $a \in \Sigma^*$, consider the subtree Λ_a of the Cayley graph $\Gamma(F(\Sigma), \Sigma)$ spanned by $a^{-1}\Sigma^*$. In the notation of the proof of Theorem 4.1, the set K consists of all such subtrees Λ_a , and J consists of all the subtrees with vertex sets of the form $w\Sigma^*$.

4.1.2 Developable complexes of groups

Let Q be a connected poset, regarded as a category with set of objects Q and with a unique arrow $p \to q$ whenever $p \geqslant q$ in Q. A *poset of groups* over Q is a functor $\mathcal{G}$ from Q to the category of groups. Equivalently, for each $q \in Q$ we have a group G_q, the *local group* at q, and for each pair (p, q) of elements of Q with $p \geqslant q$ a group homomorphism – called a *linking map* – $\varphi_{pq} : G_p \to G_q$ such that

- for each $p \in Q$ the homomorphism φ_{pp} is the identity map on G_p,
- if $p \geqslant q \geqslant r$ in Q then $\varphi_{pr} = \varphi_{pq}\varphi_{pr}$.

A poset of groups is then an example of an ordered groupoid $G = G(\mathcal{G}, Q)$. As a set we have $G = \bigsqcup_{q\in Q} G_q$, and the ordering is given by $g \leqslant g\varphi_{pq}$ for all $g \in G_p$. If Q is connected, the level groupoid $G(\mathcal{G}, Q)_\updownarrow$ is the group colimit $\widehat{G} = \operatorname{colim}(\mathcal{G}, Q)$ of the diagram of groups and homomorphisms (G_p, φ_{pq}) defined by $\mathcal{G}$.

Now G is incompressible if and only if each local group is canonically embedded into the colimit $\widehat{G}$. A necessary condition for this to occur is that all the linking maps should be injective. If this is the case, then $(\mathcal{G}, Q)$ is a *simple complex of groups* in the sense of [2], and G is incompressible if and only if $(\mathcal{G}, Q)$ is *developable.* (Of course, the injectivity of the linking maps is not sufficient to ensure that the local groups are embedded into $G(\mathcal{G}, Q)_\updownarrow$.)

Given a developable simple complex of groups $(\mathcal{G}, Q)$, fix a generating set $X_q \subset G_q$ for each G_q and let $X = \bigsqcup_{q\in Q} X_q$. Then X generates G as an ordered groupoid, and X is also a natural generating set for the colimit $\widehat{G}$ since it is a quotient of the free product $*_{q\in Q}G_q$. The Cayley graph $\Gamma(G, X)$ is the disjoint union $\bigsqcup_{q\in Q} \Gamma(G_q, \overline{X}_q)$, where $\overline{X}_q = \bigsqcup_{p\geqslant q} X_p$: for each $g \in G_q$ and $x \in \overline{X}_q$, there is an x–labelled edge from g to $g(x\varphi_{pq})$ in $\Gamma(G_q, \overline{X}_q)$.

The construction given in the proof of Theorem 4.1 now gives a $\widehat{G}$–poset J and a sub-poset and order ideal K that is order isomorphic to Q, from which the developable simple complex of groups $(\mathcal{G}, Q)$ may be recovered. This may be viewed as part of the *Basic Construction* of [2, chapter II.12].

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