

# Nonparametric GARCH Models

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## Abstract

In this paper we describe a nonparametric GARCH model of first order and propose a simple iterative algorithm for its estimation from data. We provide a theoretical justification for this algorithm and give examples of its application to stationary time series data showing stochastic volatility. We observe that our nonparametric procedure often gives better estimates of the unobserved latent volatility process than parametric GARCH(1,1) modelling, particularly when asymmetries are present in the data. We show how the basic iterative idea may be extended to more complex time series models combining ARMA or GARCH features of possibly higher order.

## 1 Introduction

Stationary time series data showing fluctuating volatility and, in particular, financial return series have provided the impetus for the study of a whole series of econometric time series models that may be grouped under the general heading of GARCH (generalized, autoregressive, conditionally heteroscedastic models). Examples include the original ARCH model of Engle (1982), the standard GARCH model, integrated GARCH (IGARCH), exponential GARCH (EGARCH) and threshold GARCH (TGARCH), to name but a very few. Two review articles giving details of these models and their many variants are Bollerslev, Chou, and Kroner (1992) and Shephard (1996). In all of these models the hidden variable volatility depends *parametrically* on lagged values of the process and lagged values of volatility. The standard approach to fitting these models to data involves nonlinear maximum likelihood estimation.

In this paper we take a nonparametric approach to GARCH modelling which is less sensitive to model misspecification. We concentrate on a model, motivated by the natural idea in finance, that the hidden volatility depends nonparametrically on one lagged volatility and one lagged value of the process; we term this model a nonparametric GARCH(1,1) process. Nonparametric estimation is a feasible alternative because, in a sense, volatility is only partially hidden. It is possible to devise an *iterative* scheme to estimate the volatility process and thus, perhaps surprisingly, to overcome the problem of *latency*. This scheme makes use of a bivariate smoother and is thus very easy to implement. Using a

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software package that provides bivariate smoothing such as S-Plus, we are simply required to program an additional loop.

We argue that such an iterative bivariate smoothing scheme has the same rates of estimation accuracy as one classical bivariate smoothing step, which is usually of the order  $n^{-2/3}$  with  $n$  denoting the sample size; see Stone (1982). Although the nonparametric GARCH(1,1)-volatility process depends on the infinite past of the process, in a certain but not fully arbitrary nonparametric way, its estimation can be achieved within reasonable accuracy. In contrast to the parametric case, one should not use an ARCH-approximation scheme in the nonparametric framework: an expansion of the nonparametric GARCH(1,1)- into a nonparametric ARCH( $\infty$ )-model and estimation of the latter with a high order nonparametric ARCH( $p$ )-model is prohibitive. The curse of dimensionality would at best lead to an estimation rate of order  $n^{-4/(4+p)}$  with  $p$  large (Stone 1982). One way to deal with the curse of dimensionality is to consider multiplicative ARCH( $p$ )-models with estimation rate usually of order  $n^{-4/5}$  (Yang, Härdle, and Nielsen 1999, Hafner 1998). But from an interpretational view, we strongly prefer the nonparametric GARCH(1,1)-model; volatility depends in a fully nonparametric way only on one-lagged values of the time series and the volatility.

The basic idea of our iterative estimation algorithm allows for many extensions to other time series models with some form of latency. These extensions include nonparametric GARCH models of higher order, nonparametric ARMA models for prediction of conditional expectations and nonparametric GARCH models with a general additive structure for the conditional variance; we give examples in Section 4. To begin with we present in Section 2 the basic first order nonparametric model and its estimation algorithm, for which we provide a justification. In Section 3 we provide examples of the use of the algorithm on both simulated and real data.

## 2 The Basic Model

In this paper we consider stationary stochastic processes  $\{X_t; t \in \mathbb{Z}\}$  adapted to the filtration  $\{\mathcal{F}_t; t \in \mathbb{Z}\}$  with  $\mathcal{F}_t = \sigma(\{X_s; s \leq t\})$ , and having the form

$$\begin{aligned} X_t &= \sigma_t Z_t, \\ \sigma_t^2 &= f(X_{t-1}, \sigma_{t-1}^2), \end{aligned} \tag{1}$$

where  $\{Z_t; t \in \mathbb{Z}\}$  is an iid innovation series with zero mean, unit variance and a finite fourth moment.  $Z_t$  is assumed independent of  $\{X_s; s < t\}$ , and  $f : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a strictly positive-valued function;  $\sigma_t^2$  is then the conditional variance  $\text{Var}[X_t | \mathcal{F}_{t-1}]$  and  $\sigma_t$  is known as the volatility.

Equations (1) specify a general discrete time stochastic volatility process of first order that includes the parametric ARCH(1) and GARCH(1,1) models, but which also allows for a more complicated dependence of the present volatility on the past. For example, for the parametric GARCH(1,1) model,

$$f(x, \sigma^2) = \alpha_0 + \alpha_1 x^2 + \beta \sigma^2, \quad \alpha_0, \alpha_1, \beta > 0,$$

and for a more unusual asymmetric model we might consider

$$f(x, \sigma^2) = \alpha_0 + \alpha_1 x^2 + \beta \mathbf{1}_{\{x \leq 0\}} \sigma^2.$$

In this paper we leave the exact form of  $f$  unspecified and attempt to estimate it by nonparametric means.

To this end we observe first that the model (1) can be written in terms of an additive noise as follows;

$$\begin{aligned} X_t^2 &= f(X_{t-1}, \sigma_{t-1}^2) + V_t, \\ V_t &= f(X_{t-1}, \sigma_{t-1}^2) (Z_t^2 - 1), \end{aligned}$$

where  $V_t$  is a martingale difference series with  $\mathbb{E}[V_t] = \mathbb{E}[V_t | \mathcal{F}_{t-1}] = 0$  and  $\text{Cov}[V_s, V_t] = \text{Cov}[V_s, V_t | \mathcal{F}_{t-1}] = 0$  for  $s < t$ . It follows that

$$\mathbb{E}[X_t^2 | \mathcal{F}_{t-1}] = f(X_{t-1}, \sigma_{t-1}^2),$$

and

$$\text{Var}[X_t^2 | \mathcal{F}_{t-1}] = f^2(X_{t-1}, \sigma_{t-1}^2) (\mathbb{E}[Z_t^4] - 1).$$

This suggests we could estimate  $f$  by regressing  $X_t^2$  on the lagged variables  $X_{t-1}$  and  $\sigma_{t-1}^2$  using a nonparametric smoothing technique. The conditional heteroscedasticity of the series  $X_t^2$  suggests a weighted regression might be used. The principal problem is that the volatility  $\sigma_{t-1}$  is an unobserved latent variable. This problem is overcome in the following algorithm.

## 2.1 An estimation algorithm

Assume we have a data sample  $\{X_t; 1 \leq t \leq n\}$ , ideally from a process satisfying (1).

1. Calculate a first estimate of volatility  $\{\hat{\sigma}_{t,0}; 1 \leq t \leq n\}$  by fitting an ordinary parametric GARCH(1,1) model by standard maximum likelihood. Set  $m = 1$ .
2. Regress  $\{X_t^2; 2 \leq t \leq n\}$  against  $\{X_{t-1}; 2 \leq t \leq n\}$  and  $\{\hat{\sigma}_{t-1,m-1}^2; 2 \leq t \leq n\}$  using a nonparametric procedure to obtain an estimate  $\hat{f}_m$  of  $f$ . For a weighted regression use the regression weights  $\{\hat{\sigma}_{t,m-1}^{-2}; 2 \leq t \leq n\}$ .
3. Calculate  $\{\hat{\sigma}_{t,m}^2 = \hat{f}_m(X_{t-1}, \hat{\sigma}_{t-1,m-1}^2); 2 \leq t \leq n\}$  and impute some sensible value for  $\hat{\sigma}_{1,m}^2$ , for example  $\hat{\sigma}_{1,m-1}^2$ .
4. Increment  $m$  and return to step two if  $m < M$ .

Generally after a few iterations the algorithm has converged and there is little to pick and choose between volatility estimates  $\hat{\sigma}_{t,m}$  for various values of  $m$ . The algorithm can however often be improved by averaging over the final  $K$  such estimates to obtain

$$\hat{\sigma}_{t,*} = (1/K) \sum_{m=M-K+1}^M \hat{\sigma}_{t,m},$$

and then performing a final regression of  $X_t^2$  against  $X_{t-1}$  and  $\hat{\sigma}_{t-1,*}^2$  to get final estimates  $\hat{f}$  of  $f$  and  $\hat{\sigma}_t^2 = \hat{f}(X_{t-1}, \hat{\sigma}_{t-1,*}^2)$ ; we refer to this as a final smoothing step.

The parameters of the estimation algorithm are  $n$ , the sample size;  $M$ , the number of basic iterations and  $K$ , the number of estimates used in the final smoothing step. There will also be a bandwidth parameter to choose in the nonparametric regression method.

## 2.2 Justification of the algorithm

The following assumption is crucial for our justification of consistency of the estimation algorithm.

*Assumption A1* (contraction with respect to hidden variable).

$$\sup_{x \in \mathbb{R}} |f(x, \sigma^2) - f(x, \tau^2)| \leq D|\sigma^2 - \tau^2| \text{ for some } 0 < D < 1, \text{ for all } \sigma^2, \tau^2 \in \mathbb{R}_+.$$

A key feature of the algorithm can be understood when simplifying first to the case of no estimation error. We recursively define

$$\begin{aligned} f_{t,m}(x, \sigma^2) &= \mathbb{E}[X_t^2 | X_{t-1} = x, \sigma_{t-1,m-1}^2 = \sigma^2] \\ \sigma_{t,m}^2 &= f_{t,m}(X_{t-1}, \sigma_{t-1,m-1}^2), \quad m = 1, 2, \dots; \quad t \in \mathbb{Z}, \end{aligned}$$

where  $\sigma_{t,0}^2$  are some starting values assumed to be elements of  $\mathcal{F}_{t-1}$  for all  $t \in \mathbb{Z}$ , i.e. they are independent from  $\{X_s; s \geq t\}$ . The  $\sigma_{t,m}^2$  are the theoretical quantities corresponding to the estimates  $\hat{\sigma}_{t,m}^2$  of the algorithm. By iteration,  $\sigma_{t-1,m-1}^2 \in \mathcal{F}_{t-2}$  is independent from  $\{X_s; s \geq t-1\}$ . Since  $Z_t$  is also independent from  $\{X_s; s < t\}$ , we can write

$$\begin{aligned} \sigma_{t,m}^2 &= \mathbb{E}[X_t^2 | X_{t-1}, \sigma_{t-1,m-1}^2] \\ &= \mathbb{E}\{f(X_{t-1}, \sigma_{t-1,m-1}^2) | X_{t-1}, \sigma_{t-1,m-1}^2\}, \quad m = 1, 2, \dots; \quad t \in \mathbb{Z}. \end{aligned}$$

The following argument quantifies the error due to wrong starting values,

$$\begin{aligned} \mathbb{E}\{(\sigma_t^2 - \sigma_{t,m}^2)^2\} &= \mathbb{E}\left\{[f(X_{t-1}, \sigma_{t-1}^2) - \mathbb{E}\{f(X_{t-1}, \sigma_{t-1}^2) | X_{t-1}, \sigma_{t-1,m-1}^2\}]^2\right\} \\ &\leq \mathbb{E}\{[f(X_{t-1}, \sigma_{t-1}^2) - f(X_{t-1}, \sigma_{t-1,m-1}^2)]^2\} \\ &\leq D^{2m} \mathbb{E}\{[\sigma_{t-m}^2 - \sigma_{t-m,0}^2]^2\}. \end{aligned} \tag{2}$$

The first inequality holds since  $\mathbb{E}\{f(X_{t-1}, \sigma_{t-1}^2) | X_{t-1}, \sigma_{t-1,m-1}^2\}$  is the best mean square error predictor of  $\sigma_t^2 = f(X_{t-1}, \sigma_{t-1}^2)$  as a function of  $X_{t-1}$  and  $\sigma_{t-1,m-1}^2$ , the second inequality holds by iteration and assumption A1. Formula (2) shows that in case of no estimation error, iteration from arbitrary starting values  $\sigma_{t,0}^2$  ( $t \in \mathbb{Z}$ ) leads exponentially fast to the true squared volatility  $\sigma_t^2$ , due to the contraction property in assumption A1.

In the presence of estimation error we consider

$$\begin{aligned} \tilde{f}_{t,m}(x, \sigma^2) &= \mathbb{E}[X_t^2 | X_{t-1} = x, \hat{\sigma}_{t-1,m-1}^2 = \sigma^2], \\ \tilde{\sigma}_{t,m}^2 &= \tilde{f}_{t,m}(X_{t-1}, \hat{\sigma}_{t-1,m-1}^2), \end{aligned}$$

the latter being a true conditional expectation as a function of  $X_{t-1}$  and the estimate  $\hat{\sigma}_{t-1,m-1}^2$ . The estimation error in the  $m$ th smoothing step of the algorithm is then described by

$$\Delta_{t;n,m} = \hat{\sigma}_{t,m}^2 - \tilde{\sigma}_{t,m}^2, \quad m = 1, 2, \dots, \quad t = m+2, \dots, n.$$

Denote by

$$\Delta_n^{(\mathcal{L}_2)} = \sup_{m \geq 1} \max_{m+2 \leq t \leq n} (\mathbb{E} \Delta_{t;n,m}^2)^{1/2}.$$

We sometimes assume the following.

*Assumption A2.*

$$\begin{aligned} \mathbb{E}\{[\hat{\sigma}_{t,m}^2 - \sigma_{t,m}^2]^2\} &= \mathbb{E}\{[\tilde{f}_{t,m}(X_{t-1}, \hat{\sigma}_{t-1,m-1}^2) - f_{t,m}(X_{t-1}, \sigma_{t-1,m-1}^2)]^2\} \\ &\leq G^2 \mathbb{E}\{[\hat{\sigma}_{t-1,m-1}^2 - \sigma_{t-1,m-1}^2]^2\} \text{ for some } 0 < G < 1, \end{aligned}$$

for  $t = m + 2, m + 3, \dots$  and  $m = 1, 2, \dots$

*Assumption A3.*

$$\max_{2 \leq t \leq n} \mathbb{E}\{(\hat{\sigma}_{t,0}^2 - \sigma_{t,0}^2)^2\} \leq K_1 < \infty, \quad \max_{2 \leq t \leq n} \mathbb{E}\{(\sigma_{t,0}^2 - \sigma_t^2)^2\} \leq K_2 < \infty \text{ for all } n.$$

*Assumption A4.*

$$\Delta_n^{(\mathcal{L}_2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Theorem 1.** *Assume that  $\{X_t\}_t$  is as in (1), satisfying assumptions A1 and A2. Denote by  $\|\cdot\|_2$  the  $L_2$ -norm, i.e.  $\|Y\|_2 = (\mathbb{E}Y^2)^{1/2}$ . Then, the estimator  $\hat{\sigma}_{t,m}^2$  in the  $m$ th step of the algorithm satisfies*

$$\|\hat{\sigma}_{t,m}^2 - \sigma_t^2\|_2 \leq \sum_{j=0}^{m-1} G^j \|\Delta_{t-j,n,m-j}\|_2 + G^m \|\hat{\sigma}_{t-m,0}^2 - \sigma_{t-m,0}^2\|_2 + D^m \|\sigma_{t-m,0}^2 - \sigma_{t-m}^2\|_2,$$

$t = m + 2, m + 3, \dots, n.$

If additionally assumption A3 holds, then

$$\limsup_{m \rightarrow \infty} \max_{m+2 \leq t \leq n} \|\hat{\sigma}_{t,m}^2 - \sigma_t^2\|_2 \leq \Delta_n^{(\mathcal{L}_2)} / (1 - G).$$

If assumptions A1-A4 hold, and by choosing  $m_n = C\{-\log(\Delta_n^{(\mathcal{L}_2)})\}$  for some  $C \geq \max\{-1/\log(D), -1/\log(G)\}$ , we have

$$\max_{m_n+2 \leq t \leq n} \|\hat{\sigma}_{t,m_n}^2 - \sigma_t^2\|_2 = O(\Delta_n^{(\mathcal{L}_2)}) \text{ as } n \rightarrow \infty.$$

A proof is given in the Appendix. Note that similar asymptotic results also hold for the final smoothed estimate  $\hat{\sigma}_t^2$ .

The last statement of Theorem 1 says that the accuracy of the estimation algorithm is of the same order as the maximal error  $\Delta_n^{(\mathcal{L}_2)}$  in one smoothing step. Since the smoothing problem is bivariate, we expect  $\Delta_n^{(\mathcal{L}_2)} = O(n^{-2/3})$  with a second order kernel, provided some appropriate smoothness conditions on the function  $f$  hold and rate-optimal bandwidth choice is used (Stone 1982), although the volatility of the nonparametric GARCH model at time  $t$  is a function of the *infinite* past  $X_{t-1}, X_{t-2}, \dots$ .

According to Theorem 1, the minimal number of iteration steps is

$$m_{n;\min} = \max\{\log(\Delta_n^{(\mathcal{L}_2)})/\log(D), \log(\Delta_n^{(\mathcal{L}_2)})/\log(G)\}$$

to achieve the order of  $\Delta_n^{(\mathcal{L}_2)}$  in the last statement of the Theorem. Assuming in the remaining part of the section that  $\Delta_n^{(\mathcal{L}_2)} \sim \text{const.}n^{-2/3}$ , we obtain  $m_{n;\min} \sim \text{const.} \log(n)$ , where the constant depends on the unknown  $D$  and  $G$ . We propose to use

$$m_n = a \log(n) \text{ with } a \in \{2, 3\}.$$

This range of  $a$  covers the values  $m_{n;\min}$  for  $D, G \in \{0.72, 0.80\}$ , and  $m_n = 3 \log(n)$  is a valid choice for the last statement in Theorem 1 for all  $D, G \leq 0.8$ .

We complete this section with a discussion about assumptions and difficulties with a rigorous theoretical analysis. Assumption A1 is crucial in our reasoning and we conjecture that without a suitable contraction property of the function  $f$  with respect to the second, hidden argument, there is no hope in getting a consistent estimate with our iterative scheme. Formula (2) states a clean mathematical result when there are no estimation errors; it thus points out that it seems indeed possible to solve the problem of

nonparametric GARCH estimation with a simple iterative smoothing scheme. Assumptions A2, A3 and A4 are all exclusively needed to deal with the case of estimation error. Assumption A2 is about differences between projections: it requires that the  $L_2$ -norm of the difference between projections is strictly smaller than the  $L_2$ -norm of the difference between the functions  $\hat{\sigma}_{t-1,m-1}^2$  and  $\sigma_{t-1,m-1}^2$  (which are the spanning elements causing distinct projection spaces). Although such an assumption is uncommon, it is natural in the framework of  $L_2$ -Hilbert theory. Assumption A2 can also be replaced by

*Assumption A2'*.

$$\sup_{m \geq 1} \max_{m+2 \leq t \leq n} \mathbb{E}\{(\mathbb{E}[X_t^2 | X_{t-1}, \hat{\sigma}_{t-1,m-1}^2] - \mathbb{E}[f(X_{t-1}, \sigma_{t-1}^2) | X_{t-1}, \hat{\sigma}_{t-1,m-1}^2])^2\} \leq \Gamma_n^2,$$

$$\Gamma_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This assumption quantifies the effect that  $\hat{\sigma}_{t-1,m-1}^2 \notin \mathcal{F}_{t-1}$  so that the two conditional expectations are not the same. If assumptions A1, A2', A3 and A4 hold, then similar to Theorem 1,

$$\|\hat{\sigma}_{t,m}^2 - \sigma_t^2\|_2 \leq (\Gamma_n + \Delta_n^{(\mathcal{L}_2)}) \sum_{j=0}^{m-1} D^j + \text{const.} D^m,$$

which yields consistency; the same rate as with assumption A2 is obtained if  $\Gamma_n = O(\Delta_n^{(\mathcal{L}_2)})$ . Assumption A3 is a very mild condition regarding initial conditions. Assumption A4 is plausible: a rigorous justification under some appropriate regularity conditions for the process  $\{X_t\}_t$  seems tedious, involving the difficulty of quantifying the effect when plugging in estimated random variables. Concluding this discussion, we believe that formula (2) and Theorem 1 reflect the correct asymptotic behaviour of our estimation algorithm, although a rigorous justification of our assumptions A2 (or A2') and A4 is missing and still an open area for future research in theoretical statistics. Our belief for correctness of the asymptotics given here is also based on satisfactory results in numerical examples as discussed in the next section.

### 3 Illustrative Examples

#### 3.1 Simulated Example

We illustrate the method using simulated data from a process satisfying (1). For the volatility surface we take

$$f(x, \sigma^2) = 5 + 0.2x^2 + (0.75 \cdot \mathbf{1}_{\{x > 0\}} + 0.1 \cdot \mathbf{1}_{\{x \leq 0\}})\sigma^2,$$

which is depicted in Figure 1, and we assume that the innovation distribution is standard normal. Note the asymmetry that has been built into the GARCH effect; the strength of the GARCH effect (that is the magnitude of the  $\sigma^2$  coefficient) varies with the sign of  $x$ . A realisation of  $n = 1000$  points from this process is shown in Figure 2.

We apply the algorithm to these data, setting the number of basic iterations to be  $M = 8$  and performing a final smooth based on  $K = 5$  final iterations. We use the default value of the smoothing parameter in the S-Plus implementation of a bivariate *Loess* smoother.

In Figure 3 the estimated surfaces  $\hat{f}_m$  calculated at the first 8 iteration and the surface calculated at the final smoothing stage are shown. Comparison with Figure 1 indicates that the nonparametric algorithm is recovering the essential features of the volatility surface. This series of nine pictures illustrates the first envisaged use of the algorithm — as an exploratory graphical tool for investigating the dependence of the present volatility on the immediate past.

To provide a quantitative assessment of the accuracy of the volatility estimates we calculate both a mean squared estimation error and a mean absolute estimation error. Because estimates of volatility at the first few time points may be unreliable we generally omit  $r = 10$  values from this calculation. Our measures of estimation error are

$$\text{MSE}(\hat{\sigma}_{\cdot,m}) = \frac{1}{n-r} \sum_{t=r+1}^n (\hat{\sigma}_{t,m} - \sigma_t)^2, \quad \text{MAE}(\hat{\sigma}_{\cdot,m}) = \frac{1}{n-r} \sum_{t=r+1}^n |\hat{\sigma}_{t,m} - \sigma_t|.$$

In Table 1 we see that the parametric GARCH(1,1) estimate of volatility can be improved upon through additional iterations of the nonparametric algorithm. Even a single iteration reduces both errors by around a third and after 2 or 3 iterations no further improvement is obtained; thereafter the errors oscillate within a band. In this example the final smooth gives the lowest errors of all.

Iteration ( $m$ )	MSE( $\hat{\sigma}_{\cdot,m}$ )	MAE( $\hat{\sigma}_{\cdot,m}$ )
0 (parametric)	0.48	0.56
1	0.31	0.57
2	0.25	0.40
3	0.21	0.37
4	0.19	0.33
5	0.20	0.32
6	0.20	0.32
7	0.20	0.32
8	0.20	0.32
* (final smooth)	0.18	0.31

Table 1: Volatility estimation errors for a realisation of  $n = 1000$  points.

In Figure 4 the volatility estimates are compared with the true volatility trajectory for an arbitrary section of 70 observations. In the left-hand picture it is clear that the parametric estimate derived from GARCH(1,1) modelling is unable to follow the true volatility closely through its peaks and troughs; on the other hand, in the right-hand picture we see that the nonparametric GARCH estimate derived from the final smoothing round is fairly faithful to the true volatility.

Essentially it is the strong asymmetry of the simulated process that denigrates the performance of ordinary parametric GARCH as a volatility estimator. For more symmetric processes the difference in performance is less obvious. For *simple* additive surfaces of the form

$$f(x, \sigma^2) = g(x) + h(\sigma^2),$$

where  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  is a positive-valued function satisfying  $g(x) = g(-x)$  (such as  $g(x) = \alpha|x|$ ,  $0 < \alpha < 1$ ) and  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a positive-valued non-decreasing function (such as  $h(\sigma^2) = \beta\sigma$ ,  $0 < \beta < 1$ ), the performance of ordinary GARCH(1,1) is often as good as or slightly better than that of the nonparametric procedure, even though GARCH(1,1) assumes an erroneous parametric form. For complicated additive functions we note that our procedure can be adapted to fit general nonparametric additive models; for an outline see Section 4.2.

When the true process is exactly GARCH(1,1) then we would expect the parametric procedure to outperform the nonparametric one, and it does so. Our simulation experiments suggest that it is often in situations of asymmetry that the nonparametric procedure has advantages.

### 3.2 Real Example

For an example with real data we take a 1000-day excerpt from the time series of daily percentage returns on the BMW share price (see Figure 5). Our time series contains the period of high volatility around the 1987 stock market crash.

In Figure 6 we show again the graphical output of the algorithm for the first 8 iterations and the final smoothing step. The pictures have been rotated so that we view them along the line  $x = 0$ . These estimated volatility surfaces show some evidence of an asymmetric effect, which depends on the sign of the last observation; as long as volatility is low, a large positive return appears to have a much more modest effect on the next day's volatility than does a large negative return. This asymmetric effect of new information is a well-known phenomenon in financial time series and our method clearly picks it up.

Finally in Figure 7 we show the estimated volatilities derived from parametric modelling and the final-smooth stage of nonparametric modelling. Although we do not know the true volatility process for these data a simple comparison of the estimators can be obtained by calculating the mean squared "error" statistic

$$\frac{1}{n-r} \sum_{t=r+1}^n (\hat{\sigma}_t^2 - X_t^2)^2$$

for the two volatility estimates. This statistic is based on the additive representation of the squared series as  $X_t^2 = \sigma_t^2 + V_t$ , as introduced in Section 2; the  $V_t$  is a martingale difference series with mean zero, which we regard as the error term. The mean squared error statistic takes the values 106 and 98.3 for the parametric and nonparametric estimators respectively. Thus the error is about 8% smaller for the nonparametric estimate.

## 4 Extensions

The iterative idea of our estimation algorithm can be extended in a variety of ways and combined with other nonparametric modelling techniques.

### 4.1 Nonparametric GARCH(p,q)

The estimation algorithm in Section 2.1 and its justification easily extend to the nonparametric GARCH( $p, q$ ) model with  $0 \leq p, q < \infty$ . Equation (1) is now generalised to

$$\begin{aligned} X_t &= \sigma_t Z_t, \\ \sigma_t^2 &= f(X_{t-1}, \dots, X_{t-p}, \sigma_{t-1}^2, \dots, \sigma_{t-q}^2). \end{aligned}$$

The  $m$ th iteration step in the estimation algorithm is then

$$\hat{\sigma}_{t,m}^2 = \hat{f}_m(X_{t-1}, \dots, X_{t-p}, \hat{\sigma}_{t-1,m-1}^2, \dots, \hat{\sigma}_{t-q,m-1}^2),$$

where  $\hat{f}_m$  is based on a  $p+q$ -variate smoother of  $X_t^2$  versus  $X_{t-1}, \dots, X_{t-p}, \hat{\sigma}_{t-1,m-1}^2, \dots, \hat{\sigma}_{t-q,m-1}^2$ . Justification of such an extension can be given as in section 2.2 by replacing assumption A1 with

$$\sup_{x \in \mathbb{R}^p} |f(x_1, \dots, x_p, \sigma_1^2, \dots, \sigma_q^2) - f(x_1, \dots, x_p, \tau_1^2, \dots, \tau_q^2)| \leq \sum_{i=1}^q d_i |\sigma_i^2 - \tau_i^2|,$$

$$\text{for some } 0 < d_1, \dots, d_q < 1 \text{ with } \sum_{i=1}^q d_i = D < 1,$$

for all  $\sigma^2, \tau^2 \in (\mathbb{R}^+)^q$ .

## 4.2 Nonparametric additive GARCH

The model in (1) or the higher order model of Section 4.1 can be modified to the case where  $\sigma_t^2$  is a general additive function of lagged values and volatilities. The estimation algorithm is simply adjusted: the smoothing operations are performed according to the additive structure of  $\sigma_t^2$  by using the backfitting algorithm. For generalized additive models and their estimation see Hastie and Tibshirani (1990)

## 4.3 Nonparametric ARMA-GARCH of first order

The estimation algorithm also extends to hybrid ARMA-GARCH models; for notational simplicity we focus on the first order case. Consider

$$\begin{aligned} X_t &= \mu_t + \sigma_t Z_t, \\ \mu_t &= g(X_{t-1}, \mu_{t-1}), \\ \sigma_t^2 &= f((X_{t-1} - \mu_{t-1}), \sigma_{t-1}^2), \end{aligned}$$

where  $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a nonparametric function giving the conditional mean of the process; all other elements of the model are as in (1).

Observe that

$$\mathbb{E}[X_t | \mathcal{F}_{t-1}] = g(X_{t-1}, \mu_{t-1}),$$

and

$$\text{Var}[X_t | \mathcal{F}_{t-1}] = f((X_{t-1} - \mu_{t-1}), \sigma_{t-1}^2).$$

This suggests that a similar scheme can be applied to estimate both  $g$  and  $f$ .

We start with estimates  $\hat{\mu}_{t,0}$  and  $\hat{\sigma}_{t,0}$  of  $\mu_t$  and  $\sigma_t$ , perhaps derived by parametric means. We regress  $X_t$  against  $X_{t-1}$  and  $\hat{\mu}_{t-1,0}$ , weighting possibly with  $\hat{\sigma}_{t,0}^{-1}$ , to estimate  $g$  and hence to re-estimate  $\mu_t$  by  $\hat{\mu}_{t,1}$ . We then regress  $(X_t - \hat{\mu}_{t,1})^2$  against  $(X_{t-1} - \hat{\mu}_{t-1,1})$  and  $\hat{\sigma}_{t-1,0}^2$ , weighting possibly with  $\hat{\sigma}_{t,0}^{-2}$ , to estimate  $f$  and hence to re-estimate  $\sigma_t$  by  $\hat{\sigma}_{t,1}$ . We iterate this procedure to improve estimates of conditional mean and volatility.

## 5 Discussion

In this paper we have proposed and justified an algorithm for fitting nonparametric GARCH models of first order, and we have suggested extensions to other time series models. The first use we see for our procedure in practical applications is an *explanatory* one: assuming the availability of a bivariate smoother (such as *loess* in S-Plus) it is a simple matter to implement our iterative algorithm to investigate visually the dependence of volatility on lagged values of the time series and the volatility itself.

The second envisaged use for our method is as a volatility estimator in situations where the functional relationship between volatility and the lagged series differs markedly from standard parametric GARCH(1,1), particularly in situations where asymmetries appear to be present.

Of course the GARCH framework is not the only way to approach volatility estimation. Another stream of models are the stochastic volatility (SV) models; see Shephard (1996). In contrast to our model (1), the SV model has a fully latent volatility process with no observable structure. It can be formalized as a state space model

$$\begin{aligned} X_t &= \sigma_t Z_t, \\ \{\sigma_t^2\} &= \text{stationary stochastic process of a pre-specified form,} \end{aligned} \tag{3}$$

with  $Z_t$  as in (1), and  $\sigma_t^2$  *conditionally independent* of  $\{X_s; s < t\}$  given  $\{\sigma_s^2; s < t\}$ . Note that this conditional independence structure is not true for our nonparametric GARCH model. This is why we refer to *partially hidden* volatility in (1), since the lagged value  $X_{t-1}$  contributes to the partial observability of  $\sigma_t^2$ . The SV model in (3) cannot be fitted with our iterative procedure. If the process  $\{\sigma_t^2\}$  and the innovation distribution of  $Z_t$  are of finite parametric form, techniques from nonlinear Kalman filtering can be used, e.g. Markov chain Monte Carlo or Gibbs sampling; see for example Shephard and Pitt (1997).

A semiparametric approach for estimation of the model in (3) is given in Franke, Härdle, and Kreiss (1998). The dynamics of  $\{\sigma_t^2\}$  are assumed nonparametric, but the innovation distribution of  $Z_t$  is taken to be standard normal. They use a deconvolution estimator that relies crucially on this assumption of normality and prove that the convergence rate of the estimator is  $\log(n)^{-2}$ . Convergence is thus extremely slow, although this is a common rate with such deconvolution techniques.

A further positive feature of our model is that it is *fully* nonparametric in the sense that not only is the exact functional relationship between volatility and the one-lagged values of the time series and volatility left unspecified, but also no assumptions are made regarding the distributional form of the innovation distribution. (We assume only the existence of a finite fourth moment to justify the use of weighted nonparametric regression.)

Much work in empirical finance suggests that GARCH-type models with Gaussian innovations cannot capture the leptokurtosis and conditional heteroscedasticity of typical financial return data. This work suggests that heavier-tailed innovations are required; see McNeil and Frey (1999). It is thus attractive to look at flexible fitting methods such as our nonparametric GARCH algorithm that do not require us to fix the form of the innovation distribution.

## Appendix

*Proof of Theorem 1.* Write

$$\|\hat{\sigma}_{t,m}^2 - \sigma_t^2\|_2 \leq \Delta_{t;n,m} + \|\tilde{\sigma}_{t,m}^2 - \sigma_{t,m}^2\|_2 + \|\sigma_{t,m}^2 - \sigma_t^2\|_2.$$

The third term on the right hand side is bounded with formula (2); for the second term we obtain with assumption A2,

$$\begin{aligned} \|\tilde{\sigma}_{t,m}^2 - \sigma_{t,m}^2\|_2 &\leq G\|\hat{\sigma}_{t-1,m-1}^2 - \sigma_{t-1,m-1}^2\|_2 \leq G(\Delta_{t-1;n,m-1} + \|\tilde{\sigma}_{t-1,m-1}^2 - \sigma_{t-1,m-1}^2\|_2) \\ &\leq \dots \leq \sum_{j=1}^{m-1} G^j \Delta_{t-j;n,m-j} + G^m \|\hat{\sigma}_{t-m,0}^2 - \sigma_{t-m,0}^2\|_2. \end{aligned}$$

These bounds then yield the first assertion in Theorem 1. The other assertions follow easily by using assumptions A3 and A4, respectively.  $\square$

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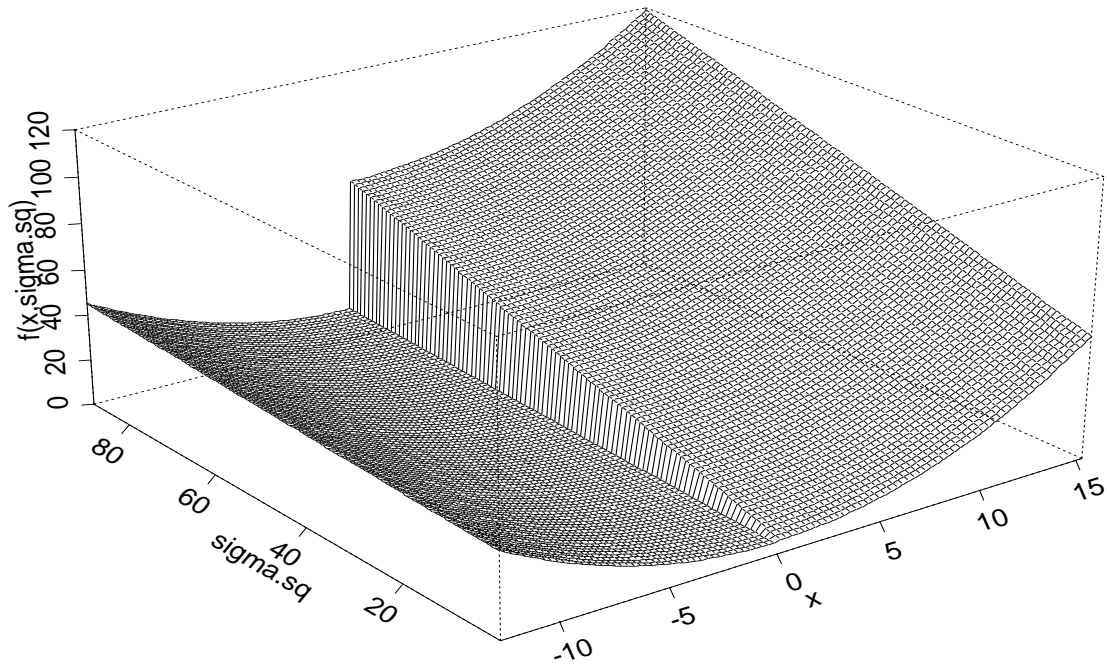


Figure 1: Simulation experiment. The volatility surface  $f(x, \sigma^2) = 5 + 0.2x^2 + (0.751_{\{x>0\}} + 0.11_{\{x\leq 0\}})\sigma^2$ ; note the asymmetry depending on the sign of  $x$ .

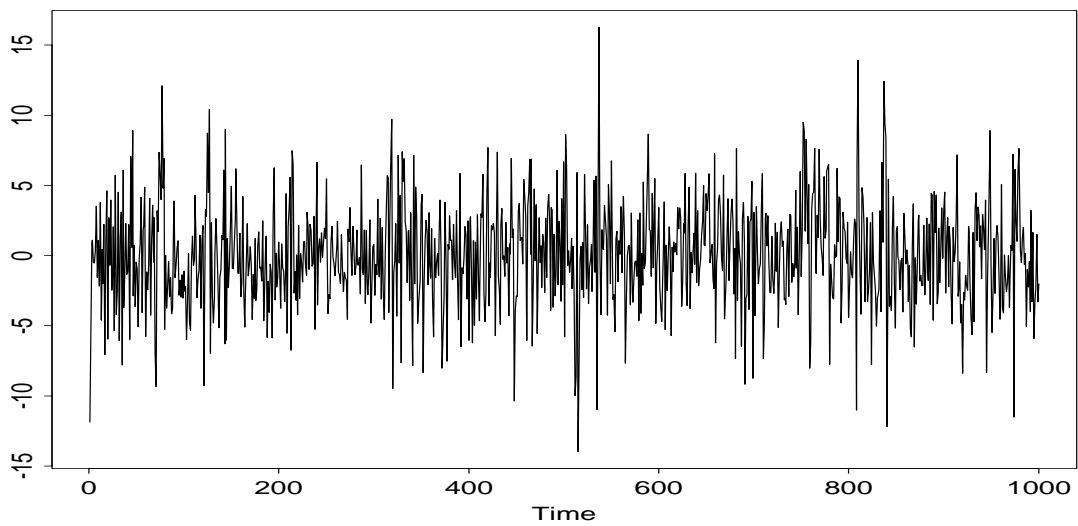


Figure 2: Simulation experiment. The data.

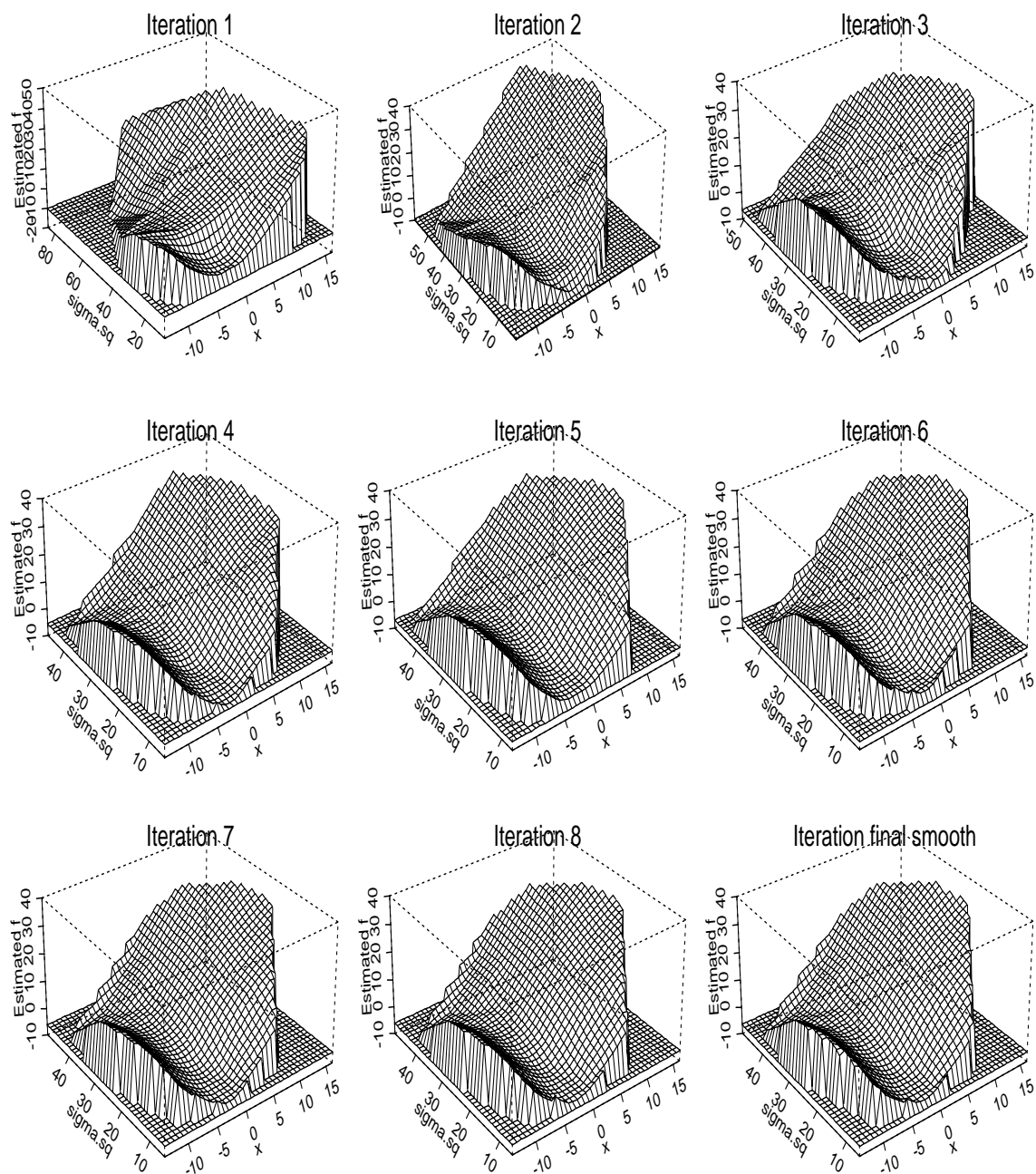


Figure 3: Simulation experiment. The estimated surfaces  $\hat{f}_m$  calculated at the first 8 iterations and the surface  $\hat{f}$  estimated at the final smoothing stage.

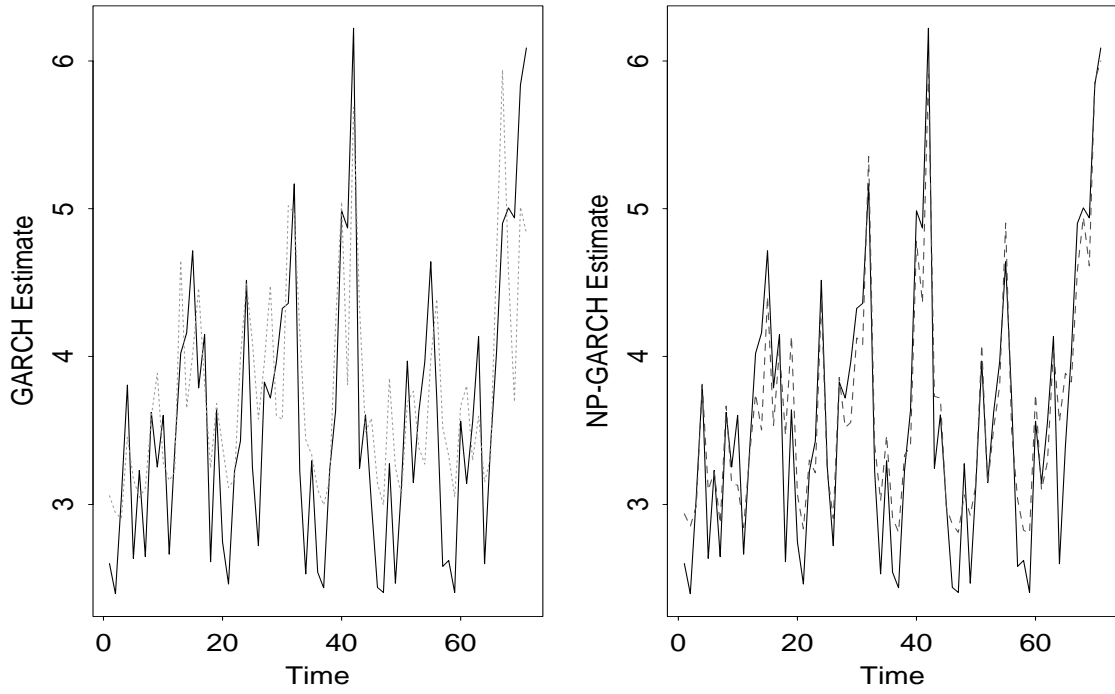


Figure 4: Simulation experiment. For an arbitrary section of 70 observations from the simulated time series, the left-hand plot shows true volatility (solid line) compared with a parametric GARCH(1,1) estimate (dotted line) and the right-hand plot shows true volatility compared with the nonparametric GARCH estimate obtained after a final smooth (dashed line).

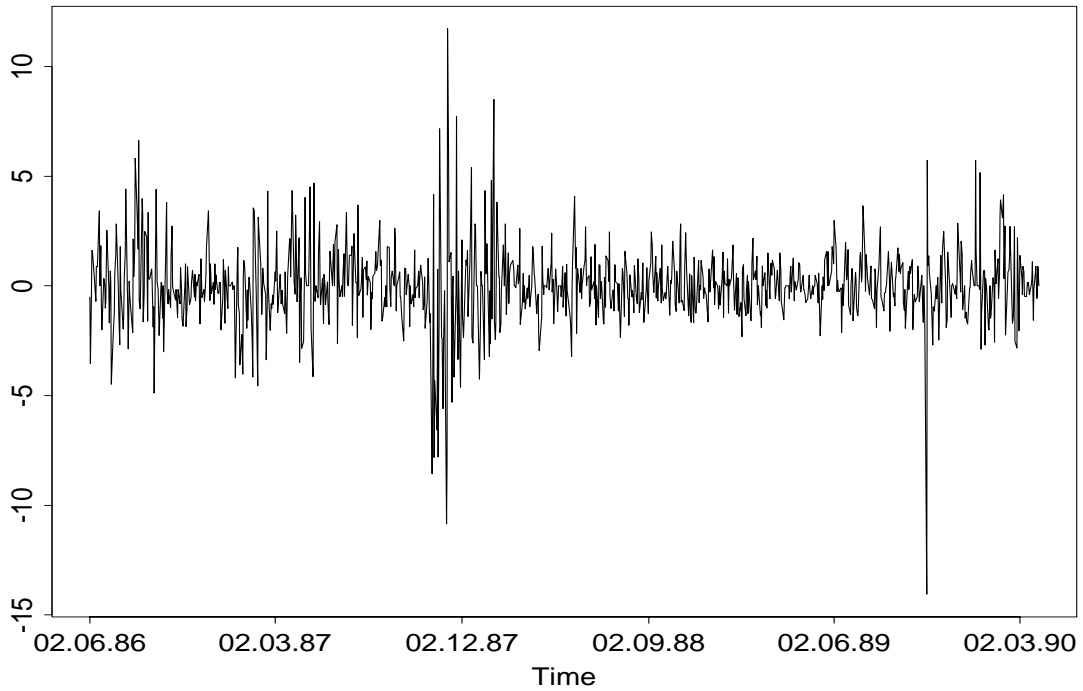


Figure 5: Real data. A 1000-day excerpt from the time series of daily returns on the BMW share price.

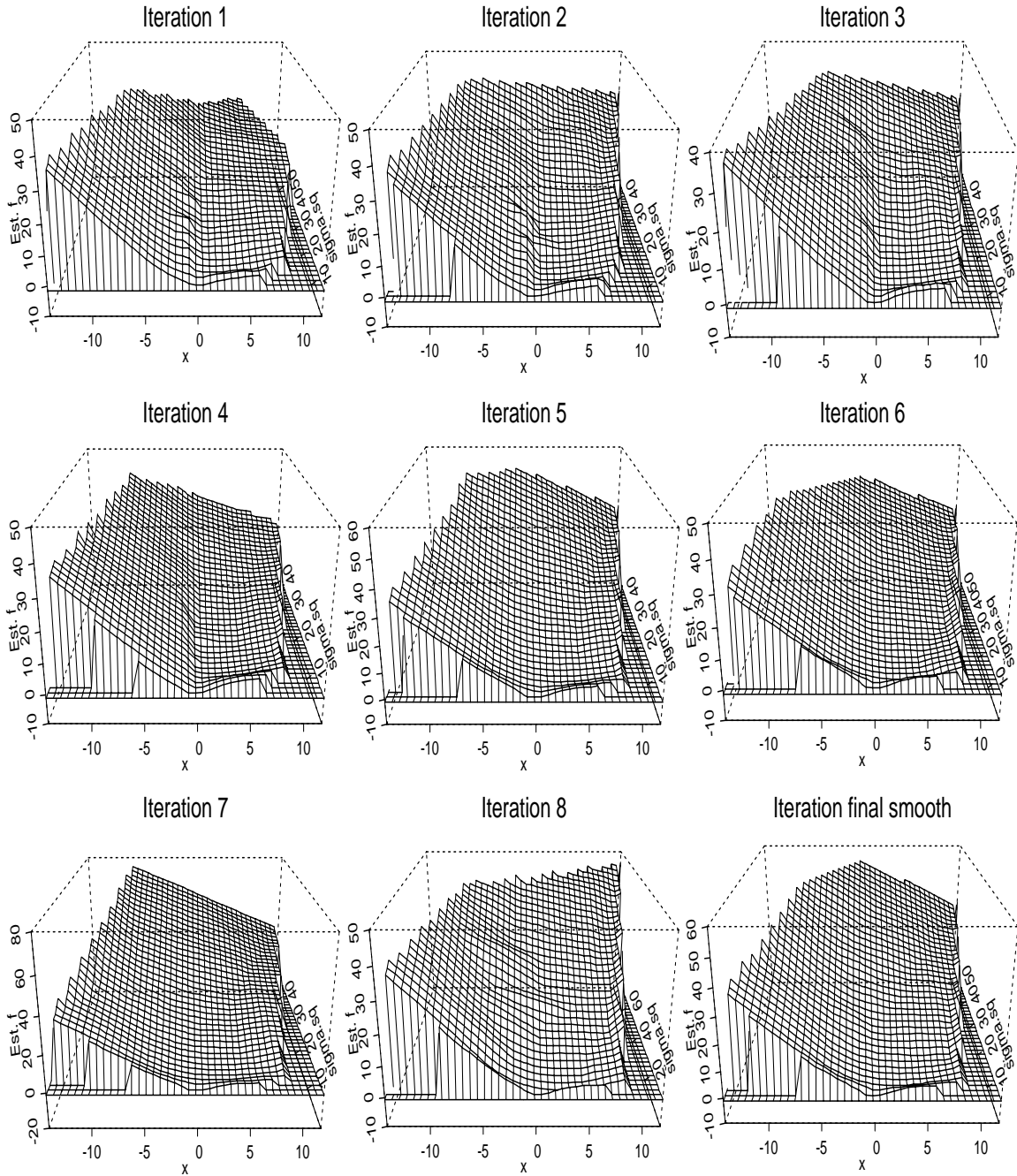


Figure 6: Real data. The estimated surfaces  $\hat{f}_m$  calculated at the first 8 iterations and the surface  $\hat{f}$  estimated at the final smoothing stage.

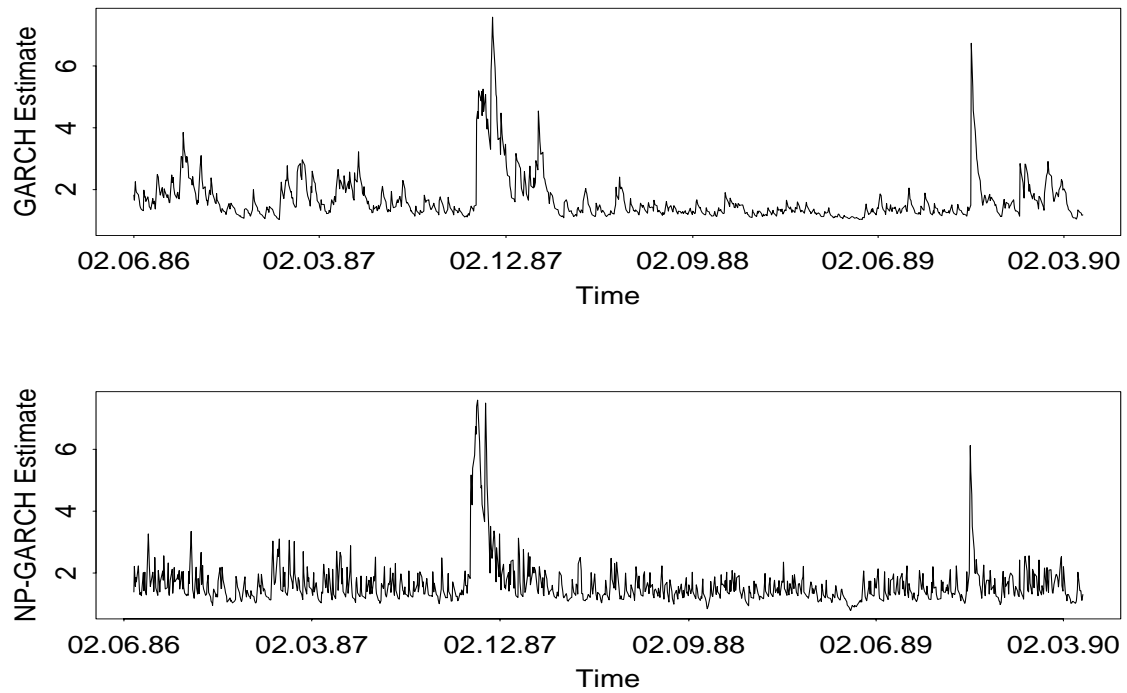


Figure 7: Real data. A comparison of volatilities estimated by ordinary parametric GARCH(1,1) modelling and nonparametric estimation (after the final smooth).