# Multivariate Stress Scenarios and Solvency

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## Abstract

We show how the probabilistic concepts of half-space trimming and depth may be used to define convex scenario sets $Q_\alpha$ for stress testing the risk factors that affect the solvency of an insurance company over a prescribed time period. By choosing the scenario in $Q_\alpha$ which minimises net asset value at the end of the time period, we propose the idea of the least solvent likely event (LSLE) as a solution to the forward stress testing problem. By considering the support function of the convex scenario set $Q_\alpha$, we establish theoretical properties of the LSLE when financial risk factors can be assumed to have a linear effect on the net assets of an insurer. In particular, we show that the LSLE may be interpreted as a scenario causing a loss equivalent to the Value-at-Risk (VaR) at confidence level $\alpha$, provided the $\alpha$-quantile is a subadditive risk measure on linear combinations of the risk factors. In this case, we also show that the LSLE has an interpretation as a per-unit allocation of capital to the underlying risk factors when the overall capital is determined according to the VaR. These insights allow us to define alternative scenario sets that relate in similar ways to coherent measures, such as expected shortfall. We also introduce the most likely ruin event (MLRE) as a solution to the problem of reverse stress testing.

*Keywords:* Stress testing, Solvency II, risk measures, convex analysis, scenario sets
JEL code: C60. Subject codes: IM10, IM13, IM43

## 1. Introduction

In this paper we consider a simple stochastic risk model for a company taking the form $V = g(\mathbf{X})$, where the random vector $\mathbf{X} \in \mathbb{R}^d$ contains the values at some future

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time point of certain fundamental *risk drivers or factors* affecting the solvency of the company, and where $V$ is the *net asset value* at the time horizon. Given knowledge of $g$ and the distribution of $\mathbf{X}$, our aim is to examine methods of constructing stress scenarios for $\mathbf{X}$.

Our interest in stress testing is prompted by the Solvency II capital adequacy proposals for insurers. In that context we consider a one-year horizon and the risk factors $\mathbf{X}$ typically represent equity prices, yields for different bond maturities, mortality rates or policy lapse rates, corporate credit spreads, or any other quantities that can affect the value of assets and liabilities. The function $g$ contains the valuation formulas that show how these factors affect net asset value over the one-year horizon.

The approach we adopt may be described as a forward stress testing procedure (to be contrasted with reverse stress testing below). We first identify appropriate sets $S$ of *plausible* multivariate scenarios; we then determine the scenario that leads to the most unfavourable outcome in terms of net asset value $V$. We refer this scenario as the *least solvent likely event* or LSLE (pronounced Leslie), a name that is used in the insurance context. In mathematical terms it is the scenario

$$\mathbf{x}_{\mathrm{LSLE}} = \arg\min_{\mathbf{x}\in S} g(\mathbf{x}) \ , \tag{1}$$

for an appropriate scenario set $S$. Of course we could also consider the equivalent maximisation problem for the loss $(v_0 - V)$, with respect to some initial net asset level $v_0$.

$$\mathbf{x}_{\mathrm{LSLE}} = \arg\max_{\mathbf{x}\in S} (v_0 - g(\mathbf{x})) \ . \tag{2}$$

This general approach may be found in other work on stress testing (Studer, 1997, 1999; Breuer et al., 2009, 2010) but we differ in this paper in our manner of constructing scenario sets.

There is a growing literature on stress testing with regard to capital adequacy requirements for banks and insurers. The Basel II regulations (Basel Committee 2004) require banks to carry out stress testing of both market and credit risk capital requirements. Early papers on stress testing theory include Berkowitz (2000) who described the essence of stress testing as "choosing scenarios that are costly and rare, and then putting them to the valuation model"; he argued for scenarios that are grounded in models and assigned probabilities rather than scenarios that are simply damaging events specified by experts without any notion of likelihood.

Studer (1997, 1999) was one of the first to develop a systematic, probabilistic approach to stress testing. He considered *trust regions* for market risk factors which were connected higher-dimensional sets with a prescribed probability and introduced the *maximum loss* risk measure, defined to be the maximum loss of a market portfolio over all scenarios in the trust region. In particular, Studer considered ellipsoidal trust regions defined by Mahalanobis distance, that is sets of the form $S = \{\mathbf{x} : (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leqslant k\}$ where $\boldsymbol{\mu}$ is a location vector, $\Sigma$ a covariance matrix describing the correlation structure of the distribution of risk factor changes, and $k$ a constant chosen to give a set with the desired probability.

Other work on stress testing in a market risk context includes Kupiec (1998), Aragonés et al. (2001) and Alexander and Sheedy (2008). Kupiec (1998) developed a multivariate stress testing procedure based on an assumption of normality with stressed values for the volatilities and correlations of individual risk factor changes. Aragonés et al. (2001) used extreme value theory to define stress events for individual risk factors (the $d$ components of $\mathbf{X}$). Alexander and Sheedy (2008) conducted an extensive investigation of both unconditional, distribution-based approaches to setting stress tests as well as conditional approaches using GARCH models; their statistical analyses focus on foreign exchange risk factors and are largely univariate.

Recently there has been a revival of interest in Studer's ellipsoidal scenario sets. Breuer et al. (2009) describe the problem of finding scenarios that are "plausible, severe and useful" and propose a number of refinements to Studer's approach. In particular, Breuer et al. point out that Studer's sets contain progressively more extreme scenarios as the dimension of the vector of risk factors is increased because the value of $k$ must be increased to ensure that the probability of the set $\{\mathbf{x} : (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leqslant k\}$ is held constant. They solve this by holding the value of $k$ fixed for all dimensions and allowing the probability of the ellipsoids to change; they also develop a method of stressing subsets of the complete risk factor vector.

In follow-up work Breuer et al. (2010) show how to apply their methodology to credit risk in a multi-period setting. More recently Breuer and Csiszár (2010) have pointed out that the Mahalanobis ellipsoid is only really a natural scenario set for elliptical distributions and have proposed a novel approach which also takes into account model risk. They redefine maximum loss as the supremum of the expected loss of a portfolio over all distributions of risk factor changes whose Kullback-Leibler distance from some reference distribution does not exceed some fixed threshold.

In Section 2 of this paper we propose a new approach to the construction of scenario sets using the concept of *half-space trimming*, a concept in multivariate statistics that has been developed by Massé and Theodorescu (1994) and Rousseeuw and Ruts (1999), among others. The concept has its origins in data-analytic ideas of Tukey (1975) and theoretical work on multivariate analogues of the quantile function by Eddy (1984) and Nolan (1992).

Using half-space trimming we obtain closed, convex sets, $Q_\alpha$ for $\alpha > 0.5$, which have the property that they are the intersection of all closed half spaces having probability at least $\alpha$. These sets are interpreted as plausible or likely sets although they are not constructed using the concept of density, but rather the concept of *depth*. Depth rather than density is used as our measure of plausibility. We show that in the case of an elliptical distribution for $\mathbf{X}$ we obtain familiar ellipsoidal scenario sets of the kind considered by many of the authors cited above (Studer, 1997; Breuer et al., 2009, 2010), but for other distributions we obtain a variety of shapes.

In Section 3 we use these scenario sets to find the worst scenario or LSLE. For simplification we consider only impacts that have a simple linear structure: $g(\mathbf{x}) = \mathbf{u}'\mathbf{x}$ for some vector $\mathbf{u} \neq \mathbf{0}$, which allows us to analyse the problem of finding a worse event by considering the so-called *support function* $\sup\{\mathbf{u}'\mathbf{x} \ : \ \mathbf{x} \in Q_\alpha\}$ of the convex set $Q_\alpha$. Essentially, the support function can be thought of as giving the LSLE for different portfolio compositions $\mathbf{u}$. Under certain circumstances the LSLEs defined in this way can have an appealing interpretation. By using results in convex analysis we show that in general the support function of $Q_\alpha$ gives a lower bound for $q_\alpha(\mathbf{u})$, the $\alpha$-quantile of $\mathbf{u}'\mathbf{X}$. However, in cases where the $\alpha$-quantile is subadditive on linear combinations of $\mathbf{X}$, the support function coincides with $q_\alpha(\mathbf{u})$.

The practical implication is that, in such cases, the LSLE scenario is a stress scenario that leads exactly to the $(1-\alpha)$-quantile of the net-asset-value distribution or, equivalently, the $\alpha$-quantile of the distribution of the loss $(v_0 - V)$, a concept well known in risk management as the the Value-at-Risk at level $\alpha$. This gives an elegant link between capital setting using Value-at-Risk and capital setting using stress testing. We can interpret Value-at-Risk as the actual loss that would be caused by the LSLE scenario. Moreover we can analyse the LSLE scenario, attempt to understand what makes it so destructive and take appropriate risk mitigating steps through hedging of risk factors or changing of the business mix.

In general the support function of the scenario set $Q_\alpha$ only provides a lower bound for $q_\alpha(\mathbf{u})$. In the case that $q_\alpha(\mathbf{u})$ is differentiable we also define an outer scenario set $O_\alpha$ which is given by the gradient vectors of $q_\alpha(\mathbf{u})$ for different values of $\mathbf{u}$. By considering $\sup\{\mathbf{u}'\mathbf{x} \ : \ \mathbf{x} \in O_\alpha\}$ we obtain an upper bound for $q_\alpha(\mathbf{u})$ in Section 4.1. This bound is also shown to coincide with $q_\alpha(\mathbf{u})$ for all $\mathbf{u}$ in the case where the $\alpha$-quantile is subadditive on linear combinations of $\mathbf{X}$. In this case, the LSLE scenario has a further interpretation as the per-unit Euler attribution of the overall risk $q_\alpha(\mathbf{u})$ to each of the risk factors and this can also give information about how to take risk mitigating actions.

Since the properties of the scenario sets $Q_\alpha$ and $O_\alpha$ are most appealing when the quantile risk measure is subadditive, in Section 5 we consider defining new scenario sets by starting with subadditive risk measures $\varrho$, such as expected shortfall, and forming convex scenario sets for which the subadditive risk measures form the support functions. The worst scenarios in these sets are always scenarios that lead to losses that attain the value of the risk measure $\varrho(\mathbf{u})$.

While there has been much work on formalizing the forward stress testing problem, there is growing interest in the so-called reverse stress testing problem, but, as yet, little theory. In the UK the Financial Services Authority (FSA) have proposed "to introduce a stress testing requirement, which would apply to banks, building societies, CRD investment firms and insurers, and would require firms to consider the scenarios most likely to cause their current business models to become unviable"(Financial Services Authority, 2008). In terms of the simple model we have set out above, we could attempt to identify the *ruin set* of all scenarios leading to insolvency $R = \{\mathbf{x} : g(\mathbf{x}) \leqslant 0\}$ and then identify within this set the scenario that is most plausible, in the sense of the depth concept described above. In forward stress testing we specify the level of plausibility of the scenarios and search for the most ruinous; in reverse stress testing we look among the ruin scenarios for the most plausible. The latter scenario has been called the *most likely ruin event* or MLRE (pronounced Mallory).

In Section 6 we define the MLRE and give examples, including the case of elliptical distributions and linear impact functions. Section 7 contains concluding practical remarks.

## 2. Scenario Sets Defined by Half-Space Probabilities

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathbf{X} : \Omega \to \mathbb{R}^d$ be a given random vector (of financial risk factors). For any $\mathbf{y} \in \mathbb{R}^d$ and any directional vector $\mathbf{u} \in \mathbb{R}^d \setminus \{0\}$, let

$$H_{\mathbf{y},\mathbf{u}} = \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{u}'\mathbf{x} \leqslant \mathbf{u}'\mathbf{y} \right\} ,$$

denote the closed half-space bounded by the hyperplane through $\mathbf{y}$ with normal vector $\mathbf{u}$. The probability that $\mathbf{X}$ lies in the half-space $H_{\mathbf{y},\mathbf{u}}$ will be written

$$P_{\mathbf{X}}(H_{\mathbf{y},\mathbf{u}}) = P(\mathbf{u}'\mathbf{X} \leqslant \mathbf{u}'\mathbf{y}) .$$

We will work with two notions of the $\alpha$-quantile. First we will consider it to be a *risk measure*, $\tilde{q}_\alpha$, defined on the vector space $\mathcal{M} = \left\{ \mathbf{u}'\mathbf{X} + c \ : \ \mathbf{u} \in \mathbb{R}^d,\, c \in \mathbb{R} \right\}$. For $Z \in \mathcal{M}$ and $\alpha \in (0, 1)$ we define

$$\tilde{q}_\alpha(Z) = \inf \left\{ z \ : \ P(Z \leqslant z) \geqslant \alpha \right\} .$$

The quantile risk measure fulfills the properties of monotonicity ($\tilde{q}_\alpha(Z_1) > \tilde{q}_\alpha(Z_2)$ if $Z_1 > Z_2$ almost surely), positive homogeneity ($\tilde{q}_\alpha(\lambda Z) = \lambda \tilde{q}_\alpha(Z)$ for $Z \in \mathcal{M}$ and $\lambda > 0$) and translation invariance ($\tilde{q}_\alpha(Z + a) = \tilde{q}_\alpha(Z) + a$ for $Z \in \mathcal{M}$ and $a \in \mathbb{R}$); see, for example, McNeil et al. (2005), page 238–241, for more information. The risk measure is not in general subadditive on $\mathcal{M}$ but may be under certain assumptions for the distribution of $\mathbf{X}$ and for certain values of $\alpha$. In the language of Artzner et al. (1999) the risk measure $\tilde{q}_\alpha$ is not coherent.

For convenience we also define an $\alpha$-quantile function on $\mathbb{R}^d$ by setting

$$q_\alpha(\mathbf{u}) = \tilde{q}_\alpha(\mathbf{u}'\mathbf{X}) , \quad \mathbf{u} \in \mathbb{R}^d .$$

Positive homogeneity now means that $q_\alpha(\lambda \mathbf{u}) = \lambda q_\alpha(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^d$ and $\lambda > 0$.

### 2.1. Definition of the scenario set

The main definition of a scenario set that we consider is

$$Q_\alpha = \bigcap \{H_{\mathbf{y},\mathbf{u}} \;:\; P_{\mathbf{X}}(H_{\mathbf{y},\mathbf{u}}) \geqslant \alpha\} \tag{3}$$

which is the intersection of closed half-spaces with probability equal to at least $\alpha$. Clearly $Q_\alpha$ is itself a *closed, convex* set (see also Rockafellar (1970), Theorem 2.1). The definition of the quantile function $q_\alpha(\mathbf{u})$ implies that

$$P(\mathbf{u}'\mathbf{X} \leqslant z) \geqslant \alpha \Longleftrightarrow z \geqslant q_\alpha(\mathbf{u}),\ \forall z \in \mathbb{R},$$

so we can also write the set as

$$Q_\alpha = \bigcap \{H_{\mathbf{y},\mathbf{u}} \;:\; \mathbf{u}'\mathbf{y} \geqslant q_\alpha(\mathbf{u})\}\ .$$

From this we can derive a more explicit representation of $Q_\alpha$ in terms of $q_\alpha(\mathbf{u})$ by observing that

$$\begin{aligned} Q_\alpha &= \{\mathbf{x} \;:\; \mathbf{u}'\mathbf{x} \leqslant \mathbf{u}'\mathbf{y}, \forall \mathbf{u}, \mathbf{y} : \mathbf{u}'\mathbf{y} \geqslant q_\alpha(\mathbf{u})\} \\ &= \{\mathbf{x} \;:\; \mathbf{u}'\mathbf{x} \leqslant q_\alpha(\mathbf{u}), \forall \mathbf{u}\}\ . \end{aligned}$$

### 2.2. Relation to depth sets

The sets $Q_\alpha$ have been studied by Massé and Theodorescu (1994) and are closely related to the depth sets $D_\alpha$ studied by Rousseeuw and Ruts (1999). The latter define the depth of the probability distribution of $\mathbf{X}$ at a point $\mathbf{x}$ by

$$\text{depth}(\mathbf{x}) = \inf_{\mathbf{u}:\mathbf{u}\neq\mathbf{0}} P_{\mathbf{X}}(H_{\mathbf{x},\mathbf{u}}), \tag{4}$$

the idea being that this is the "smallest" probability of a closed half-space containing $\mathbf{x}$. The depth set (or region) is the set

$$D_\alpha = \left\{\mathbf{x} \in \mathbb{R}^d \;:\; \text{depth}(\mathbf{x}) \geqslant 1 - \alpha\right\}, \tag{5}$$

containing points with depth at least $1-\alpha$ for $\alpha \geqslant 0.5$. (Here we differ slightly from Rousseeuw and Ruts who would call this set $D_{1-\alpha}$). The boundary of the set is known as a depth contour.

Rousseeuw and Ruts prove that the depth set may be written as

$$D_\alpha = \bigcap \{H_{\mathbf{y},\mathbf{u}} \;:\; P_{\mathbf{X}}(H_{\mathbf{y},\mathbf{u}}) > \alpha\},$$

which differs from $Q_\alpha$ only by the strict inequality for the probability of the half-spaces. It is clearly the case that $Q_\alpha \subset D_\alpha$, since $D_\alpha$ is the intersection of closed sets that are generally larger. The following simple example illustrates the difference.

**Example 2.1.** *Let the risk factor vector* $\mathbf{X}$ *consist of two independent Bernoulli variables with success probability* 0.5*. In other words a discrete probability mass of 0.25 is placed on each of the four points* $(0,0)$*,* $(0,1)$*,* $(1,0)$ *and* $(1,1)$*. Consider first* $Q_\alpha$*. For* $\alpha > 0.75$ *the set* $Q_\alpha$ *is given by the intersection of all of the closed half spaces that contain all four points, which is the square set with vertices* $(0,0)$*,* $(0,1)$*,* $(1,0)$ *and* $(1,1)$*. For* $\alpha \in (0.5, 0.75]$ *the set* $Q_\alpha$ *is the intersection of all closed half spaces containing at least three of four points and this is simply the single point* $(0.5, 0.5)$*. On the other hand, the set* $D_\alpha$ *is the square set for* $\alpha \geqslant 0.75$ *and the singleton* $(0.5, 0.5)$ *for* $\alpha \in [0.5, 0.75)$*. Thus the sets* $Q_\alpha$ *and* $D_\alpha$ *agree for all values in* $(0.5, 1)$ *except* $\alpha = 0.75$ *where* $Q_\alpha \subset D_\alpha$*.*

When $\mathbf{X}$ has a probability density the issue does not arise and we have $Q_\alpha = D_\alpha$ for all $\alpha \in (0.5, 1)$. In such cases we can use the concept of depth to describe $Q_\alpha$, as in the following example.

**Example 2.2.** *Let the risk factor vector* $\mathbf{X}$ *consist of two independent standard exponential random variables. Figure 1 shows the construction of depth sets* $Q_\alpha$ *by forming the intersection of closed half spaces with probability at least* $\alpha$*; the left picture shows* $Q_{0.95}$ *(or* $D_{0.95}$*) and the right picture shows* $Q_{0.72}$ *(or* $D_{0.72}$*). The interiors of the scenario sets are the "empty spaces" enclosed by the lines. The boundaries are points* $\mathbf{x}$ *which simultaneously satisfy* $\mathbf{x} \in Q_\alpha$ *and* $P_{\mathbf{X}}(H_{\mathbf{x},\mathbf{u}}) = \alpha$ *for some direction* $\mathbf{u} \neq \mathbf{0}$*. i.e. points where lines "touch" the scenario sets. In this case the set* $H_{\mathbf{x},\mathbf{u}}$ *is said to be a supporting half-space and its boundary a supporting hyperplane of* $Q_\alpha$*. Points in the scenario sets have depth greater or equal to* $(1-\alpha)$*.*

*There is an important distinction between* $Q_{0.95}$ *and* $Q_{0.72}$*. In the former case, for every direction* $\mathbf{u}$*, there exists some point* $\mathbf{x}$ *such that* $\mathbf{x} \in Q_{0.95}$ *and* $P_{\mathbf{X}}(H_{\mathbf{x},\mathbf{u}}) = 0.95$*. In other words, there exists a supporting half-space with probability 0.95.*

*In the latter case, for some directions* $\mathbf{u}$*, there is no point* $\mathbf{x}$ *such that* $\mathbf{x} \in Q_{0.72}$ *and* $P_{\mathbf{X}}(H_{\mathbf{x},\mathbf{u}}) = 0.72$*. There is no supporting half-space with normal vector* $\mathbf{u}$ *and probability 0.72. For example consider the half-space bounded by the line with normal vector* $\mathbf{u} = (1,1)'$*, which is shown in bold. The distinction has to do with the subadditivity properties of the function* $q_\alpha$*, as explained in Section 3.*

### *2.3. Analytical expression for elliptical distributions*

In this section we consider the case where $\mathbf{X}$ has an elliptical distribution, written $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$. The vector $\boldsymbol{\mu}$ describes the location of the distribution, the matrix $\Sigma$ describes its spread (and is a multiple of the covariance matrix when second moments are defined) and $\psi$ describes the type of distribution. In this case we can derive an analytical expression for $Q_\alpha$. The following is a more general formulation of a result by Massé and Theodorescu (1994) in the bivariate case and calls for some

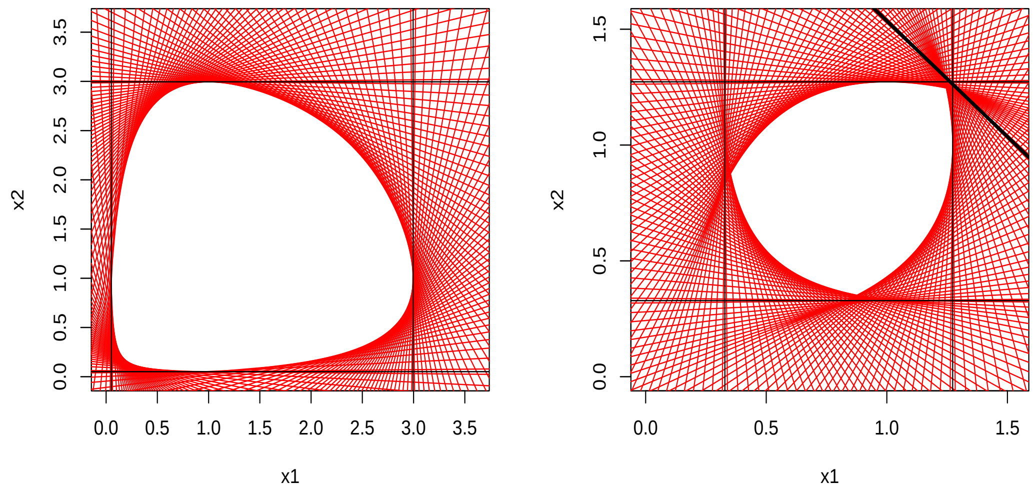


Figure 1: Scenario sets (so-called depth sets) $Q_\alpha$ in the case of the bivariate distribution of two independent standard exponential random variates; the left picture shows $Q_{0.95}$ (or $D_{0.95}$) and the right picture shows $Q_{0.72}$ (or $D_{0.72}$).

knowledge of the basic properties of elliptical distributions, which are summarised in the Appendix.

**Proposition 2.3.** *Let* $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ *be elliptically distributed with* $\Sigma$ *a positive-definite matrix. Then, for* $\alpha > 0.5$,

$$Q_\alpha = \{\mathbf{x} \ : \ (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leqslant k_\alpha^2\},$$

*where* $k_\alpha$ *denotes the* $\alpha$ *quantile of the distribution of* $(X_i - \mu_i)/\sqrt{\Sigma_{ii}}$ *for any* $i \in 1, \ldots, d$.

**Proof.** For $\alpha \in (0.5, 1)$ it follows from (A.3) and (A.4) in the Appendix, and the translation invariance and positive homogeneity of the quantile as a risk measure, that for any $\mathbf{u} \in \mathbb{R}^d$ we can write

$$q_\alpha(\mathbf{u}) = \mathbf{u}'\boldsymbol{\mu} + \sqrt{\mathbf{u}'\Sigma\mathbf{u}}k_\alpha \ . \tag{6}$$

Let $A \in \mathbb{R}^{d \times d}$ be a matrix satisfying $AA' = \Sigma$. We infer that

$$\begin{aligned} Q_\alpha &= \left\{\mathbf{x} \ : \ \mathbf{u}'\mathbf{x} \leqslant \mathbf{u}'\boldsymbol{\mu} + \sqrt{\mathbf{u}'\Sigma\mathbf{u}}\ k_\alpha, \forall \mathbf{u}\right\} \\ &= \left\{\mathbf{x} \ : \ \mathbf{u}'AA^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leqslant \|A'\mathbf{u}\|\ k_\alpha, \forall \mathbf{u}\right\} \\ &= \left\{\mathbf{x} \ : \ \mathbf{v}'\frac{A^{-1}(\mathbf{x} - \boldsymbol{\mu})}{k_\alpha} \leqslant \|\mathbf{v}\|, \forall \mathbf{v}\right\} \end{aligned}$$

where the last line follows because $\mathbb{R}^d = \{A'\mathbf{u} \;:\; \mathbf{u} \in \mathbb{R}^d\}$. By observing that the Euclidean unit ball $\{\mathbf{y} \;:\; \mathbf{y}'\mathbf{y} \leqslant 1\}$ can be written as the set $\{\mathbf{y} \;:\; \mathbf{v}'\mathbf{y} \leqslant \|\mathbf{v}\|, \forall \mathbf{v}\}$ we conclude that, for $\mathbf{x} \in Q_\alpha$, the vectors $\mathbf{y} = A^{-1}(\mathbf{x} - \boldsymbol{\mu})/k_\alpha$ describe the unit ball and therefore

$$Q_\alpha = \{\mathbf{x} \;:\; (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) \leqslant k_\alpha^2\} \,.$$

□

Thus, for elliptical distribution, the scenario sets $Q_\alpha$ are ellipsoidal sets. An example is shown in Figure 2. Note that, for elliptical distributions with densities, the sets enclosed by contours of equal density are also ellipsoidal, so scenario sets defined on the basis of half-space probabilities would coincide with scenario sets contained within contours of equal density. This is clearly not the case for all

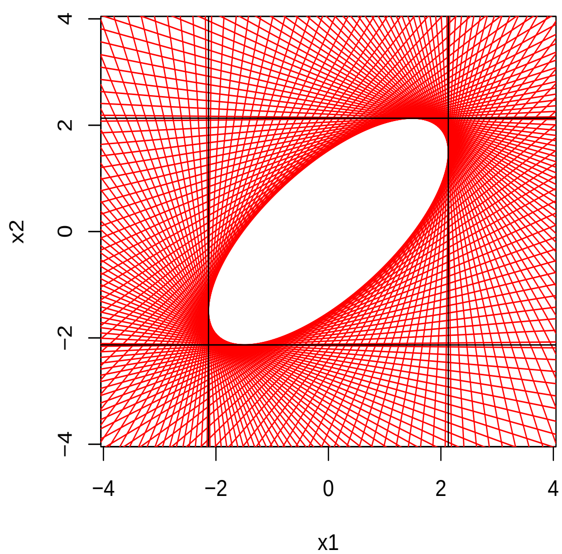


Figure 2: Scenario set $Q_{0.95}$ for the bivariate elliptical Student t distribution with degree of freedom $\nu = 4$, correlation parameter $\rho = 0.7$ and standard t marginal distributions.

multivariate distributions; in the case of the two independent exponential variates in Example 2.2 the contours of equal density are the line segments $\{\mathbf{x} : x_1 + x_2 = k, x_1 > 0, x_2 > 0\}$ for constants $k > 0$, and it is not clear how to construct scenario sets in this case.

## 3. Analysis of LSLE scenarios

To investigate the LSLE problem we will consider the linear case where $g(\mathbf{x}) = \mathbf{u}'\mathbf{x}$ for some $\mathbf{u} \in \mathbb{R}^d$. The maximisation and minimisation problems described in (1) and (2) can then be subsumed in the study of $\sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\}$, in other words, by studying the so-called *support function* of the closed convex set $Q_\alpha$.

*3.1. Convex Analysis*

For a convex set $C \in \mathbb{R}^d$ the support function is precisely the function

$$\phi(\mathbf{u}) = \sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in C\}\,.$$

The result we prove will make use of known properties of the support functions in convex analysis, for which a standard reference is Rockafellar (1970).

**Theorem 3.1.** *Let $Q_\alpha$ be the closed, convex scenario set defined in (3)and let $\phi(\mathbf{u})$ be its support function. In general we have $\phi(\mathbf{u}) \leqslant q_\alpha(\mathbf{u})$ with equality for all $\mathbf{u} \in \mathbb{R}^d$ if and only if $q_\alpha$ is subadditive, i.e. $q_\alpha(\mathbf{u}+\mathbf{v}) \leqslant q_\alpha(\mathbf{u}) + q_\alpha(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$.*

**Proof.** To establish the general inequality we note that for $\mathbf{x} \in Q_\alpha$ we have $\mathbf{u}'\mathbf{x} \leqslant q_\alpha(\mathbf{u})$ for all $\mathbf{u}$ so that

$$\phi(\mathbf{u}) = \sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\} \leqslant q_\alpha(\mathbf{u})\,.$$

If we have equality then $q_\alpha$ is a support function and its subadditivity follows from the subadditivity of support functions:

$$\sup\{(\mathbf{u}+\mathbf{v})'\mathbf{x} \,:\, \mathbf{x} \in C\} \leqslant \sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in C\} + \sup\{\mathbf{v}'\mathbf{x} \,:\, \mathbf{x} \in C\}\,.$$

Conversely assume that the quantile function is a subadditive function. A well-known result in convex analysis states that a positive homogeneous function from $\mathbb{R}^d$ to $(-\infty, \infty]$ is convex if and only if it is subadditive; see Rockafellar (1970), Theorem 4.7. Thus the quantile function is a convex function. Moreover, being finite everywhere, it is necessarily a closed convex function; see Rockafellar (1970), Corollary 7.4.2.

Such as function is the support function of a closed convex set, namely

$$\{\mathbf{x} \,:\, \mathbf{u}'\mathbf{x} \leqslant q_\alpha(\mathbf{u}), \forall \mathbf{u}\}\;;$$

see Rockafellar (1970), Corollary 13.2.1. But this is precisely the set $Q_\alpha$ which has support function $\phi$. It follows that $q_\alpha = \phi$. □

The implication of this result is that whenever $q_\alpha$ is a subadditive function and $g$ is a linear impact, the LSLE scenario as introduced in (1) is a scenario that leads to the $(1-\alpha)$-quantile of the net asset value distribution. This follows from observing that

$$\inf\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\} = -\sup\{-\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\} = -q_\alpha(-\mathbf{u}) = q_{1-\alpha}(\mathbf{u}).$$

Equivalently the LSLE is a scenario that leads to the $\alpha$-quantile of the distribution of the loss $(v_0 - \mathbf{u}'\mathbf{X})$ with respect to some initial net assets $v_0$. Or in other words, the LSLE can be thought of as a scenario attaining the Value-at-Risk.

*3.2. Subadditivity and non-subadditivity*

In any situation where

$$\sup\{\mathbf{u}'\mathbf{x} \;:\; \mathbf{x} \in Q_\alpha\} < q_\alpha(\mathbf{u}) \ . \tag{7}$$

we can construct examples to show the non-subadditivity of $q_\alpha$.

**Example 3.2.** *Consider $Q_{0.72}$ in Example 2.2. In Figure 3 (left panel) we have enlarged the picture and we now show bold lines bounding three half-spaces $H$, $H_1$ and $H_2$ each with probability $\alpha = 0.72$, such that $H_1 \cap H_2$ is a strict subset of $H$. The half-spaces have the property that their normal vectors $\mathbf{u}$, $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. The boundary of $H$ is not a supporting hyperplane for $Q_\alpha$. (Specifically we have made the choices $\mathbf{u} = (5, 8)'$, $\mathbf{u}_1 = (1, 7)'$ and $\mathbf{u}_2 = (4, 1)'$.)*

*Whenever (7) holds we can construct such a situation and in such a situation*

$$q_\alpha(\mathbf{u}_1) + q_\alpha(\mathbf{u}_2) = \sup\{\mathbf{u}_1'\mathbf{x} \;:\; \mathbf{x} \in H_1\} + \sup\{\mathbf{u}_2'\mathbf{x} \;:\; \mathbf{x} \in H_2\} = \mathbf{u}_1'\mathbf{x}_0 + \mathbf{u}_2'\mathbf{x}_0$$

*where $\mathbf{x}_0$ is the point where the lines bounding $H_1$ and $H_2$ intersect. From this it is easily seen that*

$$q_\alpha(\mathbf{u}_1) + q_\alpha(\mathbf{u}_2) = \mathbf{u}_1'\mathbf{x}_0 + \mathbf{u}_2'\mathbf{x}_0 = \mathbf{u}'\mathbf{x}_0 < \sup\{\mathbf{u}'\mathbf{x} \;:\; \mathbf{x} \in H\} = q_\alpha(\mathbf{u}_1 + \mathbf{u}_2).$$

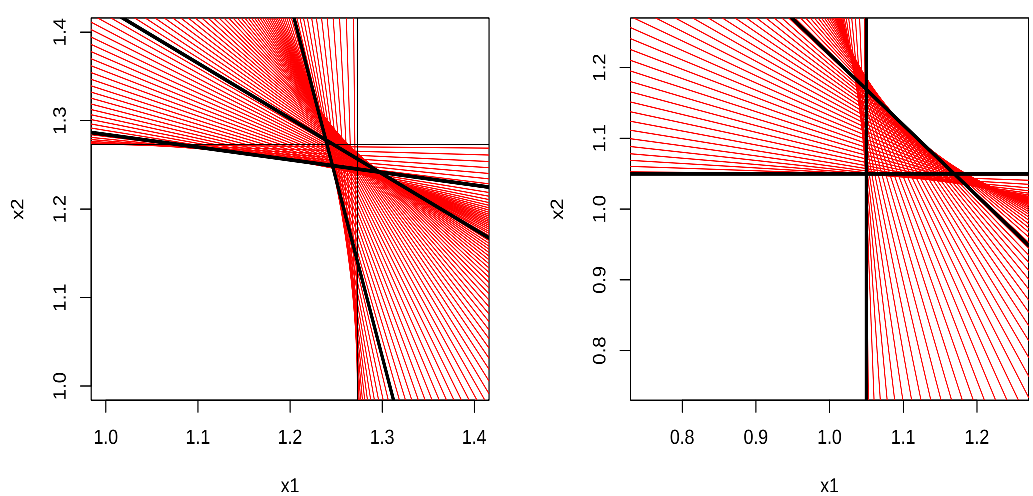


Figure 3: Example of non-subadditivity of $q_{0.72}(\mathbf{u})$ and $q_{0.65}(\mathbf{u})$ for two independent exponential variates.

*For an even easier example, when* $\alpha = 0.65$*, we can take* $\mathbf{u} = (1,1)'$, $\mathbf{u}_1 = (1,0)'$ *and* $\mathbf{u}_2 = (0,1)'$ *as shown in the right panel of Figure 3. This yields one of the simplest demonstrations of the non-subadditivity of the Value-at-Risk (VaR) risk measure: VaR is non-subadditive for two standard independent exponential risks at an* $\alpha$ *level of, say, 0.65.*

The theory clearly also applies to discrete distributions as in the next example.

**Example 3.3.** *Consider again the example of a discrete probability distribution (bivariate Bernoulli) in Example 2.1. When* $\alpha > 0.75$ *the set* $Q_\alpha$ *is the square with vertices* $(0,0)$, $(0,1)$, $(1,0)$ *and* $(1,1)$ *and for any* $\mathbf{u} \neq \mathbf{0}$ *we can find a half-plane* $H_{\mathbf{x},\mathbf{u}}$ *such that* $P(H_{\mathbf{x},\mathbf{u}}) \geqslant \alpha$ *and* $\mathbf{x} \in Q_\alpha$*. The function* $q_\alpha(\mathbf{u})$ *must be subadditive and this is easily verified because* $q_\alpha(\mathbf{u}) = u_1^+ + u_2^+$*.*

*On the other hand, when* $\alpha \in (0.5, 0.75]$ *the set* $Q_\alpha$ *is the point* $(0.5, 0.5)$ *and for certain directions* $\mathbf{u} \neq \mathbf{0}$ *it is impossible to find a half-plane* $H_{(1/2,1/2)',\mathbf{u}}$ *such that* $P(H_{(1/2,1/2)',\mathbf{u}}) \geqslant \alpha$*. For example, take* $\mathbf{u} = (1,0)'$*. The function* $q_\alpha(\mathbf{u})$ *is not subadditive in this case.*

On the other hand, whenever we know $q_\alpha$ is a subadditive function we can be certain that, for any $\mathbf{u}$, maximisation of $\mathbf{u}'\mathbf{x}$ for values of $\mathbf{x}$ in $Q_\alpha$ will yield a maximising value $\mathbf{x}_{\text{LSLE}}$ for which $\mathbf{u}'\mathbf{x}_{\text{LSLE}} = q_\alpha(\mathbf{u})$.

**Example 3.4.** *Consider the elliptical random vector* $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ *of Proposition 2.3. The quantile function is given by* $q_\alpha(\mathbf{u}) = \mathbf{u}'\boldsymbol{\mu} + \sqrt{\mathbf{u}'\Sigma\mathbf{u}}k_\alpha$ *which is clearly a subadditive function for* $\alpha > 0.5$ *because, for any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^d$*, we have*

$$\sqrt{(\mathbf{u}+\mathbf{v})'\Sigma(\mathbf{u}+\mathbf{v})} \leqslant \sqrt{\mathbf{u}'\Sigma\mathbf{u}} + \sqrt{\mathbf{v}'\Sigma\mathbf{v}}\ .$$

*For a given* $\mathbf{u} \neq \mathbf{0}$ *we can calculate the LSLE to be*

$$\mathbf{x}_{LSLE} = \arg\max\left\{\mathbf{u}'\mathbf{x}\ :\ (\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) \leqslant k_\alpha^2\right\}\ .$$

*This is a classical problem in convex optimization and is usually solved with the Kuhn-Tucker approach, i.e. the generalization of Lagrange multipliers to deal with constraints that are inequalities; see Boyd and Vandenberghe (2004). Studer (1997) provides the solution, which is*

$$\mathbf{x}_{LSLE} = \boldsymbol{\mu} + \frac{\Sigma\mathbf{u}}{\sqrt{\mathbf{u}'\Sigma\mathbf{u}}}k_\alpha\ .$$

## 4. An Outer Scenario Set when $q_\alpha$ is differentiable

We would like to be able to bound $q_\alpha$ above in situations where the subadditivity breaks down. In this section we define a further scenario set under the assumption that the quantile function $q_\alpha(\mathbf{u})$ is differentiable for all $\mathbf{u} \neq \mathbf{0}$. We write its gradient vector at $\mathbf{u}$ as $\nabla q_\alpha(\mathbf{u})$. Tasche (2000) gives sufficient conditions for the derivative to exist and to have a representation in terms of conditional expectation as

$$\nabla q_\alpha(\mathbf{u}) = E\left(\mathbf{X} \mid \mathbf{u}'\mathbf{X} = q_\alpha(\mathbf{u})\right) . \tag{8}$$

Tasche's conditions are reproduced in Assumption 1 in Appendix B and we will assume that they hold throughout this section.

We define an outer scenario set as

$$O_\alpha = \{\mathbf{x} \,:\, \mathbf{x} = \nabla q_\alpha(\mathbf{u}), \mathbf{u} \neq \mathbf{0}\} \tag{9}$$

and prove the following result.

**Theorem 4.1.** *Let $O_\alpha$ be the scenario set defined in (9) under Assumption 1 (Appendix B). Let $\psi(\mathbf{u}) = \sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\}$. In general we have $q_\alpha(\mathbf{u}) \leqslant \psi(\mathbf{u})$ with equality for all $\mathbf{u} \in \mathbb{R}^d$ if and only if $q_\alpha$ is subadditive.*

**Proof.** Since $q_\alpha$ is a positive homogeneous function, Euler's homogeneity theorem says that for all $\mathbf{u}$ at which $q_\alpha(\mathbf{u})$ is differentiable (in our case all $\mathbf{u} \neq \mathbf{0}$) we have $q_\alpha(\mathbf{u}) = \mathbf{u}'\nabla q_\alpha(\mathbf{u})$. We conclude that for all $\mathbf{u}$ the inequality

$$q_\alpha(\mathbf{u}) \leqslant \sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} = \nabla q_\alpha(\mathbf{v}), \mathbf{v} \neq \mathbf{0}\right\}$$

applies and $q_\alpha(\mathbf{u}) \leqslant \sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\right\}$ follows.

Suppose equality holds for all $\mathbf{u} \in \mathbb{R}^d$. Then $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ we have

$$\begin{aligned} q_\alpha(\mathbf{u}+\mathbf{v}) &= \sup\left\{(\mathbf{u}+\mathbf{v})'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\right\} \\ &\leqslant \sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\right\} + \sup\left\{\mathbf{v}'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\right\} = q_\alpha(\mathbf{u}) + q_\alpha(\mathbf{v}) , \end{aligned}$$

which establishes subadditivity.

Conversely, suppose $q_\alpha$ is subadditive. If we can show that $O_\alpha \subset Q_\alpha$ then clearly, by Proposition 3.1, we would have that for all $\mathbf{u} \in \mathbb{R}^d$

$$\psi(\mathbf{u}) = \sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\right\} \leqslant \sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\right\} = q_\alpha(\mathbf{u}) .$$

But we know that $q_\alpha(\mathbf{u}) \leqslant \psi(\mathbf{u})$ and so equality would follow. Suppose $\mathbf{x} \in O_\alpha$. Then $\mathbf{x} = \nabla q_\alpha(\mathbf{v})$ for some $\mathbf{v} \neq \mathbf{0}$. By Euler's homogeneity theorem, for any $\mathbf{u}$, we have

$$\mathbf{u}'\mathbf{x} = \mathbf{u}'\nabla q_\alpha(\mathbf{v}) = q_\alpha(\mathbf{v}) + (\mathbf{u}-\mathbf{v})'\nabla q_\alpha(\mathbf{v}) .$$

By the subadditivity, and hence the convexity, of $q_\alpha$ we have

$$\mathbf{u}'\mathbf{x} = q_\alpha(\mathbf{v}) + (\mathbf{u}-\mathbf{v})'\nabla q_\alpha(\mathbf{v}) \leqslant q_\alpha(\mathbf{u})$$

(see, for example, Rockafellar (1970), Theorem 25.1) which proves that $\mathbf{x} \in Q_\alpha$. □

In the proof above we have seen that, in the subadditive case, $O_\alpha \subset Q_\alpha$. In fact, in the subadditive case, it is simply an alternative description of the boundary of $Q_\alpha$. This can be seen by noting that if $\mathbf{x}$ is an interior point of $Q_\alpha$ then it satisfies $\mathbf{u}'\mathbf{x} < q_\alpha(\mathbf{u}) = \mathbf{u}'\nabla q_\alpha(\mathbf{u})$ for all $\mathbf{u}$. But if $\mathbf{x} \in O_\alpha$ then $\mathbf{x} = \nabla q_\alpha(\mathbf{u})$ for some $\mathbf{u}$, which is a contradiction. An example of the subadditive case is given by taking an elliptical distribution for $\mathbf{X}$.

**Example 4.2.** *Let $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ be elliptically distributed and assume $\mathbf{X}$ has a density. Then it is easily calculated from (6) that, for $\alpha > 0.5$,*

$$\begin{aligned} O_\alpha &= \{\mathbf{x} \,:\, \mathbf{x} = \nabla q_\alpha(\mathbf{u})\,, \mathbf{u} \neq \mathbf{0}\} \\ &= \left\{\mathbf{x} \,:\, \mathbf{x} = \boldsymbol{\mu} + \frac{\Sigma\mathbf{u}}{\sqrt{\mathbf{u}'\Sigma\mathbf{u}}}\, k_\alpha\,, \mathbf{u} \neq \mathbf{0}\right\} \end{aligned}$$

*which is simply the ellipsoid bounding $Q_\alpha$.*

The non-subadditive case is more interesting and yields sets for $O_\alpha$ which differ from the boundary of $Q_\alpha$ in interesting ways, as the following example shows.

**Example 4.3.** *In Figure 4 we show $O_\alpha$ for two independent exponential variates when $\alpha = 0.72$. In Example 3.2 we considered the half-space $\{\mathbf{x} \,:\, \mathbf{u}'\mathbf{x} \leqslant q_\alpha(\mathbf{u})\}$ with normal vector $\mathbf{u} = (5, 8)'$ and observed how $\sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\} < q_\alpha(\mathbf{u})$. Now the figure provides visual confirmation that $q_\alpha(\mathbf{u}) \leqslant \sup\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in Q_\alpha\}$.*

Returning to the subadditive case we can conclude that the outer set representation gives us both a simple way of calculating the LSLE from the quantile function and an interpretation for the LSLE.

**Corollary 4.4.** *If $q_\alpha$ is subadditive then, for given $\mathbf{u}$,*

$$\mathbf{x}_{LSLE} = \nabla q_\alpha(\mathbf{u}) = E(\mathbf{X} \mid \mathbf{u}'\mathbf{X} = q_\alpha(\mathbf{u}))\,.$$

**Proof.** If this were not true then we could find $\mathbf{x} \in O_\alpha$ such that $\mathbf{u}'\mathbf{x} > \mathbf{u}'\nabla q_\alpha(\mathbf{u}) = q_\alpha(\mathbf{u})$ which would contradict the fact that $q_\alpha(\mathbf{u}) = \sup\{\mathbf{u}'\mathbf{x} : \mathbf{x} \in O_\alpha\}$. □

The interpretation for the LSLE follows from the theory of capital allocation using the Euler principle, as developed by Denault (2001), Tasche (2004, 2008) and others. If $q_\alpha(\mathbf{u})$ is interpreted as the overall capital required by the company for solvency purposes, then the gradient vector can be interpreted as the per-unit attribution of this capital to each of the risk factors

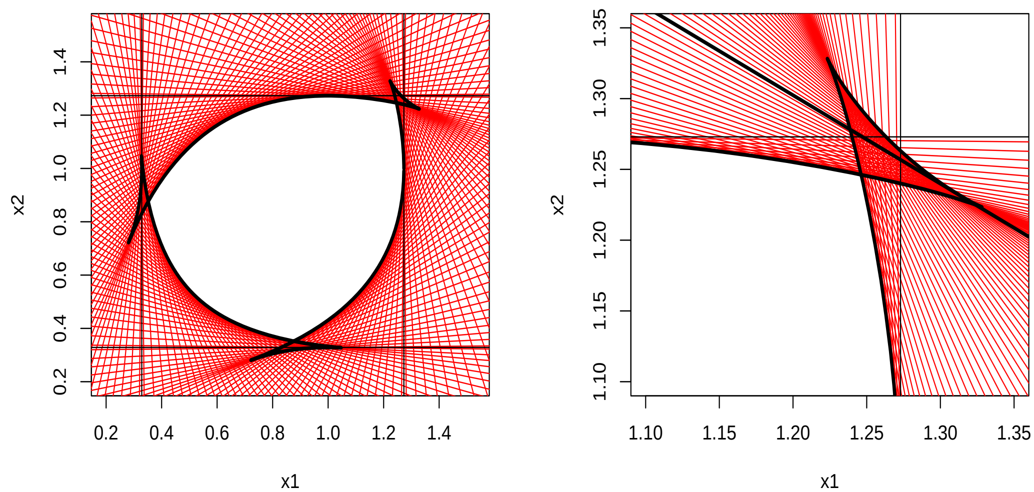


Figure 4: Scenario sets $Q_{0.72}$ and $O_{0.72}$ for two independent standard exponential random variates; the right picture magnifies the upper right corner.

## 5. Scenario Sets Defined by Coherent Risk Measures

The elegance of the theory in the case of subadditivity prompts us to consider whether we can use convex analysis to develop scenario sets that are linked to coherent risk measures. In this section we show that this is possible.

Let $\mathbf{X} : \Omega \to \mathbb{R}^d$ be given as usual and suppose $\tilde{\varrho}$ is a coherent risk measure on the vector space $\mathcal{M} = \{\mathbf{u}'\mathbf{X}+c \;:\; \mathbf{u} \in \mathbb{R}^d, c \in \mathbb{R}\}$. Since the function $\varrho(\mathbf{u}) = \tilde{\varrho}(\mathbf{u}'\mathbf{X})$ is both positive homogeneous and subadditive on $\mathbb{R}^d$ we can again use Rockafellar (1970), Corollary 13.2.1, to conclude that it is the support function of the convex set

$$C_\varrho = \{\mathbf{x} \;:\; \mathbf{u}'\mathbf{x} \leqslant \varrho(\mathbf{u}), \forall \mathbf{u}\} \ .$$

This suggests we could take $C_\varrho$ as our scenario set and be certain that the supremum of $\mathbf{u}'\mathbf{x}$ taken over this set would be equal to $\varrho(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^d$.

Moreover, if $\varrho(\mathbf{u})$ is differentiable then we can use the argumentation of the previous section to infer that the boundary of $C_\varrho$ may be written as

$$\bar{C}_\varrho = \{\mathbf{x} \;:\; \mathbf{x} = \nabla\varrho(\mathbf{u}), \mathbf{u} \neq \mathbf{0}\}$$

and that, for fixed $\mathbf{u} \neq \mathbf{0}$, the LSLE scenario for the scenario set $C_\varrho$ is given by

$$\mathbf{x}_{\mathrm{LSLE}} = \nabla\varrho(\mathbf{u}) \ ,$$

so that we can again interpret this scenario as being a per-unit capital attribution when the total capital is determined by $\varrho(\mathbf{u})$.

In the following examples we consider the function $\varrho = e_\alpha$ based on the expected shortfall risk measure, which is known to be a coherent risk measure. This function is given by

$$e_\alpha(\mathbf{u}) = \frac{\int_\alpha^1 q_\theta(\mathbf{u})d\theta}{1-\alpha}, \quad \alpha \in (0.5, 1)$$

and we write the scenario set as $E_\alpha = \{\mathbf{x} \,:\, \mathbf{u}'\mathbf{x} \leqslant e_\alpha(\mathbf{u}), \forall \mathbf{u}\}$.

Assuming again that Tasche's Assumption 1 in Appendix B holds, we have differentiability of $e_\alpha$ and the representation

$$e_\alpha(\mathbf{u}) = E\left(\mathbf{u}'\mathbf{X} \mid \mathbf{u}'\mathbf{X} \geqslant q_\alpha(\mathbf{u})\right) \ .$$

The LSLE is

$$\mathbf{x}_{\text{LSLE}} = \nabla e_\alpha(\mathbf{u}) = E\left(\mathbf{X} \mid \mathbf{u}'\mathbf{X} \geqslant q_\alpha(\mathbf{u})\right) \ .$$

**Example 5.1.** *If $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ is elliptically distributed and $\Sigma$ is a positive-definite matrix then the scenario set is simply the ellipsoidal set*

$$E_\alpha = \{\mathbf{x} \,:\, (\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) \leqslant l_\alpha^2\},$$

*where $l_\alpha$ is the result of applying the expected shortfall risk measure to $(X_i - \mu_i)/\sqrt{\Sigma_{ii}}$ for any $i \in 1, \ldots, d$. This is easily seen by noting that $e_\alpha(\mathbf{u}) = \mathbf{u}'\boldsymbol{\mu} + \sqrt{\mathbf{u}'\Sigma\mathbf{u}}l_\alpha$; see also Proposition 2.3. The LSLE is given by*

$$\mathbf{x}_{LSLE} = \boldsymbol{\mu} + \frac{\Sigma\mathbf{u}}{\sqrt{\mathbf{u}'\Sigma\mathbf{u}}}\, l_\alpha \ .$$

**Example 5.2.** *If $\mathbf{X}$ consists of two independent standard exponential variables as in Example 2.2 then, even for $\alpha = 0.65$ where the subadditivity of $q_\alpha$ broke down, we can define the set $E_\alpha$ and obtain a closed convex set with a smooth boundary that has supporting half-spaces $\{\mathbf{u}'\mathbf{x} \leqslant e_\alpha(\mathbf{u})\}$ for any direction $\mathbf{u} \neq \mathbf{0}$. See Figure 5 for an illustration.*

## 6. Most Likely Ruin Events

Let $R$ be a subset of the support of the distribution of $\mathbf{X}$ containing outcomes that will lead to insolvency (i.e. outcomes for which $g(\mathbf{x}) \leqslant 0$). We return to the ideas of half-space trimming and depth, as explained in Section 2, to give a possible definition for the most likely ruin event(s) or MLRE scenario(s). Assuming, for simplicity, that $R$ is a closed set, we define

$$\mathbf{x}_{\text{MLRE}} = \arg\max\left\{\text{depth}(\mathbf{x}) \,:\, \mathbf{x} \in R\right\} \ , \tag{10}$$

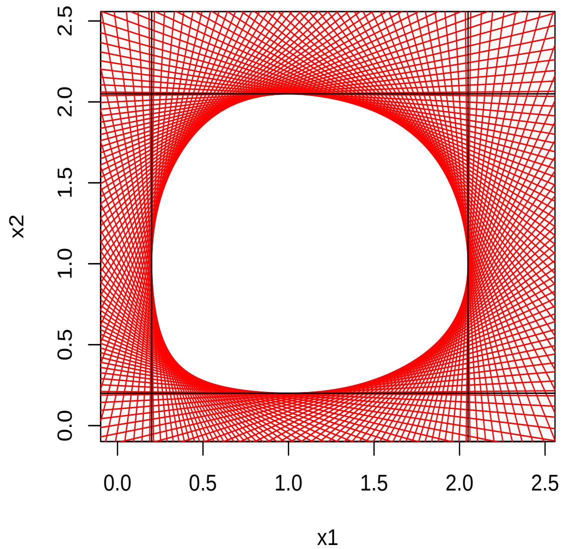


Figure 5: Scenario set $E_{0.65}$ in the case of two independent standard exponential variables.

where depth($\mathbf{x}$) is defined in (4). The MLRE scenario is the scenario in the failure region that is "deepest" into the probability distribution of $\mathbf{X}$ and which is, in that sense, most plausible; there may of course be more than one such scenario.

The problem of determining a MLRE can be thought of as finding the smallest depth set $D_\alpha$ such that $D_\alpha \cap R \neq \emptyset$. The MLRE scenarios are the scenarios in the intersection. (Recall that the sets $D_\alpha$ and $Q_\alpha$ coincide when $\mathbf{X}$ has a probability density. Henceforth, in this section, we will assume that this is the case.)

For illustration suppose that $g$ is the linear function $g(\mathbf{x}) = v_0 - \mathbf{u}'\mathbf{x}$ for some $\mathbf{u} \neq \mathbf{0}$, so that large risk factor values lead to ruin. The ruin set is

$$R = \{\mathbf{x} \,:\, v_0 - \mathbf{u}'\mathbf{x} \leqslant 0\} = \{\mathbf{x} \,:\, \mathbf{u}'\mathbf{x} \geqslant v_0\}\,, \tag{11}$$

which is the half-space $H_{\mathbf{y},-\mathbf{u}}$ for any $\mathbf{y}$ satisfying $\mathbf{u}'\mathbf{y} = v_0$. To find the MLRE we look for the set $D_\alpha$ for which $R$ is a supporting half-space and the line $\mathbf{u}'\mathbf{y} = v_0$ a supporting hyperplane. The MLRE is then the point of the hyperplane in $D_\alpha$. We give two examples of the construction.

**Example 6.1.** *The left panel of Figure 6 shows the shaded failure region* $25 - 3x_1 - 5x_2 \leqslant 0$ *and the depth set* $D_\alpha$ *for two independent exponential risk factors for which the failure region forms a supporting half-space; we obtain* $\alpha \approx 0.9835$*. The MLRE scenario is marked by a small circular point.*

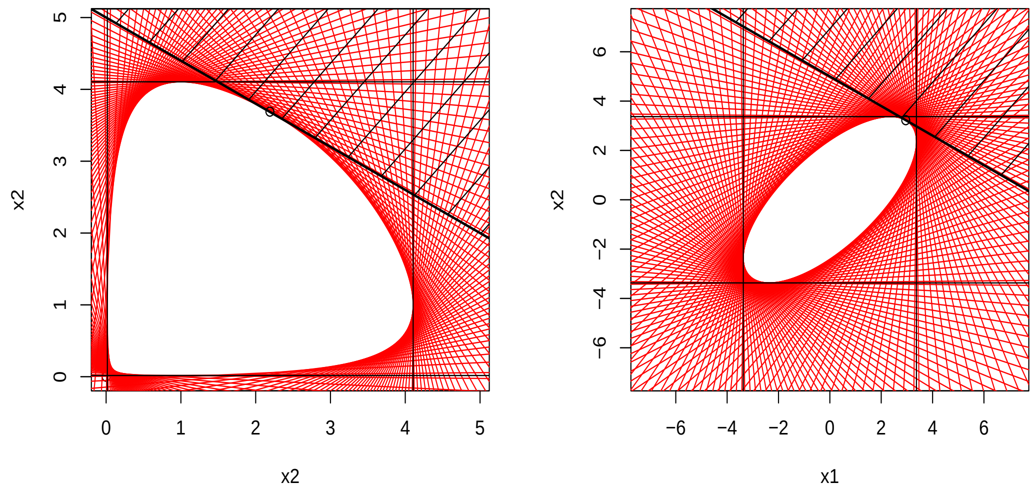


Figure 6: Both plots show the shaded failure region $R$ in (11) when $v_0 = 25$ and $\mathbf{u} = (3, 5)'$, and depth sets for which $R$ is a supporting half-space. The left panel relates to independent exponential risk factors; see Example 6.1. The right panel relates to a bivariate $t$ distribution; see Example 6.2. In both cases the MLRE scenario is marked by a small circular point.

**Example 6.2.** *We now consider the case where $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ has an elliptical distribution with a density and the failure region is the generic half-space (11). The depth maximization problem in (10) is equivalent to solving*

$$\mathbf{x}_{MLRE} = \arg\min \left\{ (\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \ : \ \mathbf{u}'\mathbf{x} \geqslant v_0 \right\} \ .$$

*This is again a simple convex optimization problem which can be solved by the method of Lagrange multipliers (or using the Kuhn-Tucker conditions) to obtain*

$$\mathbf{x}_{MLRE} = \boldsymbol{\mu} + \frac{\Sigma \mathbf{u}(v_0 - \mathbf{u}'\boldsymbol{\mu})}{\mathbf{u}'\Sigma\mathbf{u}} \ .$$

*The right panel of Figure 6 again shows the shaded failure region obtained when $v_0 = 25$ and $\mathbf{u} = (3, 5)'$. The depth set $D_\alpha$ for which the failure region is a supporting half-space is now calculated for the bivariate elliptical Student $t$ distribution underlying Figure 2; we obtain $\alpha \approx 0.9860$. The MLRE scenario is marked by a small circular point.*

## 7. Conclusion

In this paper we have set out a framework for computing forward and reverse stress tests for plausible scenario sets based on the concepts of half-space trimming and depth.

We have shown that when the impact of risk factors can be linearized there are elegant connections between stress testing and the theories of coherent risk measures and capital allocation (or attribution). These connections may hold under slightly more general assumptions for the risk impact function and this is a subject of further research.

Computation of least solvent likely events (LSLEs) and most likely ruin events (MLREs) is a simple analytical matter when distributions are elliptical and impacts are linear. In other cases numerical optimization may be required. Note, however, that the LSLE problem is a problem in convex optimization when the impact function $g$ is a convex function and the MLRE problem is a problem in convex optimization when depth is a convex function for a particular probability distribution and the insolvency region is a convex set. In such cases, efficient optimization methods are often available; see Boyd and Vandenberghe (2004).

There is an argument for working with elliptical approximations to the risk factor distribution, given the relative tractability of the elliptical case. Thus our analyses give support to the procedures developed by Studer (1997) and by Breuer et al. (2009). However, we provide clarity about how to determine the size of the elliptical sets using the concepts of depth and half-space trimming.

## Appendix A. Elliptical Distributions

If $\mathbf{X}$ has the elliptical distribution denoted

$$\mathbf{X} \sim \mathrm{E}_d(\boldsymbol{\mu}, \Sigma, \psi).$$

for some vector $\boldsymbol{\mu} \in \mathbb{R}^d$, some positive definite matrix $\Sigma \in \mathbb{R}^{d \times d}$, and some so-called generator function $\psi$, we mean by this that if $A \in \mathbb{R}^{d \times d}$ is a matrix satisfying $AA' = \Sigma$ then the random vector $\mathbf{Y} = A^{-1}(\mathbf{X} - \boldsymbol{\mu})$ has a *spherical distribution*, written $\mathbf{Y} \sim S_d(\psi)$.

This in turn means that there exists a function $\psi$ of a scalar variable, known as the characteristic generator, such that, for all $\mathbf{t} \in \mathbb{R}^d$,the characteristic function of $\mathbf{Y}$ can be written as

$$\phi_{\mathbf{Y}}(\mathbf{t}) = E\left(e^{i\mathbf{t}'\mathbf{Y}}\right) = \psi(\mathbf{t}'\mathbf{t}) = \psi(t_1^2 + \ldots + t_d^2). \tag{A.1}$$

The characteristic function of $\mathbf{X}$ is given by

$$\phi_{\mathbf{X}}(\mathbf{t}) \;=\; E\left(e^{i\mathbf{t}'\mathbf{X}}\right) = E\left(e^{i\mathbf{t}'(\boldsymbol{\mu}+A\mathbf{Y})}\right) = e^{i\mathbf{t}'\boldsymbol{\mu}} E\left(e^{i(A'\mathbf{t})'\mathbf{Y}}\right) = e^{i\mathbf{t}'\boldsymbol{\mu}}\psi(\mathbf{t}'\Sigma\mathbf{t})\,.$$

For every orthogonal mapping $U \in \mathbb{R}^{d \times d}$ (i.e. every mapping satisfying $UU' = U'U = \mathrm{I}_d$) it may be easily shown that spherically distributed random vectors satisfy

$$U\mathbf{Y} \stackrel{\mathrm{d}}{=} \mathbf{Y}\,.$$

Moreover, a further characterizing property of spherical random vectors, which we use in the paper, is the fact that for every $\mathbf{a} \in \mathbb{R}^d$,

$$\mathbf{a}'\mathbf{Y} \stackrel{\text{d}}{=} ||\mathbf{a}||Y_1, \tag{A.2}$$

where $||\mathbf{a}||^2 = \mathbf{a}'\mathbf{a} = a_1^2 + \cdots + a_d^2$. In other words linear combinations of spherical random vectors have the same distributional type as the marginal distributions.

It follows from (A.2) that, for any $\mathbf{a} \in \mathbb{R}^d$, an elliptical random vector $\mathbf{X}$ satisfies

$$\mathbf{a}'\mathbf{X} \stackrel{\text{d}}{=} \mathbf{a}'\boldsymbol{\mu} + \sqrt{\mathbf{a}'\Sigma\mathbf{a}}Y_1 \tag{A.3}$$

where $Y_1$ is a component of the spherical random vector $\mathbf{Y} = A^{-1}(\mathbf{X} - \boldsymbol{\mu})$. We can further deduce that, for any $i \in 1, \ldots, d$,

$$\frac{X_i - \mu_i}{\sqrt{\Sigma_{ii}}} \stackrel{\text{d}}{=} Y_1 \tag{A.4}$$

where $\mu_i$ is the $i$th component of $\boldsymbol{\mu}$ and $\Sigma_{ii}$ is the $i$th diagonal element of $\Sigma$.

## Appendix B. Differentiability of Quantile Function

Tasche (2000) gives a set of conditions for the derivative of the quantile function to exist and to have the conditional expectation representation in (8).

**Assumption 1 (Tasche).** *For the random vector $\mathbf{X} = (X_1, \ldots, X_d)'$ assume that the conditional density $f_{X_1|X_2,\ldots,X_d}(t, x_2, \ldots, x_d)$ of $X_1$ given $(X_2, \ldots, X_d)$ exists. Assume moreover that in an open set $U \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1}$ the following three conditions hold:*

1. *For fixed $x_2, \ldots, x_d$ the conditional density $f_{X_1|X_2,\ldots,X_d}(t, x_2, \ldots, x_d)$ is continuous in $t$.*
2. *The mapping*

$$(t, \mathbf{u}) \mapsto E\left(f_{X_1|X_2,\ldots,X_d}\left(\frac{t - \sum_{j=2}^d u_j X_j}{u_1}, X_2, \ldots, X_d\right)\right), \quad \mathbb{R} \times U \to [0, \infty)$$

*is finite-valued and continuous.*

3. *For each $i = 2, \ldots, d$ the mapping*

$$(t, \mathbf{u}) \mapsto E\left(X_i f_{X_1|X_2,\ldots,X_d}\left(\frac{t - \sum_{j=2}^d u_j X_j}{u_1}, X_2, \ldots, X_d\right)\right), \quad \mathbb{R} \times U \to \mathbb{R}$$

*is finite-valued and continuous.*

**Acknowledgement**

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