From Archimedean to Liouville Copulas

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Abstract

We use a recent characterization of the d-dimensional Archimedean copulas as the survival copulas of d-dimensional simplex distributions (McNeil and Nešlehová (2009)) to construct new Archimedean copula families, and to examine the relationship between their dependence properties and the radial parts of the corresponding simplex distributions. In particular, a new formula for Kendall's tau is derived and a new dependence ordering for non-negative random variables is introduced which generalises Laplace transform order. We then generalise the Archimedean copulas to obtain Liouville copulas, which are the survival copulas of Liouville distributions and which are non-exchangeable in general. We derive a formula for Kendall's tau of Liouville copulas in terms of the radial parts of the corresponding Liouville distributions.

Key words: Archimedean copula, simplex distribution, ℓ_1-norm symmetric distribution, Liouville distribution, Kendall's tau, Williamson d-transform, Laplace transform, stochastic ordering, dependence ordering, stochastic simulation

1 Introduction

The class of Archimedean copulas in dimension d is shown in McNeil and Nešlehová (2009) to be identical to the class of survival copulas of d-dimensional ℓ_1-norm symmetric distributions, also known as simplicially contoured or *simplex distributions*; see Fang and Fang (1988) and Fang et al. (1990). In other words Archimedean copulas are survival copulas of random vectors $\mathbf{X}$ with the stochastic structure $\mathbf{X} \stackrel{\mathrm{d}}{=} R\mathbf{S}_d$ where $\mathbf{S}_d$ is a random vector distributed uniformly on the unit simplex $\mathcal{S}_d = \{\boldsymbol{x} \in \mathbb{R}^d_+ : \|\boldsymbol{x}\|_1 = 1\}$, and R is an independent, non-negative random variable.

A simple interpretation of these models is to regard the random variable R as an amount of resources to be shared among d agents. The sharing is performed in a random but equitable way so that the proportions received by the d agents are given by the components of the random vector $\boldsymbol{S}_d$. Thus the total amount of resources received by agent i is $X_i = RS_i$ for $i = 1, \ldots, d$.

The first aim of this paper is to investigate the relationship between the distribution of R (the so-called radial part of the simplex distribution) and the dependence properties of the vector $\boldsymbol{X}$. To provide illustrations we introduce a number of new Archimedean copula families based on particular radial distributions for the "resources" R, including gamma, inverse gamma, Pareto and inverse Pareto. In particular, we derive a new expression for the Kendall's rank correlation coefficient in terms of R and give results that show how stochastic ordering relationships for a quantity we term the "ratio of radials" (R/R^* where R^* is an independent copy of R) are reflected in the ordering of Kendall's rank correlation coefficients. We also characterize the radial distributions that give rise to the mixed exponential subclass of simplex distributions, whose survival copulas are precisely those that appear in shared frailty models in survival analysis.

The second aim of the paper is to understand what happens if we relax the assumption of uniformity for the distribution of $\boldsymbol{S}_d$ and consider sharing the resources according to a more general Dirichlet distribution. In other words we consider the survival copulas induced by distributions with the structure $\mathbf{X} \stackrel{\mathrm{d}}{=} R\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$ where $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$ denotes a vector with Dirichlet distribution and parameters $(\alpha_1, \ldots, \alpha_d)$. The random vector $\mathbf{X}$ is said to have a *Liouville distribution* (see again Fang et al. (1990)) and we refer to the class of survival copulas of Liouville distributions as *Liouville copulas.* Obviously in looking at Liouville copulas we are able to move away from the exchangeable dependence structures that tend to limit the applicability of Archimedean copulas.

We pay particular attention to the case when the parameters of $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$ are positive integers since this corresponds to the idea that agents in our original equitable sharing model band together to pool their resources. For example, suppose that $\mathbf{X} \stackrel{\mathrm{d}}{=} R\mathbf{S}_3$ and agents 1 and 2 form a coalition and pool their resources. In effect we now consider the random vector $\mathbf{Y} = (Y_1, Y_2)$, where $Y_1 = X_1 + X_2$ and $Y_2 = X_3$, which has the stochastic representation $\mathbf{Y} \stackrel{\mathrm{d}}{=} R\mathbf{D}_{(2,1)}$. The Liouville distributions with integer Dirichlet parameters turn out to have survival copulas that are relatively tractable and we provide a number of examples.

The paper is structured as follows. In Section 2 we set out the main ideas of the theory developed in McNeil and Nešlehová (2009), introduce four new copula families, which will serve as illustrations in this paper, and characterise the subclass of Archimedean copulas that appear in frailty copulas. The relationship between the radial part R and the dependence properties of Archimedean copulas is explored in Section 3, which contains the new formula for Kendall's tau and a new stochastic order for non-negative random variables which we term Williamson d-transform order. Section 4 introduces Liouville copulas. We reveal a link to the accelerated failure model in survival analysis and we also obtain an explicit formula for Kendall's tau for these copulas. More involved proofs are deferred to Appendices A and B.

2 Simplex distributions and Archimedean copulas

Copulas are multivariate distribution functions with uniform margins. By the theorem of Sklar (1959) they appear implicitly in any multivariate distribution as linking functions that join the marginal distributions together to form joint distribution functions. A version of Sklar's Theorem for survival functions says that if $\bar{F}$ is a d-dimensional survival function with marginal survival functions $\bar{F}_i$, $i = 1, \dots, d$, then there exists a copula C, referred to as the *survival copula* of $\bar{F}$, such that, for any $\boldsymbol{x} \in \mathbb{R}^d$,

$$\bar{F}(\boldsymbol{x}) = C(\bar{F}_1(x_1), \dots, \bar{F}_d(x_d)).$$

Furthermore, if $\bar{F}_i$, $i = 1, \dots, d$ are continuous, C is unique and equals

$$C(\boldsymbol{u}) = \bar{F}(\bar{F}_1^{-1}(u_1), \dots, \bar{F}_d^{-1}(u_d))$$

where $\bar{F}_i^{-1}(u_i) = \inf\{x : \bar{F}_i(x) \leqslant u_i\}$, $i = 1, \dots, d$.

The Archimedean copulas are a popular family of copulas with the simple form

$$C(u_1, \dots, u_d) = \psi(\psi^{-1}(u_1) + \dots + \psi^{-1}(u_d)), \quad (u_1, \dots, u_d) \in [0,1]^d, \tag{1}$$

where $\psi : [0, \infty) \to [0, 1]$ is a *generator function* which satisfies $\lim_{x\to\infty} \psi(x) = 0$, $\psi(0) = 1$ and which is strictly decreasing on $[0, \inf\{x : \psi(x) = 0\})$.

A classical result by Kimberling (1974) shows that (1) defines a distribution in *any* dimension d if and only if ψ is a *completely monotone* function, so that is has derivatives of all orders which alternate in sign. From Bernstein's theorem (see Widder (1946)) it follows that completely monotone generators can be characterized as the Laplace-Stieltjes transforms of probability distributions on the positive real numbers that place no point mass at zero.

McNeil and Nešlehová (2009) show that (1) defines a distribution in a given dimension d if and only if ψ is *d-monotone*. A generator ψ is d-monotone if it is differentiable up to order $d-2$ on $(0, \infty)$ with derivatives satisfying

$$(-1)^k \psi^{(k)}(x) \geqslant 0, \quad k = 0, 1, \dots, d-2$$

and if $(-1)^{d-2}\psi^{(d-2)}$ is non-increasing and convex on $(0, \infty)$. The d-monotone generators can be characterized as the class of *Williamson d-transforms* of probability distributions on the positive real numbers that place no point mass at zero. This integral transform, described in Williamson (1956), is defined as follows.

Definition 1 *If R is a non-negative random variable with distribution function F_R satisfying $F_R(0) = 0$ and $d \geqslant 2$ is an integer, then the Williamson d-transform of F_R (or R) is a real function on $[0, \infty)$ given by*

$$\mathfrak{W}_d F(x) = \int_{(x,\infty)} \left(1 - \frac{x}{t}\right)^{d-1} dF_R(t) = \mathrm{E}\left(1 - \frac{x}{R}\right)_+^{d-1} \quad \text{if} \quad x \geqslant 0.$$

The distribution of a non-negative random variable is uniquely given by its Williamson d-transform, and the distribution function can be recovered using a convenient inversion formula. If $\psi = \mathfrak{W}_d F$ then, for $x \in [0, \infty)$, $F(x) = \mathfrak{W}_d^{-1}\psi(x)$ where

$$\mathfrak{W}_d^{-1}\psi(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k \psi^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} \psi_+^{(d-1)}(x)}{(d-1)!}.$$

If ψ is d-times differentiable, $F(x) = \mathfrak{W}_d^{-1}\psi(x)$ has a density is given by

$$f(x) = \frac{(-1)^d x^{d-1} \psi^{(d)}(x)}{(d-1)!}, \quad x \in (0, \infty). \tag{2}$$

The Laplace transform $\mathcal{L}$ can be thought of as a limiting Williamson d-transform as $d \to \infty$ according to the following lemma.

Lemma 1 *Let R be a positive random variable with distribution function F_R satisfying $F_R(0) = 0$. Then $\lim_{d\to\infty} \mathfrak{W}_d F_{dR}(x) = \lim_{d\to\infty} \mathfrak{W}_d F_R(x/d) = \mathcal{L}F_{1/R}(x)$.*

Proof. Fix an $x \geqslant 0$ and write

$$\mathfrak{W}_d F_{dR}(x) = E\left(1 - \frac{x}{dR}\right)_+^{d-1} = \mathfrak{W}_d F_R\left(\frac{x}{d}\right) = \int_0^\infty \left(1 - \frac{x}{rd}\right)_+^{d-1} dF_R(r)$$

For fixed $x \geqslant 0$ and $r > 0$ we have that $(1 - x/rd)_+^{d-1} \to \exp(-x/r)$ as $d \to \infty$ from which the result follows. □

The key result in McNeil and Nešlehová (2009) links simplex distributions to Archimedean copulas via the Williamson d-transform. Recall that a d-dimensional random vector $\boldsymbol{X}$ is said to have a simplex distribution if $\mathbf{X} \overset{\mathrm{d}}{=} R\mathbf{S}_d$ where $\mathbf{S}_d$ is a random vector distributed uniformly on the unit simplex $\mathcal{S}_d = \{\boldsymbol{x} \in \mathbb{R}_+^d \ : \ \|\boldsymbol{x}\|_1 = 1\}$, and R is an independent, non-negative, scalar random variable. F_R, the distribution function of R, is referred to as the radial distribution.

Theorem 1 *The following statements hold:*

(i) *If $\boldsymbol{X}$ has a simplex distribution with radial distribution F_R satisfying $F_R(0) = 0$, then $\mathbf{X}$ has an Archimedean survival copula with generator $\psi = \mathfrak{W}_d F_R$.*
(ii) *If $\boldsymbol{U}$ is distributed as an Archimedean copula C with generator ψ, then $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))$ has a simplex distribution with radial distribution $F_R = \mathfrak{W}_d^{-1}\psi$.*

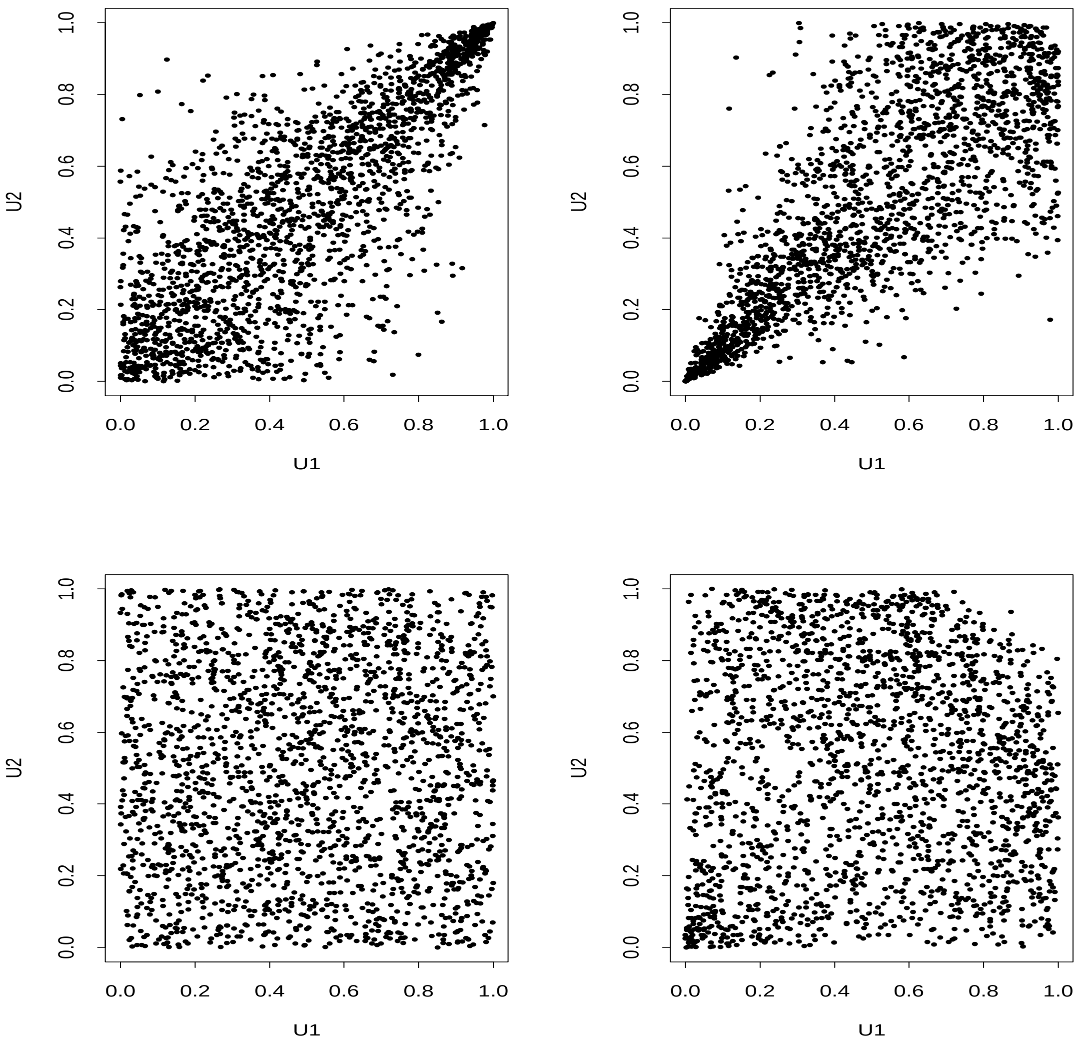

Fig. 1. Left pictures show bivariate gamma-simplex copulas while right pictures show inverse-gamma-simplex copulas. Upper copulas have $\theta = 0.3$; lower pictures have $\theta = 2$.

2.1 Some new copula families

We give examples of the use of part (i) of Theorem 1 to create new copula families that have not hitherto been considered in the literature.

Example 1 (gamma-simplex copulas) *Consider a simplex distribution with a gamma-distributed radial part, $R \sim \mathrm{Ga}(\theta)$. The radial density is $f_R(r) = r^{\theta-1}e^{-r}/\Gamma(\theta)$, for $r > 0$ and $\theta > 0$. Note that it suffices to consider a one-parameter gamma family with no scaling parameter, since, for any $k > 0$, the radial variables R and kR give rise to Williamson d-transforms that generate the same Archimedean copula.*

The generator $\psi_{\theta,d}$ of the survival copula $C_{\theta,d}$ of the simplex distribution is given by

$$\psi_{\theta,d}(x) = \mathfrak{W}_d F_R(x) = \int_x^\infty \frac{(1-x/r)^{d-1}\, e^{-r} r^{\theta-1}}{\Gamma(\theta)} dr = \int_x^\infty \frac{r^{\theta-d}(r-x)^{d-1}e^{-r}}{\Gamma(\theta)} dr$$

for $x \geqslant 0$. If $\theta = d$ this simplifies to $\psi_{d,d}(x) = e^{-x}$ so that $C_{d,d}$ is the independence copula. In the general case, applying the binomial theorem to $(r-x)^{d-1}$ yields

$$\begin{aligned}\psi_{\theta,d}(x) = \int_x^\infty \frac{r^{\theta-d}(r-x)^{d-1}e^{-r}}{\Gamma(\theta)}\, dr &= \sum_{k=0}^{d-1} \binom{d-1}{k} \int_x^\infty \frac{r^k(-x)^{d-1-k} r^{\theta-d} e^{-r}}{\Gamma(\theta)}\, dr \\ &= \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-x)^{d-1-k}}{\Gamma(\theta)} \int_x^\infty r^{k+\theta-d} e^{-r}\, dr,\end{aligned}$$

which can be rewritten as

$$\psi_{\theta,d}(x) = \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-1)^{d-1-k} x^{d-1-k}}{\Gamma(\theta)} \Gamma(k-d+\theta+1, x), \tag{3}$$

where $\Gamma(k,x) = \int_x^\infty t^{k-1} e^{-t}\, dt$ denotes the (upper) incomplete gamma function. For example, in the bivariate case we get

$$\psi_{\theta,2}(x) = \frac{\Gamma(\theta,x)}{\Gamma(\theta)} - \frac{x\Gamma(\theta-1,x)}{\Gamma(\theta)}\ ;$$

see Figure 1 for examples of random samples from this copula family.

While the generator in (3) can be evaluated as a sum involving incomplete gamma functions, it is worth noting that for large d it can also be approximated by a Laplace transform. The implication of Lemma 1 is that, for d large enough,

$$\psi_{\theta,d}(x) = \mathfrak{W}_d F_R(x) \approx \mathcal{L}F_{d/R}(x) = \mathcal{L}F_{1/R}(dx),$$

so that we can approximate the generator by computing the Laplace transform of $1/R$ at (dx). The Laplace transform of inverse-gamma can be computed by treating it as a special case of a generalised inverse Gaussian (GIG) distribution (see Appendix of McNeil et al. (2005), for example). We obtain, for d large,

$$\psi_{\theta,d}(x) \approx \frac{2(dx)^{\theta/2} K_\theta\left(2\sqrt{dx}\right)}{\Gamma(\theta)},$$

where K_θ is a modified Bessel function of the third kind. ■

Remark 1 (Erlang-simplex copulas and independence) *We observed in Example 1 that when $R \sim \mathrm{Ga}(d)$, then the generator of the gamma-simplex distribution is $\psi_{d,d}(x) = \exp(-x)$, so that the survival copula is the independence copula. Recall that R is also said to have an Erlang distribution. In general, it is* *easy to see that if* $\psi = \mathfrak{W}_d F_R$ *for some positive random variable* R *satisfying* $F_R(0) = 0$*, then* ψ *generates the independence copula if and only if* $R \stackrel{d}{=} (Z_d/k)$ *where* $Z_d \sim \mathrm{Erlang}(d)$ *and* $k > 0$ *is a constant: if* $R \stackrel{d}{=} (Z_d/k)$ *we easily calculate that* $\psi(x) = \psi_{d,d}(kx) = \exp(-kx)$*; on the other hand, the only Archimedean generators that yield the independence copula are given by* $\psi(x) = \exp(-kx)$ *for* $k > 0$*. The uniqueness of the Williamson d transform means that these can only correspond to radial distributions that are scaled Erlang(d) distributions.*

Example 2 (inverse-gamma-simplex copulas) *Suppose* $1/R \sim \mathrm{Ga}(\theta)$ *for some* $\theta > 0$*; this means that* R *is inverse-gamma with density* $f_R(r) = r^{-\theta-1}e^{-1/r}/\Gamma(\theta)$*, for* $r > 0$*. The generator of the survival copula of the simplex distribution is*

$$\psi_{\theta,d}(x) = \mathfrak{W}_d F_R(x) = \int_x^\infty \frac{(1-x/r)^{d-1}\, e^{-1/r} r^{-\theta-1}}{\Gamma(\theta)} dr = \int_x^\infty \frac{r^{-\theta-d}(r-x)^{d-1}e^{-1/r}}{\Gamma(\theta)} dr\,.$$

Applying the binomial theorem to $(r-x)^{d-1}$ *and changing variables to* $s = 1/r$ *yields*

$$\begin{aligned}
\psi_{\theta,d}(x) &= \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-x)^{d-1-k}}{\Gamma(\theta)} \int_x^\infty r^{k-\theta-d} e^{-1/r}\, dr \\
&= \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-x)^{d-1-k}}{\Gamma(\theta)} \int_0^{1/x} s^{(d+\theta-k-1)-1} e^{-s}\, ds \\
&= \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-1)^{d-1-k} x^{d-1-k}}{\Gamma(\theta)} \gamma(d+\theta-k-1, 1/x), \qquad (4)
\end{aligned}$$

where $\gamma(k,x) = \int_0^x t^{k-1}e^{-t}\,dt$ *denotes the (lower) incomplete gamma function. For example, in the bivariate case we get*

$$\psi_{\theta,2}(x) = \frac{\gamma(\theta, 1/x)}{\Gamma(\theta)} - \frac{x\gamma(\theta+1, 1/x)}{\Gamma(\theta)}\;;$$

see Figure 1 for examples of random samples from this copula family. ■

Example 3 (Pareto-simplex copulas) *Suppose* R *is Pareto with distribution function* $F_R(r) = 1 - r^{-\kappa}$ *for* $r \geqslant 1$ *and* $\kappa > 0$*. The survival copula of the d-dimensional simplex distribution has generator*

$$\begin{aligned}
x) = \mathfrak{W}_d F_R(x) &= \kappa \int_x^\infty \left(1 - \frac{x}{r}\right)^{d-1} r^{-(\kappa+1)} I(r \geqslant 1) dr \\
&= \kappa x^{-\kappa} \int_0^1 z^{\kappa-1}(1-z)^{d-1} I(z \leqslant x) dz \\
&= \kappa x^{-\kappa} B(\min(x,1), \kappa, d)\;, \qquad (5)
\end{aligned}$$

where $B(x, \alpha, \beta)$ *denotes the incomplete beta function; see Figure 2 for examples of samples from this copula family.* ■

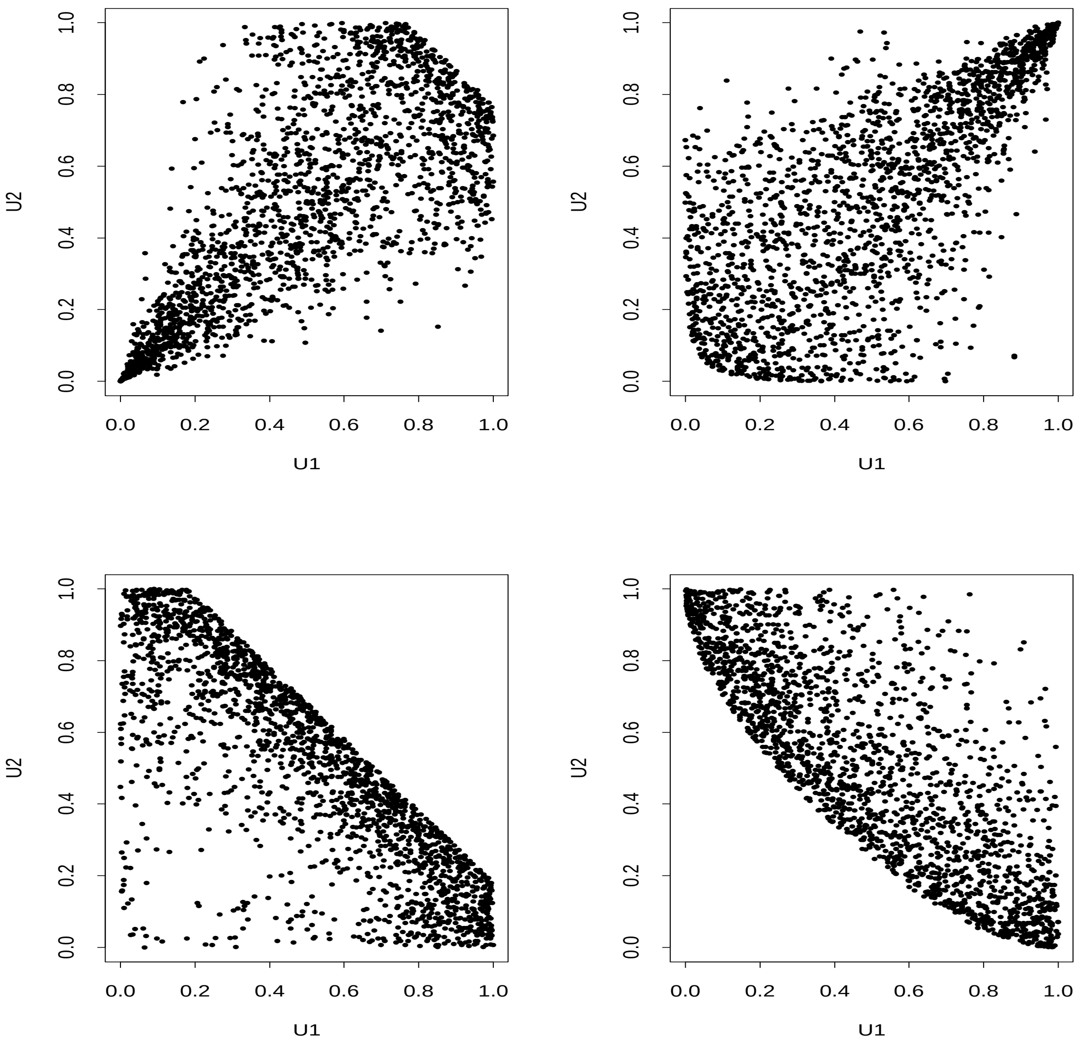

Fig. 2. Left pictures show bivariate Pareto-simplex copulas while right pictures show inverse-Pareto-simplex copulas. Upper copulas have $\theta = 0.3$; lower pictures have $\theta = 4.5$.

Example 4 (inverse-Pareto-simplex copulas) *Suppose the reciprocal of R has the Pareto distribution of Example 3 so that the density of R is $f_R(r) = \kappa r^{\kappa-1}$ on the interval $0 < r \leqslant 1$. Making the substitution $s = x/r$ the survival copula of the d-dimensional simplex distribution can be shown to have generator $\psi_{\kappa,d}$ given by*

$$\psi_{\kappa,d}(x) = \mathfrak{W}_d F_R(x) = \kappa x^{\kappa} \int_x^1 (1-s)^{d-1} s^{-(\kappa+1)} ds, \quad x < 1$$

and $\psi_{\kappa,d}(x) = 0$ *for* $x \geqslant 1$*. The binomial theorem gives, for* $x < 1$*,*

$$\psi_{\kappa,d}(x) = \kappa \sum_{i=0}^{d-1} \binom{d-1}{i} S_i \,, \quad S_i = \begin{cases} (-1)^i \left(\frac{x^\kappa - x^i}{i-\kappa}\right) & i \neq \kappa \\ (-1)^{i+1} x^\kappa \ln(x) & i = \kappa \,; \end{cases}$$

see Figure 2 for examples of random samples from this copula family. ■

2.2 Mixed-Erlang-simplex copulas and frailty models

In this section suppose that $R \stackrel{\mathrm{d}}{=} Z_d/W$ where $Z_d \sim \mathrm{Erlang}(d)$ and W is an almost surely positive random variable, independent of Z_d. We will refer to the simplex distributions generated by R as mixed-Erlang-simplex distributions and their survival copulas as mixed-Erlang-simplex copulas. We will show that this family of copulas constitutes the copulas of multiplicative frailty models in survival analysis, so that we will also refer to them as frailty copulas. The following lemma is key.

Lemma 2 *Let* $R \stackrel{\mathrm{d}}{=} Z_d/W$ *where* $Z_d \sim \mathrm{Erlang}(d)$ *and* W *is an almost surely positive random variable, independent of* Z_d*. Then* $\mathfrak{W}_d F_R = \mathcal{L} F_W$*.*

Proof. Under the conditions of the lemma R has a probability density given by

$$f_R(r) = \int_0^\infty w f_{Z_d}(rw) dF_W(w) = \int_0^\infty \frac{w^d \exp(-rw) r^{d-1}}{\Gamma(d)} dF_W(w).$$

The Williamson d-transform is the double integral

$$\mathfrak{W}_d F_R(x) = \int_{r=x}^\infty r^{1-d}(r-x)^{d-1} \int_{w=0}^\infty \frac{w^d \exp(-rw) r^{d-1}}{\Gamma(d)} dF_W(w) dr \,.$$

Changing the order of integration and making a change of variable $s = r - x$ yields

$$\begin{aligned} \mathfrak{W}_d F_R(x) &= \int_0^\infty \int_x^\infty \frac{w^d (r-x)^{d-1} \exp(-rw)}{\Gamma(d)} dr dF_W(w) \\ &= \int_0^\infty \exp(-wx) \int_{s=0}^\infty \frac{w^d s^{d-1} \exp(-sw)}{\Gamma(d)} ds dF_W(w) \\ &= \int_0^\infty \exp(-wx) dF_W(w) \,. \qquad \square \end{aligned}$$

Lemma 2 shows that the generator ψ of the Archimedean survival copula of the resulting simplex distribution is identical to the Laplace transform of W, which implies that ψ must belong to the class of completely monotone generators. In fact, we have the following characterization of the radial distributions that lead to survival copulas with completely monotonic generators.

Proposition 2 *Let $\psi = \mathfrak{W}_d F_R$ for some random variable R satisfying $F_R(0) = 0$ and let Ψ_∞ denote the class of completely monotone generators. Then*

$$\psi \in \Psi_\infty \iff R \stackrel{d}{=} Z_d/W$$

where W is an almost surely positive random variable, independent of $Z_d \sim$ Erlang(d).

Proof. If $R \stackrel{d}{=} Z_d/W$ we use Lemma 2 to conclude that $\psi = \mathcal{L}F_W$ so that $\psi \in \Psi_\infty$. Conversely, if $\psi \in \Psi_\infty$ then by Bernstein's theorem there must exist a random variable W such that $\psi = \mathcal{L}F_W$. If we now take an independent Erlang(d)-distributed random variable Z_d then Lemma 2 shows that $\mathfrak{W}_d F_{Z_d/W} = \mathcal{L}_W = \psi$. By the uniqueness of the Williamson d-transform we must have $R \stackrel{d}{=} Z_d/W$. □

The family of simplex distributions generated by mixed-Erlang radial distributions can be thought of as mixtures of independent exponential variates, so that they have a particular kind of *conditional independence* structure. Consider a vector $\mathbf{X}$,

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}/W \tag{6}$$

where $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ is a vector of independent standard exponential random variables and W is an almost surely positive scalar random variable independent of $\boldsymbol{Y}$. It is well known (and it follows easily from Lemma 3 later in this paper) that for independent standard exponential variables $\|\boldsymbol{Y}\|$ and $\boldsymbol{Y}/\|\boldsymbol{Y}\|$ are independent and that the former quantity has an Erlang(d) distribution and the latter quantity has a uniform distribution on $\mathcal{S}_d$. In other words we have a simplex distribution satisfying

$$\boldsymbol{X} \stackrel{d}{=} \frac{Z_d}{W}\boldsymbol{S}_d$$

where $Z_d \sim$ Erlang(d) and W, Z_d and $\boldsymbol{S}_d$ are independent.

It is also well known that the Archimedean copulas with completely monotone generators are identical to the class of survival copulas of multiplicative frailty models. This connection forms the basis of algorithms for generating Archimedean copulas with completely monotone generators given in Marshall and Olkin (1988), McNeil (2008) and Hofert (2008); see also the textbook by Hougaard (2000).

Lifetimes $T_1, \ldots, T_d$ are said to follow a shared multiplicative frailty model if they are independent given the so-called frailty W and have conditional hazard functions

$$\lambda_{T_i|W}(t \mid w) = w\lambda_i(t)$$

where $\lambda_1(t), \ldots, \lambda_d(t)$ are arbitrary underlying hazard functions. It is easily verified that the survival copula of the distribution of $(T_1, \ldots, T_d)$ is the Archimedean copula generated by $\psi = \mathcal{L}F_W$. In the case where $\lambda_i(t) \equiv 1$ for all i, the lifetimes have the mixed exponential distribution given in (6).

3 Dependence properties of Archimedean copulas

Consider a family $\{C_\theta : \theta \in \Theta\}$ of Archimedean copulas whose generators are given by $\psi_\theta = \mathfrak{W}_d F_{R_\theta}$, where R_θ belongs to some parametric family $\{R_\theta : \theta \in \Theta\}$, $\Theta \subseteq \mathbb{R}$. In order to interpret θ as a dependence parameter, we should have

$$\theta \leqslant \theta' \quad \Rightarrow \quad C_\theta \preccurlyeq C_{\theta'} \qquad \text{or} \qquad \theta \geqslant \theta' \quad \Rightarrow \quad C_\theta \preccurlyeq C_{\theta'}$$

where $\preccurlyeq$ denotes some stochastic ordering indicating that C_θ is less positively dependent than $C_{\theta'}$. In view of the fact that $\mathfrak{W}_d F_{R_\theta}$ may not be explicit or tractable, it is of interest to derive conditions ensuring $C_\theta \preccurlyeq C_{\theta'}$ in terms of the distribution of the radial variables R_θ and $R_{\theta'}$. This task seems to be rather difficult. Nonetheless, Section 3.1 derives a new formula for Kendall's rank correlation coefficient in terms of the radial distribution, while Section 3.2 gives conditions for $\tau(C_\theta) \leqslant \tau(C_{\theta'})$.

3.1 A new formula for Kendall's tau

Let $\boldsymbol{X}$ be a d-dimensional random vector with continuous margins and a copula C. A possible extension of Kendall's tau in dimension $d \geqslant 2$ proposed by (Joe, 1990) is

$$\tau(\boldsymbol{X}) = \tau(C) = \frac{2^d}{2^{d-1}-1} \int_{[0,1]^d} C(u_1, \ldots, u_d)\, dC(u_1, \ldots, u_d) - \frac{1}{2^{d-1}-1}.$$

When C is the independence copula, $\int C dC = 2^{-d}$ and $\tau(C) = 0$; when $C = M$, the Fréchet-Hoeffding upper bound copula, then $\int M dM = 2^{-1}$ and $\tau(M) = 1$; when $C = C_d^{\mathbf{L}}$, the lower bound Archimedean copula (see McNeil and Nešlehová (2009)), then $\int C dC = 0$ and $\tau(C_d^{\mathbf{L}}) = -1/(2^{d-1}-1)$.

If C is an Archimedean copula with generator ψ and radial part R, McNeil and Nešlehová (2009, Proposition 4.7) show that

$$\tau(C) = \frac{2^d \operatorname{E} \psi(R) - 1}{2^{d-1}-1}. \tag{7}$$

The next proposition gives a formula for $\tau(C)$ in terms of the corresponding radial distribution rather than the generator.

Proposition 1 *Let C be an Archimedean copula with generator ψ and radial part R. Further, let R^* denote an independent copy of R and set $Y = R/R^*$. Then*

$$\tau(C) = \frac{2^d \operatorname{E}\left\{(1-Y)_+^{d-1}\right\} - 1}{2^{d-1}-1}.$$

Proof. Denote by F_R the distribution function of R and observe that

$$\mathrm{E}\,(1-Y)_+^{d-1} = \int_0^\infty \int_0^\infty \left(1-\frac{r}{s}\right)_+^{d-1} dF_R(s)\,dF_R(r) = \int_0^\infty \psi(r)\,dF_R(r) = \mathrm{E}\,\psi(R).$$

The claim now follows immediately from (7). □

The remarkable observation is that Kendall's tau depends on R through the "ratio of radial variables" Y. This must mean that the same formula is obtained for the gamma- and inverse-gamma-simplex copulas, or for the Pareto- and inverse-Pareto-simplex copulas. In particular, the samples shown in the same row in Figures 1 and 2 were drawn from copulas with the same value of Kendall's τ, although they may look quite different. We begin with the slightly easier case of Pareto/inverse-Pareto.

Example 5 *For the Pareto-simplex and inverse-Pareto-simplex copulas defined in Examples 3 and 4, one easily finds that the density of* $Y_\kappa = R_\kappa/R_\kappa^*$ *is*

$$g_\kappa(y) = \begin{cases} (\kappa y^{\kappa-1})/2 & \text{if} \quad y \in [0,1], \\ (\kappa y^{-\kappa-1})/2 & \text{if} \quad y > 1. \end{cases} \tag{8}$$

Since

$$\mathrm{E}\left\{(1-Y_\kappa)_+^{d-1}\right\} = \frac{1}{2}\int_0^1 (1-y)^{d-1}\kappa y^{\kappa-1} dy = \frac{\kappa B(\kappa,d)}{2},$$

we obtain

$$\tau(C_{\kappa,d}) = \frac{2^{d-1}\kappa B(\kappa,d)-1}{2^{d-1}-1},$$

and, in particular, $\tau(C_{\kappa,2}) = (1-\kappa)/(1+\kappa)$ *for both the Pareto-simplex and inverse-Pareto-simplex families.*

Note that $\kappa B(\kappa,d) = \prod_{j=1}^{d-1}\frac{j}{j+\kappa}$ *is a strictly decreasing, continuous function of* κ *for fixed* d*, with limits given by* $\lim_{\kappa\to 0}\kappa B(\kappa,d) = 1$ *and* $\lim_{\kappa\to\infty}\kappa B(\kappa,d) = 0$*. Thus these copula families are comprehensive families giving all Kendall's tau values in the interval* $(-(2^{d-1}-1)^{-1}, 1)$*. Moreover, there will always be a* κ *for which* $\tau(C_{\kappa,d}) = 0$ *(for example, for* $d=2$*, we have* $\tau(C_{1,2}) = 0$*). However, this will not correspond to independence, since* R *does not follow an* Erlang(d) *distribution; see Remark 1.* ■

Example 6 *For the gamma-simplex and inverse-gamma-simplex copulas defined in Examples 1 and 2, the ratio* Y_θ *between* R_θ *and its independent copy* R_θ^* *is a ratio of two independent* Ga(θ) *random variables. Hence in both cases,* Y_θ *follows the Beta-prime distribution with density*

$$g_\theta(y) = \frac{y^{\theta-1}(1+y)^{-2\theta}}{\mathrm{B}(\theta,\theta)}, \quad y > 0. \tag{9}$$

Kendall's tau may be calculated by observing that

$$\mathrm{E}\left\{(1-Y_\theta)_+^{d-1}\right\} = \int_0^1 (1-y)^{d-1} g_\theta(y)\,dy = \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k s(k,\theta)$$

where

$$s(k,\theta) = \frac{1}{\mathrm{B}(\theta,\theta)} \int_0^1 y^{k+\theta-1}(1+y)^{-2\theta}\,dy = \frac{1}{\mathrm{B}(\theta,\theta)} \int_0^{1/2} x^{k+\theta-1}(1-x)^{\theta-k-1}\,dx$$

by substituting $x = y/(1+y)$. Proposition 1 thus yields that

$$\tau(C_{\theta,d}) = \frac{1}{2^{d-1}-1}\left(2^d \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k s(k,\theta) - 1\right)$$

which simplifies, for $d = 2$, to $\tau(C_{\theta,2}) = 1 - 4s(1,\theta)$. As shown in Appendix A,

$$\lim_{\theta\to\infty} \tau(C_{\theta,d}) = -\frac{1}{2^{d-1}-1}, \quad \lim_{\theta\to 0} \tau(C_{\theta,d}) = 1,$$

so that both families are again comprehensive. It can also be verified that $\tau(C_{d,d}) = 0$ when $\theta = d$, as expected: in the gamma case $C_{d,d}$ is the independence copula, although this is not true in the inverse-gamma case; see Remark 1. ■

3.2 Williamson d-transform order

Next, we derive conditions under which two Archimedean copulas C_1 and C_2 satisfy $\tau(C_1) \leqslant \tau(C_2)$ and show that the families introduced in Section 2 are ordered by their parameter. The first observation is an easy consequence of Proposition 1:

Proposition 2 *Let C_1 and C_2 be d-dimensional Archimedean copulas with radial parts R_1 and R_2. Let also R_i^* be an independent copy of R_i and G_i denote the distribution function of the ratio $Y_i = R_i/R_i^*$, for $i = 1, 2$. Then*

$$G_1(x) \leqslant G_2(x) \quad \text{for} \quad x \in [0,1] \qquad \Rightarrow \qquad \tau(C_1) \leqslant \tau(C_2).$$

Proof. First, define $\bar{F}(r) = (1-x)_+^{d-1}$ for $r \in \mathbb{R}_+$ and observe that $\bar{F}$ is a survival function on $\mathbb{R}$. Now, by partial integration,

$$\mathrm{E}\,(1-Y_i)_+^{d-1} = \int_0^1 \bar{F}(x) dG_i(x) = \int_0^1 G_i(x) dF(x), \quad i = 1, 2.$$

Thus $G_1(x) \leqslant G_2(x)$ for all $0 \geqslant x \geqslant 1$ immediately implies that $\mathrm{E}\,(1-Y_1)_+^{d-1} \leqslant \mathrm{E}\,(1-Y_2)_+^{d-1}$. Proposition 1 thus gives $\tau(C_1) \leqslant \tau(C_2)$. □

Example 7 *For the Pareto and inverse-Pareto families, we may calculate the distribution function G_κ of R_κ/R_κ^* easily using* (8)*. Indeed,*

$$G_\kappa(x) = \frac{1}{2}\int_0^x \kappa y^{\kappa-1} dy = x^\kappa/2, \quad x \in [0,1].$$

Thus if $\kappa > \kappa'$, $G_\kappa(x) \leqslant G_{\kappa'}(x)$ for all $x \in [0,1]$, implying that $\tau(C_\kappa) \leqslant \tau(C_{\kappa'})$. ■

The condition in Proposition 2 seems less tractable for the gamma and inverse-gamma families. However, the latter can be relaxed in the following sense. Because the function $(1-y)_+^{d-1}$ is d-monotone, $\tau(C_1) \leqslant \tau(C_2)$ follows from Proposition 1 if

$$\mathrm{E}\{f(Y_1)\} \leqslant \mathrm{E}\{f(Y_2)\} \tag{10}$$

holds for every d-monotone function f for which the expectations exist. This motivates the definition of the following stochastic order, which generalizes the Laplace order; see Shaked and Shanthikumar (2007) or Müller and Stoyan (2002).

Definition 2 *Let X_1 and X_2 be non-negative random variables. Then X_1 is smaller than X_2 in the Williamson d-transform order, denoted $X_1 \preccurlyeq_{\mathfrak{W}_d} X_2$, if and only if*

$$\mathfrak{W}_d G_1(x) \leqslant \mathfrak{W}_d G_2(x)$$

holds for all $x \in (0,\infty)$, where G_i denotes the distribution function of $1/X_i$, $i = 1,2$.

From the representation of a d-monotone function f due to Williamson (1956), viz.

$$f(x) = \int_0^\infty \left(1 - \frac{x}{u}\right)_+^{d-1} d\nu_f(u)$$

for some measure ν_f on $(0,\infty)$ (see also McNeil and Nešlehová (2009)), it follows that if X is a non-negative random variable with distribution function F,

$$\mathrm{E}\, f(X) = \int_0^\infty \int_0^\infty \left(1 - \frac{x}{u}\right)_+^{d-1} d\nu_f(u) dF(x) = \int_0^\infty \mathrm{E}\left(1 - \frac{X}{u}\right)_+^{d-1} d\nu_f(u).$$

However, $\mathrm{E}\,(1 - X/u)_+^{d-1}$ is $\mathfrak{W}_d G(1/u)$ where G is the distribution function of $1/X$. Taking into account that the function $(1 - x/u)_+^{d-1}$ is d-monotone for every $u \in (0,\infty)$, these observations can be summarized by the following result.

Proposition 3 *Let Y_1 and Y_2 be non-negative random variables. Then $Y_1 \preccurlyeq_{\mathfrak{W}_d} Y_2$ if and only if* (10) *holds for every d-monotone function f.*

In the context of Archimedean copulas, this has the following implication.

Corollary 1 *Under the hypothesis of Proposition 2, let ψ_i denote the generator of C_i, $i = 1,2$. Then $Y_1 \preccurlyeq_{\mathfrak{W}_d} Y_2$ implies $\tau(C_1) \leqslant \tau(C_2)$. More generally, we have*

$$Y_1 \preccurlyeq_{\mathfrak{W}_d} Y_2 \quad \Leftrightarrow \quad \mathrm{E}\{\psi_1(xR_1)\} \leqslant \mathrm{E}\{\psi_2(xR_2)\} \quad \textit{for all} \quad x \in (0,\infty).$$

Proof. First, note that $Y_i \stackrel{\mathrm{d}}{=} 1/Y_i$ for $i = 1, 2$. The result follows easily from

$$\mathfrak{W}_d G_i(x) = \int_0^\infty \int_0^\infty \left(1 - \frac{xs}{r}\right)_+^{d-1} dF_i(r) dF_i(s) = \int_0^\infty \psi_i(xs) dF_i(s) = \mathrm{E}\{\psi_i(xR_i)\}$$

for $i = 1, 2$. Here, F_i denotes the distribution function of R_i for $i = 1, 2$. $\square$

To verify the condition that $Y_1 \preccurlyeq_{\mathfrak{W}_d} Y_2$, it is useful to observe that a d-monotone function is k-monotone for any $1 \leqslant k \leqslant d$. Thus, Proposition 3 implies that

$$Y_1 \preccurlyeq_{\mathfrak{w}_k} Y_2 \quad \Rightarrow \quad Y_1 \preccurlyeq_{\mathfrak{W}_d} Y_2$$

for any $1 \leqslant k \leqslant d$. This means that for $d = 1$ and $d = 2$, the Williamson d-transform order is linked to the usual stochastic order and the second order stochastic dominance respectively:

$$Y_1 \succcurlyeq_{st} Y_2 \quad \Rightarrow \quad Y_1 \preccurlyeq_{\mathfrak{w}_1} Y_2, \qquad Y_1 \succcurlyeq_{SSD} Y_2 \quad \Rightarrow \quad Y_1 \preccurlyeq_{\mathfrak{w}_2} Y_2. \tag{11}$$

This is because $Y_1 \preccurlyeq_{st} Y_2$ is equivalent to the fact that (10) holds for any f increasing, while $Y_1 \preccurlyeq_{SSD} Y_2$ means that (10) is true for any f increasing and concave for which the expectations exist.

Example 8 *Let Z_θ be a Beta distributed random variable with parameters (θ, θ). As is well known, $Y_\theta = Z_\theta/(1 - Z_\theta)$ then follows the Beta-prime distribution with parameters (θ, θ). Now if $\theta \leqslant \theta'$, $Z_\theta \preccurlyeq_{SSD} Z_{\theta'}$, see e.g. Müller and Stoyan (2002, Section 1.12). Since the function $g(z) = z/(1 - z)$ is increasing and concave on $(0, 1)$, it follows that $Y_\theta = g(Z_\theta) \preccurlyeq_{SSD} g(Z_{\theta'}) = Y_{\theta'}$ and hence $Y_{\theta'} \preccurlyeq_{\mathfrak{w}_2} Y_\theta$ by (11). Corollary 1 thus implies that $\tau(C_{\theta,d}) \geqslant \tau(C_{\theta',d})$ for the gamma and inverse-gamma families, whenever $\theta \leqslant \theta'$.* ■

4 Liouville distributions and their copulas

In this section, we relax the condition that the distribution of $\boldsymbol{S}_d$ is uniform on $\mathcal{S}_d$. More specifically, we will assume that $\boldsymbol{S}_d$ follows the so-called *Dirichlet distribution.*

Lemma 3 *Let $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ be a vector of independent random variables such that $Z_i \sim \mathrm{Ga}(\alpha_i)$ for positive parameters $\alpha_1, \ldots, \alpha_d$. Write $\alpha = \sum_{i=1}^d \alpha_i$, $\|\boldsymbol{Z}\| = \sum_{i=1}^d Z_i$, $D_i = Z_i/\|\boldsymbol{Z}\|$ for $1 \leqslant i \leqslant d$ and $\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)} = (D_1, \ldots, D_d)$. Then*

(1) $\|\boldsymbol{Z}\|$ and $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$ are independent;
(2) $\|\boldsymbol{Z}\| \sim \mathrm{Ga}(\alpha)$;
(3) the joint density of $(D_1, \ldots, D_{d-1})$ is given by

$$f(x_1, \ldots, x_{d-1}) = \frac{\Gamma(\alpha)}{\prod_{i=1}^d \Gamma(\alpha_i)} \prod_{i=1}^{d-1} x_i^{\alpha_i - 1} \left(1 - \sum_{j=1}^{d-1} x_j\right)^{\alpha_d - 1},$$

where $\sum_{i=1}^{d-1} x_i \leqslant 1$ *and* $x_i \geqslant 0$ *for* $i = 1, \ldots, d-1$.

Proof. This is easily shown by writing down the joint density of $(Z_1, \ldots, Z_d)$ and calculating from this the joint density of $(D_1, \ldots, D_{d-1}, \|\boldsymbol{Z}\|)$. See, for example, Fang et al. (1990), pages 17–18. $\square$

The distribution of $(D_1, \ldots, D_{d-1})$, or equivalently of $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$, is known as a Dirichlet distribution. We will use the notation $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)} \sim D(\alpha_1, \ldots, \alpha_d)$. Mixtures of Dirichlet distributions, formally defined below, are known as Liouville distributions. The latter have been studied by Marshall and Olkin (1979), Gupta and Richards (1987, 1991, 1992, 1997), Song and Gupta (1997) and Fang et al. (1990).

Definition 3 *A random vector* $\boldsymbol{X}$ *on* $\mathbb{R}_+^d = [0, \infty)^d$ *is said to follow a Liouville distribution if it permits the stochastic representation*

$$\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)}$$

where $\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)} \sim D(\alpha_1, \ldots, \alpha_d)$ *is a random vector with a Dirichlet distribution on the unit simplex* $\mathcal{S}_d$. *The random variable* R *is referred to as the* radial part *of* $\boldsymbol{X}$ *and its distribution as the* radial distribution. *Furthermore, the survival copula of* $\boldsymbol{X}$ *will be called* Liouville copula *with radial part* R *and parameters* $(\alpha_1, \ldots, \alpha_d)$.

Clearly the simplex distributions form a subclass of the Liouville distributions since $\mathbf{S}_d \sim D(1, \ldots, 1)$. They are characterized by a rather simple form for their survival function. The following result, whose proof may be found in Appendix B, gives the survival function in the general Liouville case with integer parameters.

Theorem 3 *Let* $\boldsymbol{X}$ *be a Liouville distributed random vector with radial part* R *and parameters* $(\alpha_1, \ldots, \alpha_d)$ *such that* $\alpha_i \in \mathbb{N}$ *for* $i = 1, \ldots, d$. *Furthermore, set* $\alpha = \sum_{i=1}^d \alpha_i$ *and* $\psi(x) = \mathfrak{W}_\alpha F_R(x)$. *Then the survival function of* $\boldsymbol{X}$ *is given by*

$$\bar{H}(\boldsymbol{x}) = \sum_{i_1=0}^{\alpha_1-1} \cdots \sum_{i_d=0}^{\alpha_d-1} (-1)^{i_1+\cdots+i_d} \frac{\psi^{(i_1+\cdots+i_d)}(x_1+\cdots+x_d)}{i_1!\cdots i_d!} \prod_{j=1}^d x_j^{i_j}, \quad \boldsymbol{x} \in \mathbb{R}_+^d. \tag{12}$$

Theorem 3 in particular implies that the margins of a Liouville random vector are again Liouville distributed. In fact, the distribution function of X_i is

$$F_i(x) = 1 - \sum_{j=0}^{\alpha_i-1} \frac{(-1)^j x^j \psi^{(j)}(x)}{j!}, \quad x \in \mathbb{R}_+$$

for all $1 \leqslant i \leqslant d$ which means that $F_i(x) = \mathfrak{W}_{\alpha_i}^{-1}\psi(x)$.

In the case when ψ is α-times differentiable almost everywhere, $\boldsymbol{X}$ has density

$$f(\boldsymbol{x}) = (-1)^\alpha \psi^{(\alpha)}(\|\boldsymbol{x}\|) \prod_{i=1}^d \frac{x_i^{\alpha_i-1}}{\Gamma(\alpha_i)}, \quad \boldsymbol{x} \in \mathbb{R}_+^d. \tag{13}$$

This follows readily upon differentiating (12). It is worth noting that in this case, the radial part R has a density f_R given by (2). One may thus write

$$f(\boldsymbol{x}) = \frac{\Gamma(\alpha)}{\|\boldsymbol{x}\|^{\alpha-1}} f_R(\|\boldsymbol{x}\|) \prod_{i=1}^{d} \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad \boldsymbol{x} \in \mathbb{R}_+^d.$$

This is in agreement with Fang et al. (1990, Theorem 6.1). Note that the form of the density also holds in the general Liouville case, as can be seen from (16) by the change of variable $s_d = r - (s_1 + \cdots + s_{d-1})$.

Remark 4 *In Fang et al. (1990), the function $g(x) = \Gamma(\alpha) f_R(x)/x^{\alpha-1}$ is referred to as the* density generator *of $\boldsymbol{X}$. Furthermore,*

$$\int_0^\infty \left(1 - \frac{x}{r}\right)_+^{\alpha-1} dF_R(r) = \int_x^\infty \frac{(r-x)^{\alpha-1}}{\Gamma(d)} g(r) dr,$$

where the right hand-side is the Weyl fractional integral transform of g, $W^\alpha g(r)$.

4.1 Some Liouville copula families

Examples below generalize the families of Archimedean copulas introduced in Section 2 as well as the well-known Clayton family.

Example 9 (gamma-Liouville copulas.) *Consider a Liouville copula with integer parameters $(\alpha_1, \ldots, \alpha_d)$ and radial part $R \sim \mathrm{Ga}(\theta)$. We will refer to this copula as a gamma-Liouville copula.*

The Williamson α-transform of F_R, where $\alpha = \alpha_1 + \cdots + \alpha_d$ and its derivatives may be computed as follows. By Example 1,

$$\psi_{\theta,\alpha}(x) = \mathfrak{W}_\alpha F_R(x) = \int_x^\infty r^{1-\alpha} (r-x)^{\alpha-1} f_R(r;\theta) dr\,,$$

where f_R denotes the density of R. Differentiating under the integral yields

$$\psi_{\theta,\alpha}^{(i)}(x) = (-1)^i \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)} \int_x^\infty r^{1-\alpha} (r-x)^{\alpha-i-1} f_R(r;\theta) dr\,, \quad i = 0, 1, \ldots, \alpha-2, \tag{14}$$

where $\psi_{\theta,\alpha}^{(0)} = \psi_{\theta,\alpha}$; see also Lemma 5. We introduce the notation

$$g_{\alpha,k,\theta}(x) = \int_x^\infty r^{1-\alpha} (r-x)^{k-1} f_R(r;\theta) dr\,, \quad k = 2, 3, \ldots, \alpha,$$

so that

$$\psi_{\theta,\alpha}^{(i)}(x) = (-1)^i \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)} g_{\alpha,\alpha-i,\theta}(x)\,, \quad i = 0, 1, \ldots, \alpha-2.$$

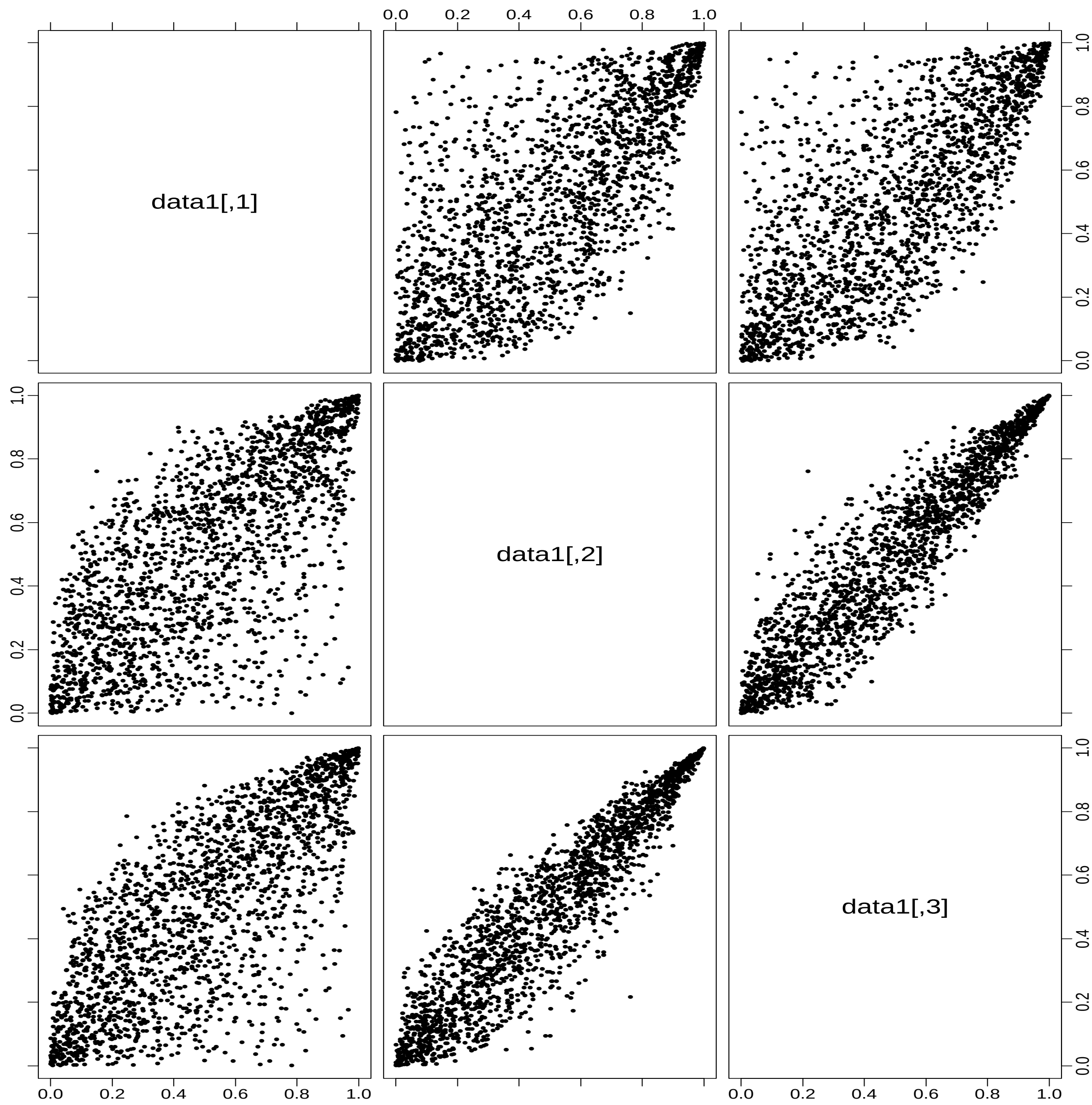

Fig. 3. Realisation of 2000 points from a 3-dimensional gamma-Liouville copula with $\theta = 0.6$ and $(k_1, k_2, k_3) = (1, 5, 20)$.

By the arguments of Example 1 we have that

$$\begin{aligned} g_{\alpha,k,\theta}(x) &= \sum_{n=0}^{k-1} \binom{k-1}{n} \int_x^\infty \frac{(-x)^{k-1-n} r^{n+\theta-\alpha} e^{-r}}{\Gamma(\theta)} \, dr \\ &= \sum_{n=0}^{k-1} \binom{k-1}{n} \frac{(-1)^{k-1-n} x^{k-1-n}}{\Gamma(\theta)} \Gamma(n+\theta-\alpha+1, x), \end{aligned}$$

where $\Gamma(k,x) = \int_x^\infty t^{k-1} e^{-t} \, dt$ *for* $k \in \mathbb{R}$ *is the incomplete gamma function.*

Using Theorem 3, we can evaluate $\bar{H}(\mathbf{x})$ and its margins

$$\bar{H}_j(x) = \sum_{i=0}^{\alpha_j-1} \frac{(-1)^i \psi_{\theta,\alpha}^{(i)}(x) x^i}{i!} = \sum_{i=0}^{\alpha_j-1} \binom{\alpha-1}{i} x^i g_{\alpha,\alpha-i,\theta}(x).$$

We thus have all the elements we need to sample $(U_1,\ldots,U_d) = (\bar{H}_1(X_1),\ldots,\bar{H}_d(X_d))$ distributed as the gamma-Liouville copula. A realisation from this copula when $\theta = 0.6$ and $(\alpha_1,\alpha_2,\alpha_3) = (1,5,20)$ is shown in Figure 3. ∎

Example 10 (inverse-gamma-Liouville copulas.) *Next, consider a Liouville copula with integer parameters $(\alpha_1,\ldots,\alpha_d)$ and radial part $1/R \sim \mathrm{Ga}(\theta)$. We will refer to this copula as an inverse-gamma-Liouville copula.*

We proceed in a similar way to Example 9. We have to be able to evaluate (14) when $f_R(r;\theta)$ is the density of inverse-gamma. We obtain

$$\begin{aligned}
\psi_{\theta,\alpha}^{(i)}(x) &= (-1)^i \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)} \int_x^\infty \frac{r^{1-\alpha}(r-x)^{\alpha-i-1} r^{-\theta-1} e^{-1/r}}{\Gamma(\theta)} dr \\
&= (-1)^i \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)} \frac{\Gamma(\theta+i)}{\Gamma(\theta)} \int_x^\infty \frac{r^{1-(\alpha-i)}(r-x)^{\alpha-i-1} r^{-(\theta+i)-1} e^{-1/r}}{\Gamma(\theta+i)} dr \\
&= (-1)^i \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)} \frac{\Gamma(\theta+i)}{\Gamma(\theta)} \psi_{\theta+i,\alpha-i}(x)
\end{aligned}$$

Since the generator can be evaluated in terms of (lower) incomplete gamma functions using (4) we again have all the elements we need to evaluate $\bar{H}$ in Theorem 3 or to sample the copula. The survival margins of $\bar{H}$ are given by

$$\bar{H}_j(x) = \sum_{i=0}^{\alpha_j-1} \binom{\alpha-1}{i} x^i \psi_{\theta+i,\alpha-i}(x) \frac{\Gamma(\theta+i)}{\Gamma(\theta)}\,.$$

Figure 4 provides an illustration when $\theta = 0.6$ and $(\alpha_1,\alpha_2,\alpha_3) = (1,5,20)$. ∎

Clearly we can also define Pareto–Liouville and inverse-Pareto–Liouville copulas which generalise the Archimedean copulas of Examples 3 and 4. These are also relatively tractable and generator derivatives can be calculated by the same approach that we have used in Examples 9 and 10. In the next example we construct a family that generalises the well known Clayton family of Archimedean copulas.

Example 11 (Clayton–Liouville copulas.) *Consider a Liouville distributed random vector with integer parameters $\alpha_1,\ldots,\alpha_d$ and a radial part with a distribution F_R whose Williamson α-transform is given by*

$$\psi_\theta(x) = \mathfrak{W}_\alpha F_R(x) = (1+\theta x)_+^{-1/\theta},$$

with $\theta \geqslant -1/(\alpha-1)$ and $\alpha = \alpha_1 + \cdots + \alpha_d$.

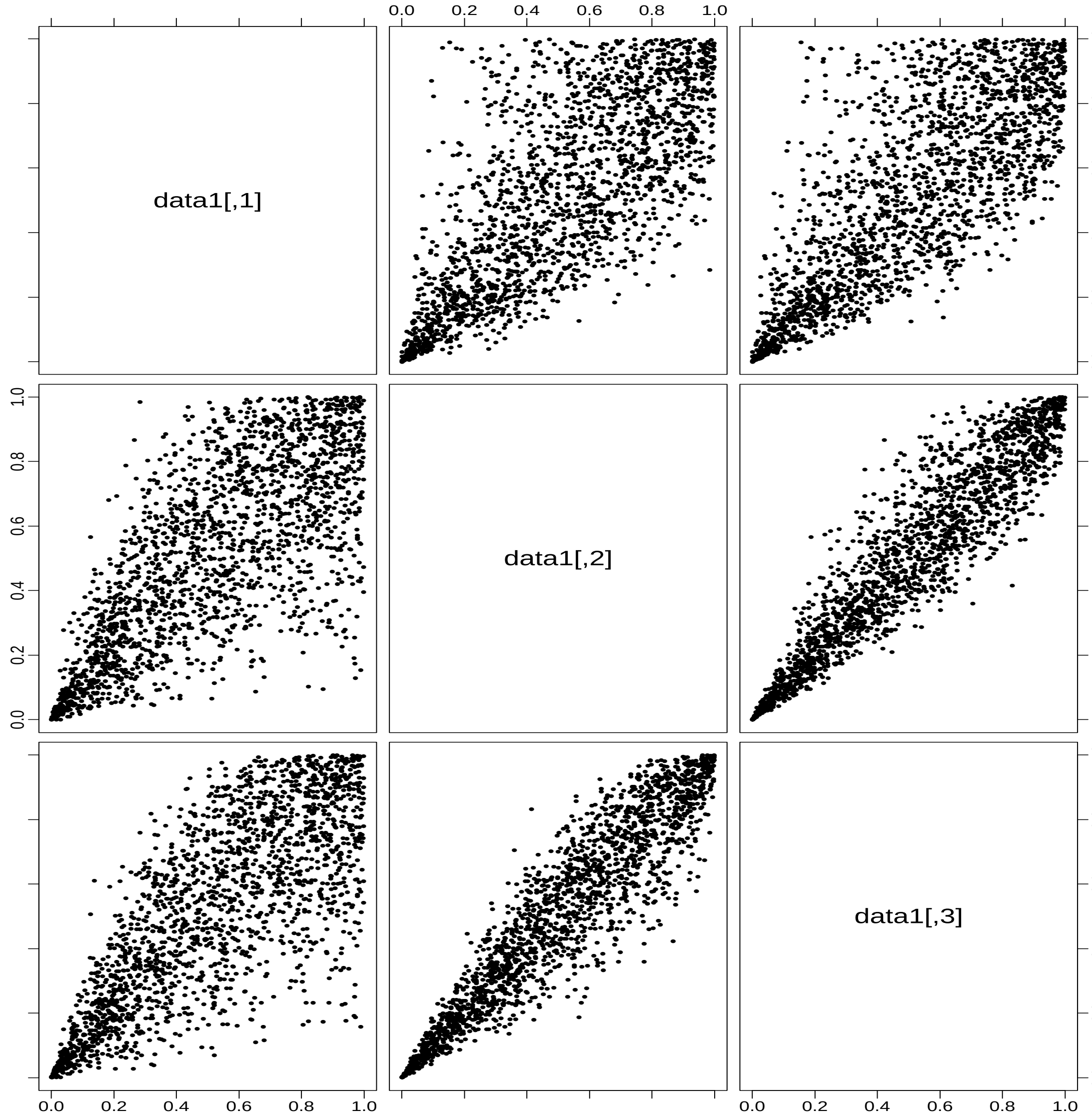

Fig. 4. Realisation of 2000 points from a 3-dimensional inverse-gamma-Liouville copula with $\theta = 0.6$ and $(\alpha_1, \alpha_2, \alpha_3) = (1, 5, 20)$.

The random vector $\tilde{\mathbf{X}} := R\mathbf{S}_\alpha$ *would have a* α*-dimensional simplex distribution whose survival copula is Clayton with parameter* θ*. The Liouville random vector* $\mathbf{X} = R\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)}$ *has a survival copula that we call a Clayton–Liouville copula.*

Derivatives of the Clayton generator take the form

$$\psi_\theta^{(k)}(x) = \begin{cases} (-1)^k c(\theta, k)(1+\theta x)^{-(1/\theta+k)} & (1+\theta x) > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

where $c(\theta,k) = \theta^{k-1}\Gamma(1/\theta+k)/\Gamma(1/\theta+1)$*. The survival margins for* $\mathbf{X}$ *thus are*

$$\bar{H}_j(x) = \psi_\theta(x) \sum_{k=0}^{\alpha_j-1} \frac{c(\theta,k)}{k!} \left(\frac{x}{1+\theta x}\right)^k .$$

As an example consider the bivariate with $\alpha_1 = 1$ *and* $\alpha_2 = 2$ *and assume* $\theta \geqslant -1/2$*. This has bivariate survival function*

$$\bar{H}(x_1,x_2) = \psi_\theta(x_1+x_2)\left(1 + \frac{x_2}{1+\theta(x_1+x_2)}\right)$$

with survival margins $\bar{H}_1(x) = \psi_\theta(x)$ *and* $\bar{H}_2(x) = \psi_\theta(x)\{1 + x/(1+\theta x)\}$*.*

Random sampling from the Clayton–Liouville copula requires the following steps:

(1) *Generate a vector* $(U_1,\ldots,U_\alpha)$ *from the Clayton copula with parameter* θ*.*
(2) *Construct* $\mathbf{X} = (X_1,\ldots,X_d)$ *where*

$$X_j = \sum_{i=S_{j-1}+1}^{S_j} \psi^{-1}(U_i), \quad j = 1,\ldots,d,$$

where $S_0 = 0$ *and* $S_j = \sum_{k=1}^{j} \alpha_k$ *for* $j = 1,\ldots,d$ *denote partial sums of the* α_j*.*
(3) *Return* $(\bar{H}_1(X_1),\ldots,\bar{H}_d(X_d))$*.*

■

4.2 Mixed-Erlang-Liouville copulas

In this section, which forms a counterpart to Section 2.2, we consider a Liouville distributed random vector $\mathbf{X} = R\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$, where $\alpha_i \in \mathbb{N}$ for $i = 1,\ldots,d$. Let us suppose that R has a mixed Erlang distribution so that $R \stackrel{\mathrm{d}}{=} Z_\alpha/W$ where $Z_\alpha \sim \mathrm{Erlang}(\alpha)$ for $\alpha = \sum_{i=1}^{d} \alpha_i$ and W is an almost surely positive random variable, independent of Z_α.

Clearly, Lemma 2 implies that $\psi = \mathfrak{W}_\alpha F_R = \mathcal{L} F_W$ is a completely monotone generator. The distribution of $\mathbf{X}$ again has a conditional independence structure. To see this, let $\boldsymbol{Y} = (Y_1,\ldots,Y_d)$ be a vector of independent gamma random variables with distributions $Y_i \sim \mathrm{Ga}(\alpha_i)$. Then we have that $\boldsymbol{X} \stackrel{\mathrm{d}}{=} W^{-1}\boldsymbol{Y}$ since, by Lemma 3,

$$W^{-1}\boldsymbol{Y} \stackrel{\mathrm{d}}{=} \frac{Z_\alpha}{W}\boldsymbol{D}_{\alpha_1,\ldots,\alpha_d}$$

with W, Z_α and $\boldsymbol{D}_{\alpha_1,\ldots,\alpha_d}$ independent.

This distribution of $\boldsymbol{X}$ also has an interpretation in terms of survival modelling. It describes conditionally independent lifetimes where the conditional survival functions are given by

$$\bar{F}_{X_i|W}(x \mid w) = \bar{F}_{Y_i}(wx) = \frac{\Gamma(\alpha_i, wx)}{\Gamma(\alpha_i)}$$

and the conditional hazard functions by $\lambda_{X_i|W}(x \mid w) = w\lambda_{Y_i}(wx)$, where $\lambda_{Y_i}(t) = t^{\alpha_i - 1}e^{-t}/\Gamma(\alpha_i, t)$ and $\Gamma(\alpha, t)$ denotes the upper incomplete gamma function as before. Thus the vector $\boldsymbol{X}$ can be thought of as a vector of lifetimes which are conditionally independent given W and which follow an *accelerated failure model* with baseline gamma hazard functions; W is the time acceleration variable.

4.3 Kendall's tau for Liouville copulas

Similarly as for Archimedean copulas, Kendall's tau for Liouville copulas can be expressed in terms of the ratio $Y = R/R^*$ between the radial distribution R and its independent copy R^*. Though this is true for any dimension d, Proposition 4 below gives the formula for the bivariate case only, for the sake of simplicity.

Proposition 4 *Let C be a bivariate Liouville copula with radial part R and parameters $\alpha_i \in \mathbb{N}$, $i = 1, 2$. Further, set $\alpha = \alpha_1 + \alpha_2$ and let Y be the ratio R/R^* where R^* is an independent copy of R. Then*

$$\tau(C) = 4 \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} \frac{\mathrm{B}(\alpha_1 + i, \alpha_2 + j)\Gamma(\alpha)}{\mathrm{B}(\alpha_1, \alpha_2) i! j! \Gamma(\alpha - i - j)} \mathrm{E}\left\{(Y)^{i+j} (1 - Y)_+^{\alpha - i - j - 1}\right\} - 1.$$

Proof. Let (X_1, X_2) be a Liouville random vector with radial part R and parameters (α_1, α_2). As is well known, $\tau = 4\,\mathrm{E}\{\bar{H}(X_1, X_2)\} - 1$. Thus, by means of (12),

$$\mathrm{E}\{\bar{H}(X_1, X_2)\} = \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} \frac{(-1)^{i+j}}{i! j!} \mathrm{E}\{\psi^{(i+j)}(R) R^{i+j} U^i (1 - U)^j\}$$

where $U \sim \mathrm{Be}(\alpha_1, \alpha_2)$. Because U and R are independent,

$$\mathrm{E}\{\bar{H}(X_1, X_2)\} = \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} \frac{(-1)^{i+j}}{i! j!} \mathrm{E}\{\psi^{(i+j)}(R) R^{i+j}\} \mathrm{E}\{U^i (1 - U)^j\} =$$

$$= \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} \frac{(-1)^{i+j}}{i! j!} \mathrm{E}\{\psi^{(i+j)}(R) R^{i+j}\} \frac{\mathrm{B}(\alpha_1 + i, \alpha_2 + j)}{\mathrm{B}(\alpha_1, \alpha_2)}.$$

Furthermore, $\mathrm{E}\{\psi^{(i+j)}(R)R^{i+j}\}$ may be calculated using (15) as follows.

$$\mathrm{E}\{\psi^{(i+j)}(R)R^{i+j}\} = \frac{(-1)^{i+j}\Gamma(\alpha)}{\Gamma(\alpha-i-j)}\int_0^\infty\int_0^\infty \frac{s^{i+j}}{r^{i+j}}\left(1-\frac{s}{r}\right)_+^{\alpha-i-j-1} dF_R(s)dF_R(r) =$$
$$= \frac{(-1)^{i+j}\Gamma(\alpha)}{\Gamma(\alpha-i-j)}\,\mathrm{E}\left\{(Y)^{i+j}\,(1-Y)_+^{\alpha-i-j-1}\right\}.$$

Gathering the terms concludes the proof. □

Example 12 (Kendall's tau for Pareto–Liouville copulas.) *Consider a bivariate Liouville distributed random vector of the form $\mathbf{X} = R\mathbf{D}_{(\alpha_1,\alpha_2)}$, where R has a Pareto distribution with distribution function $F_R(r) = 1 - r^{-\kappa}$ for $r \geqslant 1$ and $\kappa > 0$ as in Examples 3 and 5. We may compute that*

$$\mathrm{E}\left\{(Y)^{i+j}\,(1-Y)_+^{\alpha-i-j-1}\right\} = \frac{\kappa}{2}\int_0^1 y^{i+j+\kappa-1}(1-y)^{\alpha-i-j-1}dy$$
$$= \frac{\kappa}{2}\mathrm{B}(i+j+\kappa,\alpha-i-j),$$

which yields the formula

$$\tau(C_{\kappa,(\alpha_1,\alpha_2)}) = 2\kappa\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{\alpha_2-1}\frac{\mathrm{B}(\alpha_1+i,\alpha_2+j)\Gamma(\alpha)\mathrm{B}(i+j+\kappa,\alpha-i-j)}{\mathrm{B}(\alpha_1,\alpha_2)i!j!\Gamma(\alpha-i-j)} - 1\,.$$

It may be verified that this reduces to $\tau(C_{\kappa,(1,1)}) = 2\kappa B(\kappa,2) - 1 = (1-\kappa)/(1+\kappa)$, the formula for the bivariate Pareto-simplex copula in Example 5. Figure 5 displays the curves for various values of κ, α_1 and α_2. Note how for fixed κ the level of dependence is higher in the general Liouville case ($\alpha_1 > 1$ and $\alpha_2 > 1$) than in the simplex case ($\alpha_1 = \alpha_2 = 1$); moreover it seems to increase as we increase one or both of α_1 and α_2. ■

Appendix A: Limiting cases in Example 6

Observe first that $s(0,\theta) = 1/2$. For $k \geqslant 1$, use Lebesgue's Dominated Convergence Theorem to see that

$$\lim_{\theta\to 0}\int_0^{1/2} x^{k+\theta-1}(1-x)^{\theta-k-1}\,dx = \int_0^{1/2} x^{k-1}(1-x)^{-k-1}\,dx$$

which is certainly finite. At the same time, $\mathrm{B}(\theta,\theta)\to\infty$ as $\theta\to 0$ so that $s(k,\theta)\to 0$ for $k \geqslant 1$ as $\theta \to 0$. It follows that

$$\lim_{\theta\to 0}\tau(C_{\theta,d}) = \frac{1}{2^{d-1}-1}\left(2^d\frac{1}{2}-1\right) = 1.$$

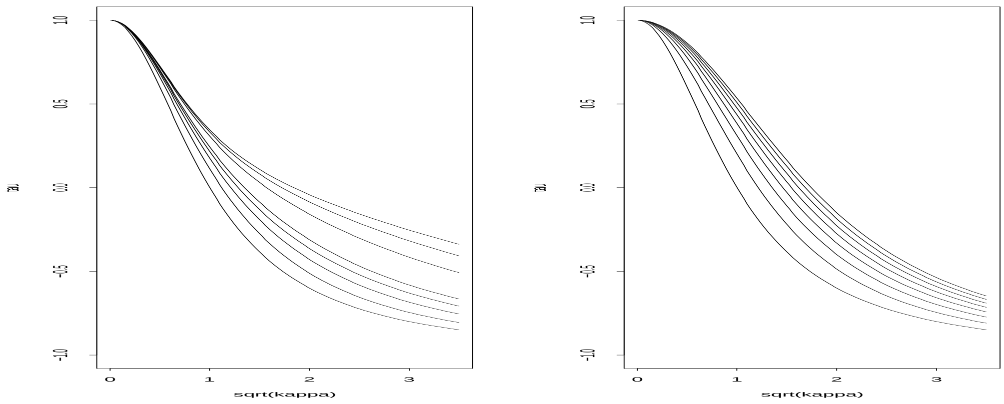

Fig. 5. Left plot shows $\tau(C_{\kappa,(1,\alpha)})$ as a function of $\sqrt{\kappa}$ for $\alpha \in \{1, 2, 3, 4, 5, 10, 15, 20\}$; for fixed κ the τ values increase with α. Right plot shows $\tau(C_{\kappa,(\alpha,\alpha)})$ as a function of $\sqrt{\kappa}$ for $\alpha \in \{1, 2, 3, 4, 5, 6, 7, 8\}$; for fixed κ the τ values again increase with α.

For the limit as $\theta \to \infty$, suppose first that θ takes integer values n. In that case the ratio of radial variables determining $\tau(C_{n,d})$ can be written

$$Y_n = \frac{R_n}{R_n^*} = \frac{\sum_{i=1}^n Z_{n,i}}{n} \frac{n}{\sum_{i=1}^n Z_{n,i}^*},$$

where the $Z_{n,i}$ and $Z_{n,i}^*$ are iid standard exponential variables. By the strong law of large numbers we must have $\lim_{n\to\infty} Y_n = 1$, almost surely. Since $(1-y)_+^{d-1}$ is a continuous bounded function it follows that $\lim_{n\to\infty} \tau(C_{n,d}) = -(2^{d-1}-1)^{-1}$. But Kendall's tau can be shown to be strictly ordered by θ (see Example 8), so we must have $\lim_{\theta\to\infty} \tau(C_{\theta,d}) = -(2^{d-1}-1)^{-1}$.

Appendix B : Proof of Theorem 3

First, note two supplementary results, first of which follows easily upon differentiating under the integral sign.

Lemma 5 *Let $\psi = \mathfrak{W}_d F_R$ for some non-negative random variable $R \sim F_R$. Then*

$$\psi_d^{(\ell)}(x) = (-1)^\ell \frac{\Gamma(d)}{\Gamma(d-\ell)} \int_x^\infty \frac{1}{r^\ell} \left(1 - \frac{x}{r}\right)^{d-\ell-1} dF_R(r), \quad x \in (0,\infty). \tag{15}$$

for $0 \leqslant \ell \leqslant d-2$. In addition, $\psi_+^{(d-1)}(x)$ equals $(-1)^{d-1}(d-1)! \int_0^\infty \mathbf{1}_{(r>x)} r^{1-d} dF_R(r)$.

Lemma 6 *Let ψ be a d-monotone Archimedean generator. Then for all $x, y > 0$ and all integers $0 \leqslant j \leqslant d-3$,*

$$\int_y^\infty s^{m-1} \psi^{(m+j)}(x+s) ds = \sum_{i=0}^{m-1} \frac{(m-1)!}{i!} (-1)^{m+i} y^i \psi^{(j+i)}(x+y),$$

whenever m is an integer such that $1 \leqslant m \leqslant d-2-j$. Furthermore, for $0 \leqslant j \leqslant d-2$,

$$\int_y^\infty s^{d-2-j}\psi_+^{(d-1)}(x+s)ds = \sum_{i=0}^{d-2-j} \frac{(d-2-j)!}{i!}(-1)^{d-j-1+i}y^i\psi^{(j+i)}(x+y).$$

Proof. The first part of the lemma follows by induction in m. Fix a $j \in \{0,\dots,d-3\}$ and assume that $m=1$. Clearly, $\int_y^\infty \psi^{(j+1)}(x+s)ds = -\psi^{(j)}(x+y)$ because $\psi^{(j)}(t) \to 0$ as $t \to \infty$. Furthermore, when $2 \leqslant m \leqslant d-2-j$,

$$\begin{aligned}
&\int_y^\infty s^{m-1}\psi^{(m+j)}(x+s)ds \\
&= -y^{m-1}\psi^{(m-1+j)}(x+y) - (m-1)\int_y^\infty s^{m-2}\psi^{(m-1+j)}(x+s)ds \\
&= -y^{m-1}\psi^{(m-1+j)}(x+y) + \sum_{i=0}^{m-2}\frac{(m-1)!}{i!}(-1)^{m+i}y^i\psi^{(j+i)}(x+y) \\
&= \sum_{i=0}^{m-1}\frac{(m-1)!}{i!}(-1)^{m+i}y^i\psi^{(j+i)}(x+y)
\end{aligned}$$

by partial integration and the induction assumption. Further, use Lemma 5 to write

$$\begin{aligned}
&\int_y^\infty s^{d-j-2}\psi_+^{(d-1)}(x+s)ds \\
&= (-1)^{d-1}(d-1)!\int_y^\infty\int_0^\infty \mathbf{1}_{(r>x+s)}r^{1-d}s^{d-j-2}dF_R(r)ds \\
&= (-1)^{d-1}(d-1)!\int_0^\infty\int_y^{r-x} \mathbf{1}_{(r>x+s)}r^{1-d}s^{d-j-2}ds\,dF_R(r) \\
&= \frac{(-1)^{d-1}(d-1)!}{d-j-1}\int_0^\infty \mathbf{1}_{(r>x+y)}\frac{1}{r^j}\left\{\left(1-\frac{x}{r}\right)^{d-j-1} - \frac{y^{d-j-1}}{r^{d-j-1}}\right\}dF_R(r).
\end{aligned}$$

The stated formula now follows by binomial expansion of $(1-x/r)^{d-j-1}$ and (15):

$$\begin{aligned}
&\frac{(-1)^{d-1}(d-1)!}{d-j-1}\sum_{i=0}^{d-j-2}\binom{d-j-1}{i}y^i\int_{x+y}^\infty \frac{1}{r^{i+j}}\left(1-\frac{x+y}{r}\right)^{d-j-i-1}dF_R(r) \\
&= \frac{(-1)^{d-1}(d-1)!}{d-j-1}\sum_{i=0}^{d-j-2}\binom{d-j-1}{i}y^i\frac{(-1)^{i+j}(d-j-i-1)!}{(d-1)!}y^i\psi^{(i+j)}(x+y) \\
&= \sum_{i=0}^{d-j-2}\frac{(d-j-2)!}{i!}(-1)^{d-1-j+i}y^i\psi^{(i+j)}(x+y). \qquad \square
\end{aligned}$$

Proof of Theorem 3. Write $\boldsymbol{X} \stackrel{\mathrm{d}}{=} (RD_1,\dots,RD_d)$, where $(D_1,\dots,D_d)$ is Dirichlet with parameters $(\alpha_1,\dots,\alpha_d)$. The survival function of $\boldsymbol{X}$ thus equals

$$\bar{H}(\boldsymbol{x}) = \int_0^\infty \Pr\left(D_1 > \frac{x_1}{r}, \ldots, D_{d-1} > \frac{x_{d-1}}{r}, 1 - \sum_{i=1}^{d-1} D_i > \frac{x_d}{r}\right) dF_R(r)$$
$$= \int_{x_1+\cdots+x_d}^\infty \Pr\left(D_1 > \frac{x_1}{r}, \ldots, D_{d-1} > \frac{x_{d-1}}{r}, 1 - \sum_{i=1}^{d-1} D_i > \frac{x_d}{r}\right) dF_R(r)\,.$$

Because $(D_1, \ldots, D_{d-1})$ has density f_D as given by Lemma 3,

$$\bar{H}(\boldsymbol{x}) = \int_{x_1+\cdots+x_d}^\infty \int_{\frac{x_1}{r}}^{a_1} \cdots \int_{\frac{x_{d-1}}{r}}^{a_{d-1}} f_D(t_1, \ldots, t_{d-1}) d\,t_{d-1} \ldots d\,t_1 d\,F_R(r),$$

where $a_i = 1 - \sum_{j=1}^{i-1} t_j - \sum_{j=i+1}^{d} x_j/r$ for $i = 1, \ldots, d-1$. Substituting $s_i = rt_i$ for $i = 1, \ldots, d-1$ gives

$$\bar{H}(\boldsymbol{x}) = \int_{x_1+\cdots+x_d}^\infty \frac{1}{r^{d-1}} \int_{\frac{x_1}{r}}^{b_1} \cdots \int_{\frac{x_{d-1}}{r}}^{b_{d-1}} f_D\left(\frac{s_1}{r}, \ldots, \frac{s_{d-1}}{r}\right) ds_{d-1} \ldots ds_1 dF_R(r),$$

where $b_i = r - \sum_{j=1}^{i-1} s_j - \sum_{j=i+1}^{d} x_j$ for $i = 1, \ldots, d-1$. By Fubini's Theorem and Lemma 3,

$$\bar{H}(\boldsymbol{x}) = \int_{x_1}^\infty \cdots \int_{x_{d-1}}^\infty \int_{s_1+\cdots+s_{d-1}+x_d}^\infty \frac{1}{r^{\alpha-1}} \frac{\Gamma(\alpha)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_d)} \times$$
$$\times \prod_{i=1}^{d-1} s_i^{\alpha_i - 1} \{r - (s_1 + \cdots + s_{d-1})\}^{\alpha_d - 1} dF_R(r) ds_{d-1} \ldots ds_1. \quad (16)$$

Next, the inner integral may be evaluated as follows.

$$\int_{s_1+\cdots+s_{d-1}+x_d}^\infty \frac{1}{r^{\alpha-1}} \{r - (s_1 + \cdots + s_{d-1})\}^{\alpha_d - 1} dF_R(r)$$
$$= \int_{s_1+\cdots+s_{d-1}+x_d}^\infty \frac{1}{r^{\alpha-1}} \{r - (s_1 + \cdots + s_{d-1} + x_d) + x_d\}^{\alpha_d - 1} dF_R(r)$$
$$= \sum_{j_d=0}^{\alpha_d-1} \binom{\alpha_d - 1}{j_d} x_d^{j_d} \int\limits_{\sum_{i=1}^{d-1} s_i + x_d}^{\infty} \frac{1}{r^{\alpha-\alpha_d+j_d}} \left(1 - \frac{\sum_{i=1}^{d-1} s_i + x_d}{r}\right)^{\alpha_d - j_d - 1} dF_R(r)$$
$$= \sum_{i_d=0}^{\alpha_d-1} \binom{\alpha_d - 1}{i_d} x_d^{i_d} \int\limits_{\sum_{i=1}^{d-1} s_i + x_d}^{\infty} \frac{1}{r^{\alpha-\alpha_d+i_d}} \left(1 - \frac{\sum_{i=1}^{d-1} s_i + x_d}{r}\right)^{\alpha - (\alpha-\alpha_d+i_d) - 1} dF_R(r)$$
$$= \sum_{i_d=0}^{\alpha_d-2} \binom{\alpha_d - 1}{i_d} \frac{(\alpha_d - 1 - i_d)!}{\Gamma(\alpha)} x_d^{i_d} (-1)^{\alpha-\alpha_d+i_d} \psi^{(\alpha-\alpha_d+i_d)}(s_1 + \cdots + s_{d-1} + x_d)$$
$$+ \frac{1}{\Gamma(\alpha)} x_d^{\alpha_d - 1} (-1)^{\alpha-1} \psi_+^{(\alpha-1)}(s_1 + \cdots + s_{d-1} + x_d),$$

where the last step follows from Lemma 5. Hence, (16) becomes

$$
\begin{aligned}
\bar{H}(\boldsymbol{x}) = &\sum_{i_d=0}^{\alpha_d-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{d-1})i_d!}\, x_d^{i_d}(-1)^{\alpha-\alpha_d+i_d} \times \\
&\quad\times \int_{x_1}^{\infty}\cdots\int_{x_{d-1}}^{\infty}\prod_{i=1}^{d-1} s_i^{\alpha_i-1}\psi^{(\alpha-\alpha_d+i_d)}(s_1+\cdots+s_{d-1}+x_d)ds_{d-1}\ldots ds_1 \\
&+\frac{\Gamma(\alpha)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{d-1})\Gamma(\alpha_d)}\, x_d^{\alpha_d-1}(-1)^{\alpha-1} \times \\
&\quad\times \int_{x_1}^{\infty}\cdots\int_{x_{d-1}}^{\infty}\prod_{i=1}^{d-1} s_i^{\alpha_i-1}\psi_+^{(\alpha-1)}(s_1+\cdots+s_{d-1}+x_d)ds_{d-1}\ldots ds_1
\end{aligned}
$$

Successive application of Lemma 6 and gathering terms yields

$$
\begin{aligned}
\bar{H}(\boldsymbol{x}) &= \sum_{i_1=0}^{\alpha_1-1}\cdots\sum_{i_d=0}^{\alpha_d-1}(-1)^{2(\alpha_1+\cdots+\alpha_{d-1})+i_1+\cdots+i_d}\frac{\psi^{(i_1+\cdots+i_d)}(x_1+\cdots+x_d)}{i_1!\cdots i_d!}\prod_{j=1}^{d} x_j^{i_j} \\
&= \sum_{i_1=0}^{\alpha_1-1}\cdots\sum_{i_d=0}^{\alpha_d-1}(-1)^{i_1+\cdots+i_d}\frac{\psi^{(i_1+\cdots+i_d)}(x_1+\cdots+x_d)}{i_1!\cdots i_d!}\prod_{j=1}^{d} x_j^{i_j}
\end{aligned}
$$

for all $\boldsymbol{x} \in \mathbb{R}_+^d$, and the proof is complete. □

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