Groups with Poly-Context-Free Word Problem

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$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$X^\star = \{x_1 \ldots x_n \mid x_i \in X, n \in \mathbb{N}_0\}$

G a group with finite generating set X. The *word problem* $W(G, X)$ of G w.r.t. X is

$$\{w \in (X \cup X^{-1})^\star \mid w =_G 1\}.$$

A *language* over an alphabet X is a subset of $X^\star$.

What does the formal language type of $W(G, X)$ tell us about the structure of G?

A *finite automaton* consists of:

- A finite state set Q, with initial state $q_0 \in Q$.
- A finite input alphabet Σ.
- A final state set $F \subseteq Q$.
- A transition function $\delta : (Q, \Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$.

A *regular language* is a language recognised by a finite automaton.

$W(G, X)$ is regular iff G is finite.

A *pushdown automaton* consists of:

- Finite state set Q, initial state q_0, alphabet Σ, final state set $F \subseteq Q$.
- A finite stack alphabet Σ'.
- A transition function δ, which takes as input a triple in $(Q, \Sigma \cup \{\epsilon\}, \Sigma')$ and outputs a set of pairs (q, γ), where $q \in Q$, $\gamma \in (\Sigma')^\star$.

Can accept by final state or by empty stack.

A *context-free language* is a language recognised by a pushdown automaton.

k-*CF* language: an intersection of k CF languages.

poly-CF language: k-CF for some $k \in \mathbb{N}$.

Proposition The class of k-CF languages is closed under inverse homomorphisms and intersection with regular sets.

Proposition $W(G, X)$ is k-CF iff $W(G, Y)$ is k-CF for all finite generating sets Y of G.

A group G is k-*CF* if its word problem is a k-CF language.

Proposition Finitely generated subgroups and finite index overgroups of k-CF groups are k-CF.

Observation If G is virtually a direct product of k free groups, then G is k-CF.

Theorem (Muller and Schupp) G is CF iff G is virtually free.

Conjecture G is k-CF iff G is virtually a direct product of k free groups.

Proposition Let G be a nilpotent or polycyclic group which is not virtually abelian. Then G is neither poly-CF nor co-CF.

Proposition

(i) $\mathbb{Z}^k$ is k-CF but not $(k-1)$-CF.

(ii) $\mathbb{Z} \wr \mathbb{Z}$ is not poly-CF, since $\mathbb{Z}^k \leq \mathbb{Z} \wr \mathbb{Z}$ for all $k \in \mathbb{N}$.

Proposition For any prime p, $C_p \wr \mathbb{Z}$ is not poly-CF.

For $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{Z}^{n+1}$ define

$$G(\mathbf{c}) = \left\langle a, b \mid [b, b^{a^i}] \ (i \in \mathbb{Z}),\ b^{c_0}(b^a)^{c_1} \cdots (b^{a^n})^{c_n} \right\rangle.$$

Lemma Let $\mathbf{c} \in \mathbb{Z}^{n+1}$ with $c_0, c_n \neq 0$ and $\gcd(c_0, \ldots, c_n) = 1$. Then $G(\mathbf{c})$ can be embedded in $\mathbb{Q}^n \rtimes \mathbb{Z}$.

Proposition If $c_0, c_n \neq 0$ and $\gcd(c_0, \ldots, c_n) = 1$, then $G(\mathbf{c})$ is not poly-CF.

Theorem (TB) Let G be a f.g. metabelian group which is not virtually abelian. Then G has at least one of the following:

(i) a polycyclic subgroup which is not virtually abelian,

(ii) a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$,

(iii) a subgroup isomorphic to $C_p \wr \mathbb{Z}$ for some prime p,

(iv) a $G(\mathbf{c})$ subgroup with $c_0, c_s \neq 0$ and $\gcd(c_0, \ldots, c_s) = 1$.

Theorem (TB) Let G be a f.g. torsion-free soluble group which is not virtually abelian. Then G has at least one of the following:

(i) a polycyclic subgroup which is not virtually abelian,

(ii) subgroups isomorphic to $\mathbb{Z}^k$ for all $k \in \mathbb{N}$,

(iii) a $G(\mathbf{c})$ subgroup with $c_0, c_n \neq 0$ and $\gcd(c_0, \ldots, c_n) = 1$.

Theorem (TB) A f.g. metabelian or torsion-free soluble group has poly-CF word problem iff it is virtually abelian.