A class of subgroups of Thompson's group V

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January 3, 2007

Abstract

We prove that the subgroups of the Thompson group V which are closed under a binary operation introduced by K. S. Brown are in bijective correspondence with a class of varieties of algebras with a single binary operation. In addition, we prove that there are exactly three such proper subgroups which contain the Thompson group F.

2000 AMS Subject Classification: 20F05, 20M18.

1 Introduction

This paper is both an advertisement and application of our paper [14]. We shall show how semigroup theory can be used to obtain information about an important group: the Thompson group V. The results from that paper needed here are described in Section 2.

Kenneth S. Brown proved [3] that the Thompson group F is equipped with a binary operation, additional to its multiplication, which he denotes by $*$, such that $*$ is a group homomorphism from $F \times F$ to F. This operation is 'associative upto conjugacy' meaning that there is an element $x_0 \in F$ such that for all elements $f, g, h \in F$ the following equation holds:

$$f * (g * h) = ((f * g) * h)^{x_0}.$$

In other words: the element x_0 implements the associative law by means of conjugation.[1] The group F is a subgroup of the group V, and this binary operation

[1] In fact this operation is also discussed in my book [13] (pp 297–304) where it arose in Peter

can be extended to the whole V and gives rise to an injective homomorphism from $V \times V$ to V. It is natural to ask which proper subgroups of V are closed under the operation.[2] We show that this question is equivalent to classifying certain special varieties of universal algebras. Those subgroups of this type which contain F can be described exactly: there are exactly three. The proof uses semigroup theory in two different ways: inverse semigroups are used to link subgroups of V closed under the operation to certain semigroup varieties; we then use the theory of semigroup varieties to show that only three semigroup varieties satisfy the conditions which arise. The Thompson groups F and V and the binary operation $*$ will be defined in Section 2.

2 Preliminaries

We summarise the theory developed in [14] needed to understand this paper. Our reference for inverse semigroup theory is [13].

Patrick Dehornoy proved that the Thompson groups F and V are closely related to varieties of semigroups: the group F is the geometry group of the variety of all semigroups whereas the group V is the geometry group of the variety of commutative semigroups [5, 6, 7, 8]. In [14], we established a general result linking certain kinds of universal algebra varieties to a special class of inverse semigroups. To state this result we need some definitions.

Remark Although the results can be stated for arbitrary universal algebras, I shall state them here for the case where there is exactly one binary operation symbol, which I shall denote by $\times$.

We shall assume that there is a countably infinite set of variables X, and we shall work with terms $T(X)$ over these variables. A term is *linear* if any variable that occurs occurs exactly once. An *identity* $s \approx t$ is simply an ordered pair of terms. We say that it is *linear* if the same variables occur on each side, and no variable appears more than once on each side. A variety of algebras is *linear* if it can be defined by a set of linear identities. A *substitution* maps a finite number of variables to the set of terms. A *relabelling substitution* maps a finite number of variables bijectively to a finite number of variables. An identity $s' \approx t'$ is an *instance of* an identity $s \approx t$ if there is a substitution f such that $s' = f(s)$ and $t' = f(t)$. We say that it is a *proper instance* if f is not a relabelling substitution. The book by Jorge Almeida [1] is a good reference for the theory of varieties. An inverse semigroup S equipped with an additional binary operation $*$ such that $*\colon S \times S \to S$ is a homomorphism is called an

Hines PhD work [11] on analysing Girard's geometry of interaction program [10], although at that time we did not know about the connection with V.

[2]The subgroups of F closed under $*$ are considered in [9]. I am grateful to my colleague Nick Gilbert here at Heriot-Watt for first alerting me to this paper, and the referee for reminding me of it.

inverse algebra.[3] Inverse subsemigroups of S which contain all the idempotents of S are said to be *wide*.[4] An observation we shall use on a number of occasions is that wide inverse submonoids are also order ideals. We can now state a special case of Theorem 5.7 [14].

Theorem 2.1 *There is an inverse algebra $\mathcal{LCM}_2$, called the* linear clause monoid*, such that there is a bijection between the linear varieties and the wide inverse subalgebras of $\mathcal{LCM}_2$.* ∎

In fact, the binary operation on $\mathcal{LCM}_2$, which I shall denote by $\otimes$, is an injective homomorphism. We shall give two different, but equivalent, descriptions of $\mathcal{LCM}_2$.

Description 1: This works with equivalence classes of linear identities. We say that two linear identities are *equivalent* if they differ only by a relabelling substitution. We denote the equivalence class of the linear identity $s \approx t$ by $[s,t]$, and the set of equivalence classes by $\mathcal{LCM}_2$. The product of $[s,t]$ and $[u,v]$ is defined as follows. Without loss of generality, we can assume that t and u have no variables in common. It is possible to find substitutions f and g such that the following two properties hold:

(i) $f(t) = g(u)$.

(ii) If f' and g' are such that $f'(t) = g'(u)$ then there is a substitution h such that $f' = hf$ and $g' = hg$.

The proof of this is Theorem 3.4 and Proposition 8.3 of [14] We accordingly define the product of $[s,t]$ and $[u,v]$ to be $[f(s), g(v)]$. With respect to this operation, $\mathcal{LCM}_2$ is an inverse monoid: the identity is $[x,x]$, the idempotents are $[s,s]$, and the inverse of $[s,t]$ is $[t,s]$. The additional binary operation on $\mathcal{LCM}_2$, which I shall denote by $\otimes$, is defined as follows. Given $[s_1, t_t]$ and $[s_2, t_2]$ assume, without loss of generality, that the variables in s_1 are disjoint from those in s_2. Define

$$[s_1, t_1] \otimes [s_2, t_2] = [s_1 \times s_2, t_1 \times t_2].$$

Description 2: This will enable us to make the explicit connection with the Thompson group V. A good reference for this approach is [2]. Let M be a monoid. A *right ideal* R of M is a subset such that $RM \subseteq R$. A bijective function θ: $R \to R'$ is called a *right ideal isomorphism* if R and R' are right ideals and $\theta(rm) = \theta(r)m$ for all $r \in R$ and $m \in M$. A right ideal R is said to be *essential* if $R \cap R' \neq \emptyset$ for all right ideals R'. We define the inverse monoid T_2 to be the set of all right ideal isomorphisms between essential finitely generated right ideals of the free monoid on 2 generators; it can be shown that such right

[3]For the purposes of this paper. Observe that 'algebra' here is being used in the sense of 'universal algebra'.

[4]The term 'full' is often used in semigroup theory, but this conflicts with the usage in category theory.

ideals are precisely those generated by maximal finite prefix codes.[5] It remains to describe how T_2 can be viewed as an inverse algebra. The free monoid on two generators consists of all finite strings over an alphabet with two letters. Let this alphabet consist of p and q, and denote the free monoid on this alphabet by M_2. If Z and Z' are maximal prefix codes so is $pZ \cup qZ'$. Let $\beta_1 \colon Z_1 M_2 \to Z'_1 M_2$ and $\beta_2 \colon Z_2 M_2 \to Z'_2 M_2$ be two elements of T_2. Define $\beta_1 \circ \beta_2$ as follows: its domain is $(pZ_1 \cup qZ_2)M_2$, its codomain is $(pZ'_1 \cup qZ'_2)M_2$, and the rule is $pz \mapsto p\beta_1(z)$ and $qz \mapsto q\beta_2(z)$.

The following was proved in a more general frame as Theorem 6.10 of [14].

Theorem 2.2 *The inverse algebras $\mathcal{LCM}_2$ and T_2 are isomorphic.* ■

The connection between the two inverse monoids above and Thompson's group V follows from the work of Section 8 of [14], but can be seen most easily using the following ideas. Let S be an inverse semigroup. Define the relation σ on S by $s \,\sigma\, t$ iff $u \leq s, t$ for some u, where the order is the natural partial order on the inverse semigroup. Then σ is a congruence and S/σ is a group. In addition, if ρ is any congruence on S such that S/ρ is a group then $\sigma \subseteq \rho$. For this reason, σ is called the *minimum group congruence*. The following can be deduced from [2], and Proposition 8.7 of [14].

Theorem 2.3 *The group T_2/σ is isomorphic to the Thompson group V. In addition, the injective binary operation $\circ$ on T_2 induces an injective binary operation $*$ on V when we define $*$ on V by*

$$\sigma(s) * \sigma(t) = \sigma(s \circ t).$$

■

It is not hard to prove that the binary operation we have defined is essentially the same as the one given by Brown.

Remark Theorem 2.3 can be taken as the definition of the Thompson group V. It is identical to the one given by Scott in [16], although this needs a little decoding. The monoid T_2 is the inverse semigroup of all right ideal isomorphisms between essential finitely generated right ideals. The group V consists of the maximal elements of T_2; the composition of two maximal elements need not be maximal, but does lie beneath a unique maximal element: this is defined to be the product in the group V. The Thompson group F is that subgroup of V consisting of all elements that preserve the 'dictionary order'. We shall not use this description here, but [2] contains a lot more on the Thompson groups F and V viewed in this way.

[5] A 'prefix code' is simply a subset of a free monoid in which no two elements are proper prefixes of each other.

3 Results

Our first main result, Proposition 3.2, tells us that the operation $*$ defined on the group V is closely related to the group-theoretic structure of V. It generalises Brown's observation mentioned in the second paragraph of this paper.

Let $u(x_1, x_2) = x_1 \times x_2$. Then the definition of $\otimes$ can be stated in the form

$$u([s_1, t_1], [s_2, t_2]) = [u(s_1, s_2), u(t_1, t_2)].$$

The following lemma generalises this to arbitrary linear terms u.

Lemma 3.1 *Let $u(x_1, \ldots, x_n)$ be a linear term in $T(X)$ where the variables occur in the order indicated from left-to-right. Let $[s_i, t_i]$ be n elements of $\mathcal{LCM}_2$ where, without loss of generality, we assume that the sets of variables of the s_i are disjoint. Then*

$$u([s_1, t_1], \ldots, [s_n, t_n]) = [u(s_1, \ldots, s_n), u(t_1, \ldots, t_n)].$$

Proof We prove the result by induction on n. The case $n = 1$ is immediate. Assume the result holds for all $n < m$. We prove the result for $n = m$. The term $u(x_1, \ldots, x_m)$ is equal to $u_1(x_1, \ldots, x_r) \times u_2(x_{r+1}, \ldots, x_m)$ for some r and some terms u_1 and u_2. By the induction hypothesis, we therefore can write

$$u_1([s_1, t_1], \ldots, [s_r, t_r]) = [u_1(s_1, \ldots, s_r), u_1(t_1, \ldots, t_r)]$$

and

$$u_2([s_{r+1}, t_{r+1}], \ldots, [s_m, t_m]) = [u_2(s_{r+1}, \ldots, s_m), u_2(t_{r+1}, \ldots, t_m)].$$

The result now follows from the definition of the operation $\otimes$ on $\mathcal{LCM}_2$. ■

A linear identity $s \approx t$ will be called *reducible* if there is a linear identity $s' \approx t'$ and a substitution f, which is not a relabelling substitution, such that $f(s') = s$ and $f(t') = t$. A linear identity which is not reducible will be called *irreducible*. Thus the irreducible linear identities are not proper instances of any other linear identity.

An element $g \in V$ is said to *implement the linear identity $u \approx v$ (by conjugation)* iff

$$u(h_1, \ldots, h_n) = gv(h_1, \ldots, h_n)g^{-1}$$

for all $h_i \in V$.

Proposition 3.2 *Each linear identity is implemented by an element of V, and each element of V implements a linear identity.*

Proof Let $u \approx v$ be any linear identity, where u contains n variables. Then $[u, v]$ is an element of in $\mathcal{LCM}_2$. We calculate

$$[u, v]v([s_1, t_1], \ldots, [s_n, t_n])[v, u].$$

By Lemma 3.1, this is equal to

$$[u, v][v(s_1, \ldots, s_n), v(t_1, \ldots, t_n)][v, u].$$

We now apply the definition of multiplication via unification to get

$$[u(s_1, \ldots, s_n), u(t_1, \ldots, t_n)],$$

which, by Lemma 3.1 again, gives

$$u([s_1, t_1], \ldots, [s_n, t_n]).$$

Both claims now follow since every element of V is determined by a linear identity, and every linear identity gives rise to an element of V. ■

Remark There is a bijection between the elements of V and the irreducible linear identities (upto the choice of variables). This follows from the observation that on the inverse monoid $\mathcal{LCM}_2$ each σ-class contains a unique maximum element.[6] This is proved in [2] as Proposition 2.1, and is equivalent to the result stated in [4] that each element is represented by exactly one reduced tree diagram. Thus each element of V implements an irreducible linear identity.

A subgroup of V closed under $*$ will be called a *subalgebra*. The goal of the remainder of this paper is to classify the subalgebras of V that contain F.

Let S be an inverse semigroup. Inverse subsemigroups of S are mapped to subgroups of S/σ. We need a slightly stronger result.

Lemma 3.3 *Let T be a wide inverse submonoid of the inverse monoid S. Denote the minimum group congruence on S by σ_S and the one on T by σ_T. Then the image of T in S/σ_S is isomorphic to T/σ_T.*

Proof For each $t \in T$ we have that $\sigma_T(t) \subseteq \sigma_S(t)$. We define a function from T/σ_T to S/σ_S by $\sigma_T(t) \mapsto \sigma_S(t)$. This is clearly a homomorphism. It is injective because T is wide in S and so is an order ideal. The image of this map is precisely the image of T under the natural map determined by σ_S. ■

An inverse submonoid T of an inverse monoid S is said to be *closed* if $a \in T$ and $a \leq b$ implies that $b \in T$.

Lemma 3.4 *Let S be an inverse monoid. Then the wide closed inverse submonoids of S are in bijective correspondence with the subgroups of $G = S/\sigma$.*

Proof For each wide inverse submonoid U of S let $G(U)$ be the image of U in G. Observe that by Lemma 3.3, $G(U)$ is isomorphic to U/σ_U. Suppose that $G(U) = G(V)$. Let $a \in U$. Then there exists $b \in V$ such that $\sigma(a) = \sigma(b)$. Thus there exists $c \in S$ such that $c \leq a, b$. Now V is an order ideal and so $c \in V$.

[6] Inverse monoids with this property are known as *F-inverse*.

But V is closed and so $a \in V$. Thus $U \subseteq V$. By symmetry $U = V$. Now let H be a subgroup of G. Let $U = \{a \in S \colon \sigma(a) \in H\}$. It is routine to check that U is a wide full inverse submonoid of S. ∎

An inverse semigroup S is said to be *E-unitary* if $e \leq s$ where e is an idempotent implies that s is an idempotent. The linear clause monoid is E-unitary; see Theorem 8.4 of [14].

Lemma 3.5 *Let S be an E-unitary inverse algebra. Then $G = S/\sigma$ is an inverse algebra. In addition, the closed wide inverse subalgebras of S are in bijective correspondence with the subalgebras of G.*

Proof Denote the algebra operation on S by $\otimes$. We define

$$\sigma(s) \bullet \sigma(t) = \sigma(s \otimes t).$$

The proof that this operation is well-defined and leads to an algebra structure is straightforward, once it is noted that in an E-unitary inverse semigroup $a \, \sigma \, b$ iff $a^{-1}b$ and ab^{-1} are both idempotents. A complete proof is written out as Proposition 8.7 in [14]. Observe that by construction the natural map from S to G is also an algebra homomorphism. By Lemma 3.4, there is a bijection between the wide closed inverse submonoids of S and the subgroups of G. By our observation above, and the definition of the algebra operation on G, this bijection restricts to a bijection between subalgebras in S and subalgebras of G. ∎

We say that a linear variety is *closed* if the following condition holds: if $s \approx t$ is a linear identity that holds in every element of the variety, and if $s' \approx t'$ is a linear identity such that $s = f(s')$ and $t = f(t')$ for some substitution f then $s' \approx t'$ holds in every element of the variety.

Lemma 3.6 *There is a bijection between closed linear varieties and the closed wide inverse subalgebras of the linear clause monoid.*

Proof This follows from Theorem 2.1, and the observation that the linear identity $s' \approx t'$ is an instance of the linear identity $s \approx t$ iff $[s', t'] \leq [s, t]$ in the linear clause monoid. ∎

The following is the key to classifying the subgroups of V closed under the operation $*$.

Theorem 3.7 *There is a bijection between the closed linear varieties and the subalgebras of V. This bijection restricts to one between the closed linear varieties that satisfy the associativity law and the subalgebras of V that contain the Thompson group F.*

Proof By Lemma 3.5, there is a bijection between the subalgebras of V and the closed wide inverse subalgebras of the linear clause monoid. By Lemma 3.6, there is a bijection between the closed wide inverse subalgebras of the linear clause monoid and the closed linear varieties. Thus we have a bijection between the closed linear varieties and the subalgebras of V.

By work of Dehornoy, we know that the (linear) variety of semigroups corresponds to the Thompson group F [6, 8]. Thus the subalgebras of V that *contain* F correspond to the closed linear varieties that *are contained in* the variety of semigroups; the reversal in order comes about because of Birkhoff's variety theorem [1]. ■

By the second half of Theorem 3.7, to classify the subalgebras of V that contain F we have to classify the closed linear varieties of semigroups. Clearly, we need only the ones that are strictly contained in the variety of semigroups; I shall refer to these as 'proper' varieties. In what follows, I shall denote the binary operation symbol $\times$ by concatenation. We shall be working with linear semigroup identities and these all have the form

$$x_1 \dots x_n \approx x_{\tau(1)} \dots x_{\tau(n)}$$

where τ is a permutation of $1, \dots, n$. Such an identity is said to be *non-trivial* if τ is not the identity. Observe that brackets can be omitted because we are assuming that we are always working with semigroups. A linear identity is said to be $\mathsf{P}_{p,q}$ if τ fixes the first p subscripts, but not the first $p+1$, and τ fixes the last q subscripts, but not the last $q+1$. Because of closure, we shall only be interested in linear identities which are either $\mathsf{P}_{0,0}$, $\mathsf{P}_{1,0}$, $\mathsf{P}_{0,1}$ or $\mathsf{P}_{1,1}$.

Proposition 3.8 *There are exactly four closed linear proper varieties of semigroups:*

(1) *The variety of commutative semigroups: the variety of semigroups defined by $xy \approx yx$.*

(2) *The variety of semigroups defined by $xyz \approx xzy$.*

(3) *The variety of semigroups defined by $xyz \approx yxz$.*

(4) *The variety of semigroups defined by $xuvz \approx xvuz$.*

Proof We begin with a number of observations which will be useful throughout our proof.

(a) A theorem of Putcha and Yaqub (see Exercise 6.3.9 of [1]) states that a variety of semigroups satisfying a non-trivial linear identity must satisfy an identity of the form

$$x_1 \dots x_p y z t_1 \dots t_q \approx x_1 \dots x_p z y t_1 \dots t_q.$$

Thus a closed linear variety of semigroups satisfying a non-trivial linear identity must satisfy

$$xuvz \approx xvuz.$$

(b) All the linear semigroup identities deducible from identities in $\mathsf{P}_{1,0}$ belong to $\mathsf{P}_{p,q}$ where $p \geq 1$.

(c) All the linear semigroup identities deducible from identities in $\mathsf{P}_{0,1}$ belong to $\mathsf{P}_{p,q}$ where $q \geq 1$.

(d) All the linear semigroup identities deducible from identities in $\mathsf{P}_{1,1}$ belong to $\mathsf{P}_{p,q}$ where $p, q \geq 1$.

The four varieties of semigroups (1)–(4) are distinct. This is well-known but for completeness I shall give an argument. The variety (1) is distinguished from the remaining three because they each contain non-commutative semigroups. The varieties (2),(3)and (4) are distinguished from each other by the bands[7] they contain: variety (2) contains the left normal bands, variety (3) the right normal bands, and variety (4) the normal bands.

Next we prove that each of these varieties is closed. For (1), all linear semigroup identities hold and so this is closed. For (2), we note by observation (b) that the only linear semigroup identities which can be deduced belong to $\mathsf{P}_{p,q}$ where $p \geq 1$. But using $xyz \approx xzy$ and $xuvz \approx xvuz$ (observation (a)) we can deduce *every* such linear semigroup identity in $\mathsf{P}_{p,q}$ where $p \geq 1$. Suppose now that $s \approx t$ is in $\mathsf{P}_{p,q}$ where $p \geq 1$ and so in the variety and that $s' \approx t'$ is such that there is a substitution f such that $s = f(s')$ and $t = f(t')$. Then in fact $s' \approx t'$ belongs to $P_{p',q'}$ for some $p' \geq 1$ and so $s' \approx t'$ belongs to the variety. Thus (2) is closed. The proof that (3) is closed is similar to the proof for (2) and uses observation (c). To prove that (4) is closed, we apply observation (d) that the only linear semigroup identities we can deduce from $xuvz \approx xvuz$ are in $\mathsf{P}_{p,q}$ where $p, q \geq 1$. But every such linear semigroup identity can be deduced using $xuvz \approx xvuz$. Suppose now that $s \approx t$ is in $\mathsf{P}_{p,q}$ where $p, q \geq 1$ and so in the variety and that $s' \approx t'$ is such that there is a substitution f such that $s = f(s')$ and $t = f(t')$. Then in fact $s' \approx t'$ belongs to $P_{p',q'}$ for some $p', q' \geq 1$ and so $s' \approx t'$ belongs to the variety. Thus (4) is closed.

It remains to show that a closed linear semigroup variety must be equal to one of these four. By a theorem of Putcha and Yaqub [15] (see Proposition 6.3.8 of [1]), any semigroup variety satisfying a non-trivial linear identity of type $\mathsf{P}_{0,0}$ must satisfy an identity of the form

$$x_1 \ldots x_r y_1 \ldots y_r \approx y_1 \ldots y_r x_1 \ldots x_r$$

for r large enough. We deduce that a closed linear semigroup variety satisfying a non-trivial linear identity of type $\mathsf{P}_{0,0}$ is the variety of commutative semigroups. Taking into account the closure assumption, it follows that we can restrict our attention to closed linear semigroup varieties defined by semigroup identities of types $\mathsf{P}_{1,0}$, $\mathsf{P}_{0,1}$ or $\mathsf{P}_{1,1}$ only. However, if a closed linear semigroup variety satisfies a non-trivial identity from $\mathsf{P}_{1,0}$ and one from $\mathsf{P}_{0,1}$ then it satisfies a non-trivial identity from $\mathsf{P}_{0,0}$ and so is the variety of commutative semigroups.

[7]Idempotent semigroups. See [12] for more on these classes of bands.

We deduce that a closed linear semigroup variety that is not the variety of semigroups or the variety of commutative semigroups is defined by: identities from $\mathsf{P}_{1,1}$ alone; identities from $\mathsf{P}_{1,0}$ alone; or identities from $\mathsf{P}_{0,1}$ alone. The first case can be quickly dealt with. By observation (d), the only linear semigroup identities satisfied are those in $\mathsf{P}_{p,q}$ where $p, q \geq 1$. However, by observation (a) the linear semigroup identity $xuvz \approx xvuz$ holds. Thus every linear semigroup identity in $\mathsf{P}_{p,q}$ where $p, q \geq 1$ can be deduced. It follows that the linear semigroup variety in question is (4). To complete the proof, it is sufficient to show that a closed linear semigroup variety satisfying a non-trivial identity from $\mathsf{P}_{1,0}$ must satisfy $xyz \approx xzy$ and so be variety (2); it will then follow that an analogous argument will show that a closed linear semigroup variety satisfying a non-trivial identity from $\mathsf{P}_{0,1}$ must satisfy $xyz \approx yxz$ and so be variety (3). Let the linear identity satisfied be

$$xx_1 \ldots x_n \approx xx_{\tau(1)} \ldots x_{\tau(n)}$$

where $n \geq 2$. If $n = 2$ then the only non-trivial linear identity of the required form is $xyz \approx xzy$. Induction hypothesis: all closed linear semigroup varieties satisfying a non-trivial linear identity of type $\mathsf{P}_{1,0}$ with $n < m$ satisfy $xyz \approx xzy$. Consider now a closed linear semigroup variety satisfying

$$xx_1 \ldots x_m \approx xx_{\tau(1)} \ldots x_{\tau(m)}.$$

Use the identity $xuvy \approx xvuy$, which holds by observation (a), to rearrange $x_{\tau(1)} \ldots x_{\tau(m-1)}$ into a new product, u say, such that $ux_{\tau(m)}$ has the form $u'x_{j-1}x_jv'$ for some $1 < j \leq m$. The identity

$$xx_1 \ldots x_m \approx xu'x_{j-1}x_jv'$$

is an instance of a non-trivial linear identity of length $m-1$ belonging to $P_{1,0}$, and so we may invoke our induction hypothesis. ■

Combining Theorem 3.7 and Proposition 3.8, we have proved the following.

Theorem 3.9 *There are exactly three proper subgroups of V strictly containing F and closed under the binary operation $*$. They are: a group V_L that corresponds to the semigroup variety defined by the linear semigroup identity $xyz \approx xzy$; a group V_R that corresponds to the semigroup variety defined by the linear semigroup identity $xyz \approx yxz$; and a group V_N that corresponds to the semigroup variety defined by the linear semigroup identity $xuvy \approx xvuy$. In addition, $V_L \cap V_R = V_N$ and the smallest subgroup of V containing V_L and V_R and closed under $*$ is V itself.* ■

Remark The group V_R has been studied before by Dehornoy [8].

By restricting attention to the subalgebras of V containing F, we have made life easier: much is known about linear semigroup varieties. However, the theory developed in this paper also shows that arbitrary subalgebras of V are classified by the closed linear varieties of algebras (over a single binary operation). Thus our understanding of certain kinds of subgroups of V depends on our understanding of certain kinds of varieties of universal algebras. The paper [9] adopts exactly the same approach as ours but adapted to the case of subalgebras of F.

Acknowledgements I am grateful to Jorge Almeida of the University of Porto for a very helpful correspondence on the subject of semigroup varieties. I would also like to thank the anonymous referee for their thorough reading of the original version of this paper and suggestions that led to significant improvements.

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