Orthogonal completions of the polycyclic monoids

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Abstract

We introduce the notion of an orthogonal completion of an inverse monoid with zero. We show that the orthogonal completion of the polycyclic monoid on n generators is isomorphic to the inverse monoid of right ideal isomorphisms between the finitely generated right ideals of the free monoid on n generators, and so we can make a direct connection with the Thompson groups $V_{n,1}$.

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1 Introduction

The polycyclic monoids were introduced by Nivat and Perrot [7] as generalisations of the bicyclic monoid. They can be used to study context-free languages [3, 4], they arise in the construction of the Cuntz C^*-algebras [8], and they are implicit in the definition of amenability given in [2]. They are discussed in detail in Chapter 9 of my book [5], and I outline their properties below. Prior knowledge of these semigroups is not necessary to read this paper. In [1], Birget described a connection between the Thompson group V and the polycyclic monoid on two generators: he proved that the group is a subgroup of a quotient algebra of the monoid. His calculations with the polycyclic monoids suggested the results in this paper.

2 Orthogonal completions of inverse semigroups

Throughout this paper, we shall be dealing with inverse semigroups with zero. *We shall always require that homomorphisms between such semigroups map zero* *to zero*. Multiplication in semigroups will usually be denoted by concatenation, but occasionally I shall use $\cdot$ for clarity. Inverse semigroups come equipped with their own order, called the *natural partial order*, and this will always be the order used. We write $\mathbf{d}(s) = s^{-1}s$ and $\mathbf{r}(s) = ss^{-1}$ for each element s in the inverse semigroup S. The key definition for this paper is the following. A pair of elements $s, t \in S$ is said to be *orthogonal* if

$$s^{-1}t = 0 = st^{-1}.$$

Observe that s and t are orthogonal iff $\mathbf{d}(s)\mathbf{d}(t) = 0$ and $\mathbf{r}(s)\mathbf{r}(t) = 0$. A subset of S is said to be *orthogonal* iff each pair of distinct elements in it is orthogonal. We denote by $s + t$ the join of orthogonal elements s and t if it exists. More generally, we denote by $\sum A$ the join of the orthogonal subset A if it exists. In these cases, we talk about *orthogonal joins*. The goal of this section is to construct an 'orthogonal completion' $D(S)$ of an inverse semigroup with zero S (Theorem 2.5).

Lemma 2.1 *Let A and B be orthogonal subsets containing zero of an inverse semigroup with zero.*

(i) *AB is a orthogonal subset containing zero.*

(ii) *$AA^{-1} = \{aa^{-1} \colon a \in A\}$ and $A^{-1}A = \{a^{-1}a \colon a \in A\}$.*

(iii) *$A = AA^{-1}A$ and $A^{-1} = A^{-1}AA^{-1}$.*

Proof (i) Let $ab, cd \in AB$ be distinct elements where $a, c \in A$ and $b, d \in B$. Then $(ab)^{-1}cd = b^{-1}a^{-1}cd = 0$ if $a \neq c$. If $a = c$, then $b^{-1}a^{-1}ad \leq b^{-1}d$. Now $b \neq d$ since ab and cd are distinct. Thus $b^{-1}d = 0$. It follows that in both cases $(ab)^{-1}cd = 0$. A similar argument shows that $ab(cd)^{-1} = 0$.

(ii) Let $ab^{-1} \in AA^{-1}$. If $a \neq b$ then $ab^{-1} = 0$. Thus the non-zero elements of AA^{-1} are of the form aa^{-1}. A similar argument applies to the elements of $A^{-1}A$.

(iii) This follows from (ii). ■

Let $D(S)$ denote the set of finite orthogonal subsets of the inverse semigroup S that contain zero.

Lemma 2.2 *With the above definition, $D(S)$ is an inverse semigroup with zero. If S is a monoid then $D(S)$ is a monoid.*

Proof By Lemma 2.1, $D(S)$ is a semigroup under multiplication of subsets. We now describe the idempotents. Suppose that $A^2 = A$. Let $a \in A$ be a non-zero element. Then $a = bc$ where $b, c \in A$ are non-zero. Now $b^{-1}a = b^{-1}bc$. By assumption, the righthand side is non-zero, but the lefthand side will be zero unless $b = a$. A similar argument shows that $c = a$. Hence $a = a^2$. It is now clear that the idempotents in $D(S)$ are the orthogonal subsets containing zero consisting entirely of idempotents of S. It follows immediately that idempotents commute. By Lemma 2.1, we can now deduce that $D(S)$ is inverse.

In the monoid case, if 1 is the identity of S then $\{0,1\}$ is the identity of $D(S)$. ■

Lemma 2.3 *In the inverse semigroup $D(S)$ the following hold:*

(i) *If $A, B \in D(S)$ then $A \leq B$ iff for each $a \in A$ there exists $b \in B$ such that $a \leq b$.*

(ii) *If $A, B \in D(S)$ then A and B are orthogonal iff $A \cup B$ is an orthogonal subset of S.*

(iii) *If $A, B \in D(S)$ and A and B are orthogonal then $A + B = A \cup B$.*

(iv) *Multiplication distributes over finite orthogonal joins in $D(S)$.*

Proof Observe that mutiplication of subsets of a semigroup distributes over union.

(i) Suppose first that for each $a \in A$ there exists $b \in B$ such that $a \leq b$. We prove that $A = BA^{-1}A$. Let $a \in A$. Then there is a $b \in B$ such that $a \leq b$. Thus $a = ba^{-1}a$. Hence $A \subseteq BA^{-1}A$. To prove the reverse inclusion, let $ba^{-1}a \in BA^{-1}A$ be non-zero. By assumption, there is $b_1 \in B$ such that $a \leq b_1$. Hence $ba^{-1}a \leq bb_1^{-1}b_1$. Now $bb_1^{-1}b_1 \neq 0$ and so $b = b_1$. Hence $a \leq b$ and so $a = ba^{-1}a$. Thus $BA^{-1}A \subseteq A$. We have therefore shown that $A \leq B$. Conversely, suppose that $A \leq B$. Then $A = BA^{-1}A$. Let $a \in A$ be non-zero. Then $a = ba_1^{-1}a_1$ for some $b \in B$ and $a_1 \in A$. It is immediate that $a \leq b$.

(ii) Suppose that A and B are orthogonal. Let $a \in A$ and $b \in B$. Then $a^{-1}b \in A^{-1}B$ and so by assumption $a^{-1}b = 0$. Similarly $ab^{-1} = 0$. Thus $A \cup B$ is a orthogonal subset of S. The converse is clear.

(iii) In $D(S)$, if $A \subseteq B$ then $A \leq B$ by (i). It follows that $A, B \leq A \cup B$. Let $A, B \leq C$. We calculate $C(A \cup B)^{-1}(A \cup B)$. This reduces quickly to $C(A^{-1}A \cup B^{-1}B)$, which is equal to $A \cup B$. This shows that $A \cup B \leq C$, proving the result.

(iv) Immediate by (iii), and our first stated observation. ■

An inverse semigroup with zero S will be said to be *orthogonally complete* if it satisfies the following two axioms:

(DC1) S has joins of all finite orthogonal subsets.

(DC2) Multiplication distributes over finite orthogonal joins.

Example Let X be a set. The symmetric monoid on X, denoted $I(X)$, is the inverse semigroup of all partial bijections on X, where functional composition is evaluated from right to left. If f and g are orthogonal in $I(X)$ then f and g have disjoint domains and disjoint ranges. It follows that their union $f \cup g$ also belongs to $I(X)$. It is easy to check that the symmetric inverse monoid is orthogonally complete.

The proof of the following is straightforward.

Lemma 2.4 *Let S be orthogonally complete.*

(i) *If $\sum_{i=1}^{n} a_i$ exists, then $\sum_{i=1}^{n} a_i^{-1}$ exists and*

$$(\sum_{i=1}^{n} a_i)^{-1} = \sum_{i=1}^{n} a_i^{-1}.$$

(ii) *If $\sum_{i=1}^{n} a_i$ exists, then both $\sum_{i=1}^{n} \mathbf{d}(a_i)$ and $\sum_{i=1}^{n} \mathbf{r}(a_i)$ exist and*

$$\mathbf{d}(\sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} \mathbf{d}(a_i) \text{ and } \mathbf{r}(\sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} \mathbf{r}(a_i).$$

■

Homomorphisms between inverse semigroups with zero map finite orthogonal subsets to finite orthogonal subsets. If orthogonal joins are preserved then we say that the homomorphism is *orthogonal join preserving*. Define the function $\iota\colon S \to D(S)$ by $s \mapsto \{0, s\}$. This is an injective homomorphism.

Theorem 2.5 *Let S be an inverse semigroup with zero. Then $D(S)$ is orthogonally complete. Let $\theta\colon S \to T$ be a homomorphism to an orthogonally complete inverse semigroup T. Then there is a unique orthogonal join preserving homomorphism $\phi\colon D(S) \to T$ such that $\phi\iota = \theta$.*

Proof Lemma 2.3 proves that $D(S)$ is orthogonally complete. Define

$$\phi\colon D(S) \to T$$

by $\phi(A) = \sum \theta(A)$. This is well-defined since if A is an orthogonal subset of S then $\phi(A)$ is an orthogonal subset of T. This is a homomorphism using the fact that T is orthogonally complete. The fact that $\phi\iota = \theta$ is straightforward. Let ϕ' be another homomorphism satisfying the properties. We show that $\phi' = \phi$. The key observation is that we can write finite orthogonal subsets in the form

$$\{0, a_1, \ldots, a_n\} = \sum_{i=1}^{n} \{0, a_i\}.$$

Thus

$$\phi'(\{0, a_1, \ldots, a_n\}) = \phi'(\sum_{i=1}^{n} \{0, a_i\}) = \theta(a_1) + \ldots + \theta(a_n) = \phi(\{0, a_1, \ldots, a_n\}).$$

■

The inverse monoid $D(S)$ is called the *orthogonal completion of S*.

Remark What I have called the 'orthogonal completion' I should really refer to as the 'finitary orthogonal completion'. The results of this section can all be generalised in the obvious way to the construction of 'infinitary' orthogonal completions.

Notation It is natural to write the elements of $D(S)$ as formal sums $\sum_{i=1}^{n} a_i$ of elements of S where $a_i \neq a_j$ implies a_i and a_j are orthogonal. We require that the sum operation is commutative and idempotent and that the left and right distributivity laws hold. In addition, $0 + a = a = a + 0$ for all elements $a \in S$.

Problem Given an inverse semigroup presentation of S, find an inverse semigroup presentation of $D(S)$.

In the usual way, Theorem 2.5 yields a functor from the category of inverse semigroups with zero and their homomorphisms to the category of orthogonally complete inverse semigroups and their orthogonal join preserving homomorphisms. We denote the image of $\theta\colon S \to T$ under D by $D(\theta)\colon D(S) \to D(T)$.

3 Orthogonal completions of polycyclic monoids

In this section, the orthogonal completion of the polycyclic monoid on n generators will be shown to be isomorphic to the inverse monoid of right ideal isomorphisms between the finitely generated right ideals of the free monoid on n generators (Theorem 3.6).

Put $A_n = \{a_1, \ldots, a_n\}$. A string in A_n^*, the free monoid generated by A_n, will be called *positive*. The empty string is denoted ε. If $u = vw$ are strings, then v is called a *prefix* of u, and a *proper prefix* if w is not the empty string. A pair of elements of A_n^* is said to be *prefix-comparable* if one is a prefix of the other. If x and y are prefix-comparable we define

$$x \wedge y = \begin{cases} x & \text{if } y \text{ is a prefix of } x \\ y & \text{if } x \text{ is a prefix of } y \end{cases}$$

The polycyclic monoid P_n, where $n \geq 2$, is defined as a monoid with zero by the following presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\colon a_i^{-1} a_i = 1 \,\text{and}\, a_i^{-1} a_j = 0, i \neq j \rangle.$$

Intuitively, think of $a_1, \ldots, a_n$ as partial bijections of a set X and $a_1^{-1}, \ldots, a_n^{-1}$ as their respective partial inverses. The first relation says that each partial bijection a_i has domain the whole of X and the second says that the ranges of distinct a_i are orthogonal. As a concrete example of P_2, one can take as a_1 and a_2 the two maps that shrink the Cantor set to its lefthand and righthand sides, respectively.

Every non-zero element of P_n is of the form yx^{-1} where $x, y \in A_n^*$. Identify the identity with $\varepsilon\varepsilon^{-1}$. The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable. If they are prefix-comparable then

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some string } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some string } z \end{cases}$$

The non-zero idempotents in P_n are the elements of the form xx^{-1}, where x is positive, and the natural partial order is given by $yx^{-1} \leq vu^{-1}$ iff $(y, x) = (v, u)p$ for some positive string p.

Lemma 3.1 *Let xx^{-1} and yy^{-1} be non-zero idempotents. Then $xx^{-1} \cdot yy^{-1} \neq 0$ if and only if either $xx^{-1} \leq yy^{-1}$ or $yy^{-1} \leq xx^{-1}$. When non-zero*

$$xx^{-1} \cdot yy^{-1} = (x \wedge y)(x \wedge y)^{-1}.$$

Proof Suppose that $xx^{-1} \cdot yy^{-1} \neq 0$. Then either x is a prefix of y or vice-versa. Suppose the former. Then $y = xz$ for some string z, and so $yy^{-1} \leq xx^{-1}$, as required. The proof of the last assertion is straightforward. ■

An immediate corollary of the above lemma is the following property noted by Birget [1].

Corollary 3.2 *Let u and v be positive strings. Then $u^{-1}v = 0$ iff u and v are not prefix-comparable.* ■

A *prefix code* in A_n^* is a non-empty subset C with the property that no element of C is a proper prefix of any other element of C. A prefix code is *maximal* if it is not contained in any other prefix code. The following result was inspired by reading Birget [1].

Lemma 3.3 *A subset*

$$\{y_1 x_1^{-1}, \ldots, y_m x_m^{-1}\}$$

of P_n is orthogonal iff $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ are both prefix codes.

Proof Observe first by Corollary 3.2 that $\{u_1, \ldots, u_m\}$ is a prefix code iff

$$\{u_1 u_1^{-1}, \ldots, u_m u_m^{-1}\}$$

is an orthogonal subset of P_n. Next observe that $\mathbf{d}(yx^{-1}) = xx^{-1}$ and $\mathbf{d}(yx^{-1}) = yy^{-1}$. The result is now clear. ■

A special case of the above lemma is worth stating separately.

Corollary 3.4 *The subset $\{x_1, \ldots, x_n\}$ is a prefix code in A_n^* iff*

$$\{x_1 x_1^{-1}, \ldots, x_n x_n^{-1}\}$$

is an orthogonal subset of P_n. ■

It follows that prefix codes will play an important role in our work.

We now recall some results on the structure of right ideals of free monoids [1], [9]. Proofs can be found there; alternatively, they are easy to construct directly. Let $R \subseteq A_n^*$ be a right ideal. Put $P = R \setminus RA_n$. Then it is easy to check that P is precisely the set of elements of R with the property that no proper prefix belongs to R. It follows that $R = PA_n^*$ and that P is a prefix code. It can be checked that there is exactly one prefix code P such that $R = PA_n^*$. There is therefore a bijection between the set of right ideals of A_n^* and the set of prefix codes in A_n^*. The finitely generated right ideals correspond to the finite prefix codes. The intersection of any two finitely generated right ideals is again a finitely generated right ideal (possibly empty): this follows from the fact that the intersection of two principal right ideals of A_n^* is either empty or a principal right ideal. A function $\alpha\colon R \to R'$ is a *right ideal isomorphism* if it is a bijective function such that $\alpha(rx) = \alpha(r)x$ for all $r \in R$. If $\alpha\colon PA_n^* \to P'A_n^*$ is a right ideal isomorphism where P and P' are prefix codes then α induces a bijection from P to P'. Furthermore, every bijection from P to P' induces a right ideal isomorphism between the right ideals generated by the codes. There is therefore a bijection between the set of right ideal isomorphisms between finitely generated right ideals and the set of bijective functions between finite prefix codes. Finally, if $\alpha\colon R \to R'$ is a right ideal isomorphism and $S \subseteq R$ is a right ideal then $\alpha(R)$ is a right ideal. It follows that the set of right ideal isomorphisms between the finitely generated right ideals of A_n^* is an inverse monoid. We denote this monoid by R_n.

Let Z and Z' be finite prefix codes. We shall now define a way of 'combining' them $Z \circ Z'$. If no element of Z is a prefix of an element of Z' or vice versa then define $Z \circ Z'$ to be empty, else $Z \circ Z'$ is the set of all $z \wedge z'$ where $z \in Z$ and $z' \in Z'$.

Lemma 3.5 *Let Z and Z' be prefix codes. Then $Z \circ Z'$ is either empty or a prefix code, and $ZA_n^* \cap Z'A_n^* = (Z \circ Z')A_n^*$.*

Proof Assume that $Z \circ Z'$ is non-empty. Let $u, v \in Z \circ Z'$. Then $u = z_1 \wedge z_1'$ and $v = z_2 \wedge z_2'$ where $z_1, z_2 \in Z$ and $z_1', z_2' \in Z'$. Suppose that u is a prefix of v; the case where v is a prefix of u is handled similarly. Thus $v = uw$. There are four cases to consider.

1. $u = z_1$ and $v = z_2$. Then $u = v$ since Z is a prefix code.

2. $u = z_1'$ and $v = z_2$. Then z_1' and z_2' are prefix-comparable and so are equal since Z' is a prefix code. Thus z_1 and z_2 are and so $z_1 = z_2$ since Z is a prefix code.

3. $u = z_1'$ and $v = z_2'$. Then z_1' and z_2' are prefix comparable and so are equal since Z' is a prefix code. But then it is immediate that $z_1 = z_2'$, as required.

4. $u = z_1'$ and $v = z_2'$. Then $u = v$ since Z' is a prefix code.

Thus $Z \circ Z'$ is either empty or a prefix code. The proof that

$$ZA_n^* \cap Z'A_n^* = (Z \circ Z')A_n^*$$

is straightforward. ■

Theorem 3.6 *The inverse monoid R_n is isomorphic to the orthogonal completion of the polycyclic monoid P_n.*

Proof We set up some notation we shall need. If A is a non-zero idempotent of $D(P_n)$ then we denote the corresponding finite prefix code, guaranteed by Corollary 3.4, by Z_A. If $A = \{0\}$ then Z_A is the empty set. An arbitrary non-zero element A of $D(P_n)$ consists of zero and a non-empty set $\{y_1x_1^{-1}, \ldots, y_nx_n^{-1}\}$. By Lemma 3.3, both $Z_{\mathbf{d}(A)} = \{x_1, \ldots, x_n\}$ and $Z_{\mathbf{r}(A)} = \{y_1, \ldots, y_n\}$ are (finite) prefix codes.

We shall define an isomorphism Θ from $D(P_n)$ to R_n. The zero $\{0\}$ of $D(P_n)$ is mapped to the empty function in R_n. Let A be a non-zero element of $D(P_n)$. Then $A = \{y_1x_1^{-1}, \ldots, y_nx_n^{-1}\} \cup \{0\}$. Define

$$\theta_A \colon Z_{\mathbf{d}(A)}A_n^* \to Z_{\mathbf{r}(A)}A_n^*$$

by

$$\theta_A(x_iu) = y_iu.$$

This is a well-defined right ideal isomorphism. The function Θ is a bijection since each finitely generated right ideals of a free monoid is generated by a unique prefix code, and right ideal isomorphisms between finitely generated right ideals are determined by their (bijective restrictions) to the corresponding prefix codes.

By Lemma 3.5,

$$ZA_n^* \cap Z'A_n^* = (Z \circ Z')A_n^*.$$

By Lemma 3.1, it follows that

$$Z_A \circ Z_B = Z_{AB}$$

for all idempotents A and B in $D(S)$.

We are now ready to prove that Θ is a homomorphism. First, we show that $\theta_A\theta_B = 0$ iff $AB = \{0\}$. It is enough to show that $Z_{\mathbf{d}(A)} \circ Z_{\mathbf{r}(B)}$ is empty iff $AB = \{0\}$. But by our result above $Z_{\mathbf{d}(A)} \circ Z_{\mathbf{r}(B)} = Z_{\mathbf{d}(A)\mathbf{r}(B)}$ and so the result is clear. We now look at the case where AB is non-zero. We prove that $\theta_A\theta_B = \theta_{AB}$. It is straightforward to check that the domains of the two maps agree: the prefix code that generates their common right ideal consists of elements of the form u_jv where $v_jv = x_i \wedge v_j$. We now show that $\theta_A\theta_B$ and θ_{AB} implement the same rule. Let u_jv be in the domain code where $v_jv = x_i \wedge v_j$. Then $(\theta_A\theta_B)(u_jv)$ is equal to y_i if $x_i \wedge v_j = x_i$ and is equal to y_iw if $x_i \wedge v_j = v_j$ so that $v = \varepsilon$ and $v_j = x_iw$. To calculate θ_{AB} we calculate the elements of AB, and this will yield the same map as above. ■

4 The Thompson groups $V_{n,1}$

In this section, I shall show how the Thompson groups $V_{n,1}$ can be constructed from $D(P_n)$. We begin by summarising some results to be found in [1] and [9]. Following Birget, we define a right ideal of A_n^* to be *essential* if its intersection with every other right ideal is non-empty. It can be proved that the essential right ideals are precisely those whose associated prefix codes are maximal. In addition, amongst the right ideals the essential finitely generated ones are precisely the cofinite ones.

An idempotent e in an inverse semigroup S is called *essential* if for each non-zero idempotent $f \in S$ the product ef is non-zero.

Lemma 4.1 *The following are equivalent in $D(P_n)$:*

(i) *A is an essential idempotent.*

(ii) *For each nonzero idempotent $e \in P_n$ there is an idempotent $f \in A$ such that ef is non-zero.*

(iii) *Z_A is a maximal prefix code.*

Proof The equivalence of (i) and (ii) is immediate.

The equivalence of (ii) and (iii) follows from the following argument. The set $\{x_1, \ldots, x_n\}$ is a maximal prefix code iff for each string y there exists an i such that y and x_i are prefix comparable. Thus by Lemma 3.1, the prefix code corresponding to $\{x_1 x_1^{-1}, \ldots, x_n x_n^{-1}\}$ is maximal iff for each non-zero idempotent yy^{-1} there is an i such that $yy^{-1} \cdot x_i x_i^{-1} \neq 0$. ■

Lemma 4.2 *Let S be an inverse monoid with zero. Let S^e denote the set of elements s such that both $\mathbf{d}(s)$ and $\mathbf{r}(s)$ are essential idempotents. Then S^e is an inverse submonoid of S.*

Proof The identity belongs to S since the identity is an essential idempotent. Let e and f be essential idempotents. Let k be any idempotent. Then ek is non-zero because e is essential, and $f(ek)$ is non-zero because f is essential. Thus ef is essential. Let a and b be elements of S^e. We prove that ab is an element of S^e. The idempotents $\mathbf{d}(a)$ and $\mathbf{r}(b)$ are essential and so $e = \mathbf{d}(a)\mathbf{r}(b)$ is essential. Observe that $ab = (ae)(eb)$. We shall prove that $\mathbf{d}(eb)$ and $\mathbf{r}(ae)$ are both essential. We prove that $\mathbf{d}(eb)$ is essential; the proof that $\mathbf{r}(ae)$ is essential is similar. Let f be a non-zero idempotent. Suppose that $b^{-1}ebf = 0$. Then $bb^{-1}ebf = 0$ and so $edf = 0$. Hence $e \cdot bfb^{-1} = 0$. But e is an essential idempotent and so $bfb^{-1} = 0$. Thus $b^{-1}bfb^{-1}b = 0$. Thus $b^{-1}bf = 0$. But $b^{-1}b$ is an essential idempotent and so $f = 0$, which is a contradiction. The fact that S^e is an inverse subsemigroup is now clear. ■

By Lemma 4.2, $D^e(S)$ is an inverse monoid, and by Lemma 4.1, the essential idempotents in $D(P_n)$ are those whose associated prefix codes are maximal. It follows by Lemma 3.3, that the elements of $D^e(P_n)$ are those in which the domains and ranges correspond to maximal prefix codes. Under the isomorphism of Theorem 3.6, non-zero idempotents of $D(P_n)$ correspond to prefix codes, and the essential idempotents correspond to the maximal prefix codes. We therefore have the following.

Proposition 4.3 *The inverse semigroup $D^e(P_n)$ is isomorphic to the inverse monoid of right ideal isomorphisms between the finitely generated essential right ideals of the free monoid on n generators.* ■

The *minimum group congruence* σ on an inverse semigroup is defined by $a\sigma b$ iff there exists $c \leq a, b$ with respect to the natural partial order. For any inverse semigroup S, it can be proved that S/σ is a group called the *universal group of* S. An inverse monoid is said to be F*-inverse* if each σ-class contains a maximum element. In this case, the group S/σ can also be described in the following way: it is isomorphic to the set of maximal elements of S equipped with a product $\circ$ where $a \circ b$ is the unique maximal element lying above ab. The inverse monoid R_n^e is F-inverse; this is proved, though not with this terminology, in both [1] and [9]. From Scott [9], the following is now immediate.

Theorem 4.4 *The maximum group homomorphic image of $D^e(P_n)$ is the Thompson group $V_{n,1}$.* ■

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