Mosaics and inverse semigroups

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1 Introduction

These notes are merely a slight reformulation of Ben Steinberg's preprint 'Building inverse semigroups from group actions'. Ben's 'abstract geometric representations' are essentially my 'mosaics'. The main differences are that my definition does not use zeros and so my operations are partial in general, and I use the reformulation of abstract geometric representations in terms of the relation 'is an element of'.

The aim of these notes is to provide a general enough framework for talking about inverse semigroups which are constructed in a manner analogous to tiling semigroups. In particular, we prove that the class of inverse semigroups constructible from mosaics is the same as the class of strongly E^*-unitary inverse semigroups.

2 Definitions and results

Here is some not entirely satisfactory terminology. A *presemigroup* is a set S equipped with a partial binary operation $\circ$ such that $\exists (x \circ y) \circ z \Leftrightarrow x \circ (y \circ z)$; if either product is defined then so is the other and they are equal. If we adjoin a zero to a presemigroup we get a semigroup and every presemigroup arises in this way. Thus presemigroups are just a device for handling semigroups with zero without the need for a zero. This terminology can be used in a number of places below. Definitions for semigroups can be carried over to presemigroups in the obvious way.

Definition A triple (B, P, G) is called a *mosaic* if the following conditions hold: B is a set called the set of *tiles*, P is a set called the set of *patterns*, and G is a group. The group G acts partially on both B and P, in addition it acts freely on B. There is a binary relation $\epsilon \subseteq B \times P$. If $a \, \epsilon \, x$ then we say 'a is an element of x' and 'x contains a'. If $a \in B$ and $x, y \in P$ and $a \, \epsilon \, x, y$ we say that the patterns x and y are *adjacent*. The set P is equipped with a partial binary operation $\circ$. The following axioms (M1)–(M9) have to be satisfied:

(M1) For each $x \in P$, $\exists x \circ x$ and $x \circ x = x$.

(M2) $\exists(x \circ y) \circ z \Leftrightarrow \exists x \circ (y \circ z)$; if either exists then so does the other and they are equal.

(M3) $\exists x \circ y \Leftrightarrow \exists y \circ x$; if either exists so does the other and they are equal.

If $\circ$ is globally defined then $(P, \circ)$ is just a semilattice. If $\circ$ is partially defined then the adjunction of a 0 gives us a semilattice with zero. Conversely, if we remove the zero from a semilattice with zero then the resulting structure satisfies the above axioms. Thus $(P, \circ)$ is a *presemilattice.*

(M4) For each $x \in P$ there exists $a \in B$ such that $a \epsilon x$.

(M5) If $a \epsilon x$ and $\exists x \circ y$ then $a \epsilon x \circ y$; if $b \epsilon y$ and $\exists x \circ y$ then $b \epsilon x \circ y$

Warning we do not insist that a pattern is determined by the tiles it contains. So we should visualise a pattern as 'floating above' (in a sort of Platonic heaven) the set of tiles which it contains.

(M6) If $a \epsilon x$ and $\exists g \cdot x$ then $\exists g \cdot a$ and $g \cdot a \epsilon g \cdot x$.

(M7) Let x and y be adjacent. If $\exists x \circ y$ and $g \cdot x$ and $\exists g \cdot y$ and $\exists (g \cdot x) \circ (g \cdot y)$ then $\exists g \cdot (x \circ y)$ and $g \cdot (x \circ y) = (g \cdot x) \circ (g \cdot y)$.

(M8) Let x and y be adjacent. If $\exists x \circ y$ and exists $g \cdot (x \circ y)$ then $\exists g \cdot x$ and $\exists g \cdot y$.

(M9) Let x and y be adjacent. If $\exists x \circ y$ and $\exists g \cdot (x \circ y)$ then $\exists (g \cdot x) \circ (g \cdot y)$ and $g \cdot (x \circ y) = (g \cdot x) \circ (g \cdot y)$.

Definition A *multiplicative graph* is defined as follows. Let (S, S_o) be a pair of sets equipped with functions $\partial_0, \partial_1 \colon S \to S_o$. This is just a way of describing a directed graph. Let S be equipped with a partial binary operation such that the following axioms hold:

(MG1) If $\exists x \circ y$ then $\partial_0(x) = \partial_1(y)$.

(MG2) If $\exists x \circ y$ then $\partial_0(x \circ y) = \partial_0(y)$ and $\partial_1(x \circ y) = \partial_1(y)$.

(MG3) $\exists x \circ (y \circ z) \Leftrightarrow (x \circ y) \circ z$; if either exists they are equal.

Thus a multiplicative graph is a presemigroup equipped with the structure of a directed graph with some interaction between these two structures. If axiom (MG1) is replaced by an 'if and only if' we have the definition of a semigroupoid (category without identities). We define an *inverse multiplicative graph* in the obvious way.

Definition We define what we mean by a *group acting freely and partially on an (inverse) multiplicative graph.* Let S be a multiplicative graph whose partial product we denote by concatenation. Let G be a group which acts partially on both S_o and S, and freely on S_o. In addition the following axioms should be satisfied:

(A1) If $\exists g \cdot x$ then $\exists g \cdot \partial_0(x)$ and $\exists g \cdot \partial_1(x)$ and $\partial_0(g \cdot x) = g \cdot \partial_0(x)$ and $\partial_1(g \cdot x) = g \cdot \partial_1(x)$.

(A2) If $\exists xy$ and $\exists g \cdot x$ and $\exists g \cdot y$ and $\exists (g \cdot x)(g \cdot y)$ then $\exists g \cdot (xy)$ and $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

(A3) If $\exists g \cdot (xy)$ then $\exists g \cdot x$ and $\exists g \cdot y$.

(A4) If $\exists g \cdot (xy)$ then $\exists (g \cdot x)(g \cdot y)$ and $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

Remark The above definition simplifies a little when the multiplicative graph is a semigroupoid, because in that case $\exists xy \Leftrightarrow \partial_0(x) = \partial_1(y)$.

(A2) Suppose $\exists xy$, $\exists g \cdot x$ and $\exists g \cdot y$. Then $\partial_0(g \cdot x) = g \cdot \partial_0(x)$ and $\partial_1(g \cdot y) = g \cdot \partial_1(y)$ by (A1) and $\partial_0(x) = \partial_1(y)$ since $\exists xy$. Hence $\partial_0(g \cdot x) = \partial_1(g \cdot y)$ and so $(g \cdot x)(g \cdot y)$ exists. We can therefore replace (A2) by the following axiom:

(A2)$*$ If $\exists xy$ and $\exists g \cdot x$ and $\exists g \cdot y$ then $\exists g \cdot (xy)$ and $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

A similar argument enables us to combine (A3) and (A4) into the following single axiom:

(A3/4)$*$ If $\exists xy$ and $\exists g \cdot (xy)$ then $\exists g \cdot x$ and $\exists g \cdot y$ and $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

It follows that in the case of semigroupoids we have essentially the same axioms as in Kellendonk/Lawson except there a category is involved instead of a semigroupoid.

Proposition 1 *Let S be a multiplicative graph equipped with a free partial action by a group G. Define $\sim$ on S by $x \sim y$ iff there exists $g \in G$ such that $g \cdot x = y$. Then $\sim$ is an equivalence relation. Let $S/G = \{[x] \colon x \in S\}$ be the set of equivalence classes. Define the following operation on S/G:*

$$[x][y] = [(g \cdot x)(h \cdot y)]$$

if there exist $g, h \in G$ such that $(g \cdot x)(h \cdot y)$ is defined in S. Then with respect to this operation, S/G is a presemigroup.

The map from S to S/G defined by $x \mapsto [x]$ is a surjective idempotent pure morphism.

If the multiplicative graph is inverse then so too is S/G.

Proof The relation $\sim$ is an equivalence relation using the same argument as in Kellendonk/Lawson.

We show first that the partial product is well-defined. Let $x, y \in S$ and suppose that there are elements $g, h, k, l \in G$ such that the following two products make sense

$$(g \cdot x)(h \cdot y) \text{ and } (k \cdot x)(l \cdot y).$$

We have to prove that

$$(g \cdot x)(h \cdot y) \sim (k \cdot x)(l \cdot y).$$

Because S is a multiplicative graph we have that

$$\partial_0(g \cdot x) = \partial_1(h \cdot y) \text{ and } \partial_0(k \cdot x) = \partial_1(l \cdot y).$$

Thus by axiom (A1), we have that

$$g \cdot \partial_0(x) = h \cdot \partial_1(y) \text{ and } k \cdot \partial_0(x) = l \cdot \partial_1(y).$$

Thus from the properties of partial actions we obtain

$$(h^{-1}g) \cdot \partial_0(x) = (l^{-1}k) \cdot \partial_0(x).$$

By assumption, G acts freely on S_o and so

$$h^{-1}g = l^{-1}k.$$

Put

$$m = kg^{-1} = lh^{-1}.$$

Then

$$m \cdot (g \cdot x) = k \cdot x \text{ and } m \cdot (h \cdot y) = l \cdot y.$$

We therefore have $\exists (g \cdot x)(h \cdot y)$, $\exists m \cdot (g \cdot x) = k \cdot x$, $\exists m \cdot (h \cdot) = l \cdot y$, and $\exists (h \cdot x)(l \cdot y)$. Thus by axiom (A2), $\exists m \cdot ((g \cdot x)(h \cdot y))$ and

$$m \cdot ((g \cdot x)(h \cdot y)) = (k \cdot x)(l \cdot y),$$

as required.

Next we show that S/G is a presemigroup. Suppose that $([x][y])[z]$ is defined. Then there exist $g, h \in G$ such that $(g \cdot x)(h \cdot y)$ exists. Thus $[x][y] = [(g \cdot x)(h \cdot y)]$. Thus there exist $k, l \in G$ such that $k \cdot ((g \cdot x)(h \cdot y))$ exists and $l \cdot z$ exists. Thus

$$([x][y])[z] = [k \cdot ((g \cdot x)(h \cdot y))][l \cdot z].$$

By axioms (A3) and (A4) we have

$$k \cdot ((g \cdot x)(h \cdot y)) = (k \cdot (g \cdot x))(k \cdot (h \cdot y)) = ((kg) \cdot x)((kh) \cdot y).$$

Hence

$$([x][y])[z] = ((kg) \cdot x)((kh) \cdot y)(l \cdot z).$$

But this implies that $[x]([y][z])$ is defined because $[y][z] = [(kh) \cdot y)(l \cdot z)]$ and so

$$[x]([y][z]) = [(kg) \cdot x)((kh) \cdot y)(l \cdot z)].$$

We can likewise prove that the existence of $[x]([y][z])$ implies the existence of $([x][y])[z]$.

Consider now the function π given by $x \mapsto [x]$. If xy is defined in C then $[x][y]$ is defined in S/G and $[x][y] = [xy]$. Thus π is a surjective morphism. Suppose that $[x]$ is an idempotent. Then $[x][x]$ is defined and is equal to $[x]$. Thus there exist $g, h \in G$ such that $(g \cdot x)(h \cdot x)$ is defined and a $k \in G$ such that $x = k \cdot ((g \cdot x)(h \cdot x))$. By axiom (A4) and properties of partial actions, we have that $k \cdot ((g \cdot x)(h \cdot x)) = ((kg) \cdot x)((kh) \cdot x)$. Thus $x = ((kg) \cdot x)((kh) \cdot x)$. By axiom (MG2), we have that $\partial_0(x) = \partial_0((kh) \cdot x)$ and by axiom (A1) we have that $\partial_0(x) = (kh) \cdot \partial_0(x)$. But G acts freely on S_o and so $kh = 1$. Similarly $kg = 1$. It follows that $x = xx$ and so is idempotent.

Suppose now that S is inverse. By adjoining zeros and extending π in the obvious way we have that $(S/G)^0$ is a homomorphic image of S^0. But S inverse implies S^0. Thus S/G is an inverse presemigroup. ■

Proposition 2 *Let $M = (B, P, G)$ be a mosaic. Put*

$$S(M) = \{(a, x, b) \in B \times P \times B \colon a, b \,\epsilon\, x\}$$

and $S(M)_o = B$ and define $\partial_0(a, x, b) = b$ and $\partial_1(a, x, b) = a$. Define

$$(a, x, b)(b, y, c) = (a, x \circ y, c)$$

if $\exists x \circ y$. Then $S(M)$ is an inverse multiplicative graph.

Let $g \in G$. Define $g \cdot (a, x, b) = (g \cdot a, g \cdot x, g \cdot b)$ if $\exists g \cdot x$. The partial action of G on $S_0 = B$ is just the given partial action of G on B. Then this defines a free partial group action of G on the multiplicative graph S.

Proof It is straightforward to check that the axioms (M1)–(M5) imply that $S(M)$ is an inverse multiplicative graph.

The definition of the partial action is correct by axiom (M6). It remains to check that axioms (A1)–(A4) hold. Axiom (A1) follows from axiom (A1) and the definitions of ∂_0 and ∂_1 in $S(M)$. Axioms (A2), (A3) and (A4) follow from respectively (M7), (M8) and (M9). ■

Proposition 3 *Let $M = (B, P, G)$ be a mosaic and let $S(M)$ be the associated multiplicative graph. Then $S(M)$ is a semigroupoid if and only if when patterns x and y are adjacent then $\exists x \circ y$.*

Proof Suppose that M satisfies the condition: x and y adjacent implies $\exists x \circ y$. Then in $S(M)$ the fact that $\partial_0(a, x, b) = \partial_1(b, y, c)$ implies x and y are adjacent and so $\exists x \circ y$. Hence the product $(a, x, b)(b, y, c)$ is defined.

Suppose that $S(M)$ is a semigroupoid. Let x and y be adjacent patterns. Let $b \,\epsilon\, x, y$. Then (b, x, b) and (b, y, b) are well-defined elements of $S(M)$ and since $S(M)$ is a semigroupoid the product $(b, x, b)(b, y, b)$ is defined. Thus $\exists x \circ y$ as required. ■

Let $M = (B, P, G)$ be a mosaic. The element $(a, x, b) \in S(M)$ is an idempotent if and only if $a = b$, since by axiom (M1) we always have that

$\exists x \circ x = x$. If (a, x, a) and (a, y, a) are two idempotents then $\exists(a, x, a)(a, y, a)$ iff $\exists(a, y, a)(a, x, a)$ by axiom (M3); if either exists then so does the other and their products are equal. It is now clear that the set of all elements in $S(M)$ which begin and end at the same place forms a presemilattice.

Say that two elements x and y in a multiplicative graph are *parallel* iff $\partial_0(x) = \partial_0(y)$ and $\partial_1(x) = \partial_1(y)$. The set of all elements x in a multiplicative graph such that $\partial_0(x) = \partial_1(x) = e$ where $e \in S_o$ is fixed is called a *local presemigroup at* e. If each local presemigroup is a local presemilattice we say that the multiplicative graph S is *locally idempotent.*

Let S be a locally idempotent multiplicative graph. Let G act on S partially and freely. Let $[x] \leq [y]$ in S/G where $[x]$ is an idempotent. We may assume that x is idempotent by our result earlier. We have that $[x] = [x][y]$ and so there exist elements $g, h \in G$ such that $(g \cdot x)(h \cdot y)$. Thus $[x][y] = [(g \cdot x)(h \cdot y)]$. Also there exists $k \in G$ such that $x = k \cdot ((g \cdot x)(h \cdot y))$. By axiom (A4), we have that $x = ((kg) \cdot x)((kh) \cdot y)$.

Now x an idempotent implies that $(kg) \cdot x$ is an idempotent (using (MG2) and (M1) and (A2)). Thus in the inverse presemigroup S we have that $x \leq (kh) \cdot y$. It follows that x and $(kh) \cdot y$ must be parallel and therefore by assumption $(kh) \cdot y$ is an idempotent. Thus y is an idempotent. Hence $[y]$ is an idempotent.

The above two propositions, combined with the argument above, provide a way of constructing an E^*-unitary inverse presemigroup from each mosaic.

The question now is: precisely which E^*-unitary inverse presemigroups can be constructed from mosaics? My first result generalises slightly some ideas of Ben Steinberg.

Proposition 4 *Every strongly E^*-unitary inverse presemigroup can be constructed from a mosaic*

Proof Our proof generalises an idea due to Steinberg. Let S be strongly E^*-unitary. Let $\theta \colon S \to G$ be an inverse presemigroup equipped with an idempotent pure grading θ. By the theory developed in a paper by Lawson, such a presemigroup can be constructed in the following way.

A *McAlister $*$-triple* (G, Y, X) consists of a group G, a poset X, a subset $Y \subseteq X$ which is an order ideal and presemilattice such that G acts on X by order automorphisms and $G \cdot Y = X$. Let (G, X, Y) be a McAlister $*$-triple. Put

$$P^* = P^*(G, X, Y) = \{(x, g) \in Y \times G \colon g^{-1} \cdot x \in Y\}.$$

We define a partial product as follows: $(x, g)(y, h)$ is defined iff x and $g \cdot y$ have a lower bound in X, in which case, $(x, g)(y, h) = (x \wedge g \cdot y, gh)$. The function $P(G, X, Y) \to G$ defined by $(y, g) \mapsto g$ is the *natural grading associated with* $P(G, X, Y)$. It is idempotent pure. Semigroups of the form $P(G, X, Y)$ are called *P-semigroups.* It can be shown that $\theta \colon S \to G$ is isomorphic to a P-semigroup equipped with its natural grading.

We show first that from every McAlister $*$-triple (G, X, Y) we can construct a mosaic. Let G be the group, $B = G$ and $P = X$. The group G acts on itself (globally) by left multiplication and this action is free. The action of G

on X is by order automorphisms. The partial operation on X is just $x \wedge y$ if it exists. We define $g \epsilon x$ iff $g^{-1}x \in Y$. We show that we have a mosaic. Axioms (M1)–(M3) are immediate. Axiom (M4) holds because $GY = X$. Axiom (M5) holds because Y is an order ideal of X and the action of G on X is by order automorphisms. Axiom (M6) is straightforward.

Axiom (M7): let x and y be adjacent, and suppose that both $x \wedge y$ and $gx \wedge gy$ exists. I prove that $g(x \wedge y) = gx \wedge gy$. Now $x \wedge y \leq x, y$ and so $g(x \wedge y) \leq gx \wedge gy$. On the other hand $gx \wedge gy \leq gx, gy$ and so $g^{-1}(gx \wedge gy) \leq x, y$ giving $g^{-1}(gx \wedge gy) \leq x \wedge y$ and so $gx \wedge gy \leq g(x \wedge y)$. Hence result.

Axiom (M7) needs no proof.

Axiom (M8): let x and y be adjacent and suppose $x \wedge y$ exists. I prove $gx \wedge gy$ exists and that $g(x \wedge y) = gx \wedge gy$. This is straightforward.

From this mosaic, we can construct an inverse presemigroup $S' = S(M)/G$. We show that this is isomorphic to $P(G, X, Y)$. The elements of S' have the form $[a, x, b]$ where $a, b \in G$ and $x \in X$ such that $a^{-1}x, b^{-1}x \in Y$. Observe that $[a, x, b] = [1, a^{-1}x, a^{-1}b]$ where $(a^{-1}b)^{-1}(a^{-1}x) = b^{-1}x \in Y$. and $a^{-1}x \in Y$. Thus the element $[a, x, b]$ is of the form $[1, y, g]$ where $y \in Y$ and $g^{-1}x \in Y$. There is clearly a unique element of this form in $[a, x, b]$. Define α: $S' \to P$ by $\alpha[1, y, g] = (y, g)$. This is evidently a bijection.

Suppose $[1, y, g][1, z, h]$ is defined. Then there are group elements $a, b \in G$ such that $ag = b1$ and $ay \wedge bz$ is defined. Thus $[1, y, g][1, z, h] = [a, ay \wedge bz, bh] = [1, y \wedge gz, gh]$. This implies that π is a morphism. In the other direction, if $(y, g)(z, h)$ is defined in P then $[1, y, g][1, z, h]$ is defined in S'. It is now clear that α is an isomorphism. ■

The following lemma is due to Ben Steinberg.

Lemma 5 *Let (G, B, P) be a mosaic. Let $S' = S(M)/G$ be the corresponding inverse presemigroup. Let ϕ: $B \to G$ be an injective function such that if $a \in B$ and if $g \cdot a$ exists then $\phi(g \cdot a) = g \cdot \phi(a)$. Then S' is strongly E^*-unitary.*

Proof Define Φ: $S \to G$ by $\Phi([a, x, b]) = \phi(a)^{-1}\phi(b)$. This function is well-defined by our assumption on ϕ. Idempotent purity follows from the injectivity of ϕ. It is easy to check that Φ is a morphism. ■

The following result is due to John Fountain.

Proposition 6 *The inverse semigroup constructed from a mosaic is strongly E^*-unitary.*

Proof The group G acts partially and freely on the set B. Thus B is partitioned by this partial action. Let $\{b_i : i \in I\}$ be a set of representatives of this partition. For each $a \in B$ there exists a unique (because the group acts freely) element $g \in G$ such that $g \cdot a = b_i$ for some unique b_i. Let H be the free group on I. Let $S' = S(M)/G$ be the inverse presemigroup associated with the mosaic M. Define θ: $S' \to G \times H$ as follows: $\theta[a, x, b] = (k_1^{-1}k_2, ij^{-1})$ where $a = k_1 \cdot b_i$ and $b = k_2 \cdot b_j$. We show first that θ is well-defined. Suppose that $[a, x, b] = [a', x', b']$.

Then there exists $g \in G$ such that $g \cdot a = a'$, $g \cdot x = x'$ and $g \cdot b = b'$. Now $a = k_1 \cdot b_i$ and so $g \cdot a = g \cdot (k_1 \cdot b_i) = (gk_1) \cdot b_i$. Thus $a' = (gk_1) \cdot b_i$. Similarly, $b' = (gk_2) \cdot b_j$. It's now easy to check that $\theta[a', x', b'] = \theta[a, x, b]$.

Next we show that θ is a morphism. Without loss of generality, suppose that $[a, x, b][b, y, c]$ is defined. Suppose that $a = k_1 \cdot b_i$, $b = k_2 \cdot b_j$, $c = k_3 \cdot b_l$. Then $\theta[a, x, b] = k_1^{-1}k_2, ij^{-1}$, and $\theta[b, y, c] = (k_2^{-1}k_3, jl^{-1})$. But $\theta[a, x \circ y, c] = (k_1^{-1}k_3, il^{-1})$ and so θ is a morphism.

Finally, we show that θ is idempotent pure. Suppose that $\theta[a, x, b] = (1, 1)$, where $a = k_1 \cdot b_i$ and $b = k_2 \cdot b_j/$ Then $k_1 = k_2$ and $i = j$ and so $a = b$, as required. ■

Combining the two results above we arrive at the following.

Theorem 7 *The class of inverse semigroups which can be constructed from mosaics is coextensive with the class of strongly E^*-unitary inverse semigroups.* ■

We can express the above theorem in more abstract terms which avoids the need to use mosaics.

Theorem 8 *Let G be a group acting partially and freely on inverse multiplicative graph S which is also locally idempotent. Then S/G is strongly E^*-unitary and every strongly E^*-unitary semigroup is obtained in this way.*

Proof We have already proved that every strongly E^*-unitary inverse semigroup can be constructed from a multiplicative inverse graph of the required type actually constructed from a mosaic.

It remains to prove that S/G is strongly E^*-unitary. To do this we simply adapt Fountain's argument above. The group G acts partially and freely on S_o. Thus G induces an equivalence relation on S_o. Let $\{e_i \colon i \in I\}$ be a complete set of representatives of these classes. Let H be the free group on I. Define $\theta \colon S/G \to G \times H$ as follows: let $[x] \in S/G$. Let $e = \partial_0(x)$ and $f = \partial_1(x)$. There exist unique group elements g_1 and g_2 such that $e = g_1 \cdot e_i$ and $f = g_2 \cdot f_i$. Define $\theta[x] = (g_2^{-1}g_1, ij^{-1})$. Suppose that $[x] = [x']$. Then there exists $g \in G$ such that $g \cdot x = x'$. But $\partial_0(g \cdot x) = g \cdot \partial_0(x) = \partial_0(x')$ and $\partial_1(g \cdot x) = g \cdot \partial_1(x) = \partial_1(x')$. Thus $g \cdot e = \partial_0(x')$ and $g \cdot f = \partial_1(x')$. Therefore $(gg_1) \cdot \partial_0(x') = e_i$ and $(gg_2) \cdot \partial_1(x') = e_j$. Hence $\theta[x'] = ((gg_2)^{-1})(gg_1), ij^{-1}$ which is just $\theta[x]$ as required. Suppose that $[x][y]$ is defined. Then without loss of generality we can assume xy is defined.

■

We now turn to new material. The following definition was motivated by a paper of Zhu, which in turn was motivated by a definition due to Penrose used in studying aperiodic tilings.

Definition Let G be a group acting freely and partially on the inverse semigroupoid S. Define a relation $\mathcal{E}$ on the inverse semigroupoid S/G as follows:

$[x]\,\mathcal{E}\,[y]$ iff there exists $x' \in [x]$ and $y' \in [y]$ such that $\partial_0(x') = \partial_0(y')$ and $\partial_1(x') = \partial_1(y')$ and for each $g \in G$ we have that $\exists g \cdot x' \Leftrightarrow \exists g \cdot y'$.

Lemma 9 *Let G be a group acting partially on the set X. If $\exists a \cdot y$ and $\exists (ga) \cdot y$ then $\exists g \cdot (a \cdot y)$.*

Proof $\exists (ga)^{-1}((ga) \cdot y)$ and equals y. Since $\exists a \cdot y$ we have that $\exists a \cdot ((ga)^{-1}((ga) \cdot y))$ and equals $a \cdot y$. But $a \cdot ((ga)^{-1}((ga) \cdot y))$ also equals $g^{-1} \cdot ((ga) \cdot y)$. Thus $\exists g \cdot (g^{-1} \cdot ((ga) \cdot y))$ consequently $\exists g \cdot (a \cdot y)$, as required. ∎

Proposition 10 *Let S be a locally commutative inverse semigroupoid on which G acts freely and partially. Then the relation $\mathcal{E}$ is an idempotent pure congruence on S/G.*

Proof It is clear that $\mathcal{E}$ is reflexive and symmetric. We prove that it is transitive. Let $\equiv$ be the relation defined on S by: $x \equiv y$ iff x and y are parallel and $\exists g \cdot x \Leftrightarrow g \cdot y$.

Let $[x]\,\mathcal{E}\,[y]$ and $[y]\,\mathcal{E}\,[z]$. Then there exist $g, h, k, l \in G$ such that $g \cdot x \equiv h \cdot y$ and $k \cdot y \equiv l \cdot z$. We shall prove that $[x]\,\mathcal{E}\,[z]$.

From $g \cdot \equiv h \cdot y$ we have that from $\exists h^{-1} \cdot (h \cdot x)$ that $\exists h^{-1} \cdot (g \cdot x)$. Thus $\exists (h^{-1}g) \cdot x$. Similarly, $\exists k^{-1} \cdot (l \cdot z)$ and so $\exists (k^{-1}l) \cdot z$.

From axiom (A1), it is easy to check that $(h^{-1}g) \cdot x$ and $(k^{-1}l) \cdot z$ are parallel.

We prove that $(h^{-1}g) \cdot x \equiv (k^{-1}l) \cdot z$. Suppose that $\exists((h^{-1}g) \cdot x)$. Then $\exists a \cdot (h^{-1} \cdot (g \cdot x))$. Thus $\exists (ah^{-1}) \cdot (g \cdot x)$. But $g \cdot x \equiv h \cdot y$ and so $\exists (ah^{-1}) \cdot (h \cdot y)$, which gives $\exists a \cdot y$. Clearly $\exists a \cdot (k^{-1} \cdot (k \cdot y))$ and so $(ak^{-1}) \cdot (k \cdot y)$. But $k \cdot y \equiv l \cdot z$ and so $(ak^{-1}) \cdot (l \cdot z)$. Thus $\exists (a(k^{-1})l) \cdot z$ and $(k^{-1}l) \cdot z$. It follows from the lemma that $a \cdot ((k^{-1}l) \cdot z)$, as required. The reverse direction is proved similarly.

It is now clear that $[x]\,\mathcal{E}\,[z]$, as required.

Suppose that $[x]\,\mathcal{E}\,[y]$ and $[u]\,\mathcal{E}\,[v]$ and that $[x][u]$ and $[y][v]$ are both defined. We prove that $[x][u]\,\mathcal{E}\,[y][v]$. Let $[x][u] = [(a \cdot x)(b \cdot u)]$. In addition, let $g, h \in G$ be such that $g \cdot x \equiv h \cdot y$ and $m, n \in G$ such that $m \cdot u \equiv n \cdot v$.

It is easy to check that $a \cdot x$ is parallel to $(ag^{-1}h) \cdot y$ and that $b \cdot u$ is parallel to $(bm^{-1}n) \cdot v$. In particular,

$$[c][v] = [((ag^{-1}h) \cdot y), ((bm^{-1}n) \cdot v)].$$

Suppose that $\exists w \cdot ((a \cdot x)(b \cdot u))$. Then $\exists w \cdot (a \cdot x)$ and $\exists w \cdot (b \cdot u)$. Thus $\exists w \cdot (a \cdot (g^{-1} \cdot (g \cdot x)))$ and so $\exists (wag^{-1}) \cdot (g \cdot x)$. Thus $\exists (wag^{-1}) \cdot (hy)$, and so $\exists (wag^{-1}h) \cdot y$. But from $\exists (ag^{-1}h) \cdot y$ and $\exists (wag^{-1}h) \cdot y$ we have by Lemma that $\exists w \cdot ((ag^{-1}h) \cdot y)$.

We may similarly show that $\exists w \cdot (bm^{-1}n) \cdot v$.

From $\exists w \cdot ((ag^{-1}h) \cdot y)$ and $\exists w \cdot (bm^{-1}n) \cdot v$ we have that $\exists w \cdot (((ag^{-1}h) \cdot y)((bm^{-1}n) \cdot v))$.

The above argument works backwards. Hence result.

It remains to prove that $\mathcal{E}$ is idempotent pure. Suppose that $[x]\,\mathcal{E}\,[y]$ where $[y]$ is an idempotent. Then we can without loss of generaility assume that y is an idempotent. But then for some $x' \in [x]$ and $y' \in [y]$ we have that $x' \equiv y'$. Now y' is an idempotent and x' and y' are parallel. If S is locally commutative then x' is an idempotent and so x is an idempotent, as required. ∎