Characterisations of Morita equivalent inverse semigroups

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Abstract

For a fixed inverse semigroup S, there are two natural categories of left actions of S: the category **Fact** of *unitary* actions of S on sets X meaning actions where $SX = X$, and the category **Étale** of *étale* actions meaning those unitary actions equipped with a function $p\colon X \to E(S)$, to the set of idempotents of S, such that $p(x)x = x$ and $p(sx) = ses^*$, where s^* denotes the inverse of s. The category **Étale** can be regarded as the classifying topos of S. There is a forgetful functor U from **Étale** to **Fact** that forgets étale structure and simply remembers the action. Associated with these two types of actions are appropriate notions of Morita equivalence which we term *Morita equivalence* and *strong Morita equivalence*, respectively. We prove three main results: first, strong Morita equivalence is the same as Morita equivalence; second, the forgetful functor U has a right adjoint R, and the category of Eilenberg-Moore algebras of the monad $M = RU$ is equivalent to the category of presheaves on the Cauchy completion $C(S)$ of S; third, we show that equivalence bimodules, which witness strong Morita equivalence, can be viewed as abstract atlases, thus connecting with the pioneering work of V. V. Wagner on the theory of inverse semigroups and Anders Kock's more recent work on pregroupoids.

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1 Introduction

The following definition is due to Steinberg [27]. A *Morita context* consists of a set X which is an (S,T)-bimodule equipped with surjective functions $\langle -,-\rangle\colon X \times X \twoheadrightarrow S$ and $[-,-]\colon X \times X \twoheadrightarrow T$ such that the following axioms hold, where $x, y, z \in X$ and $s \in S$ and $t \in T$:

(MC1) $\langle sx, y\rangle = s\langle x, y\rangle$.

(MC2) $\langle y, x\rangle = \langle x, y\rangle^*$.

(MC3) $\langle x, x\rangle x = x$.

(MC4) $[x, yt] = [x, y]t$.

(MC5) $[x, y] = [y, x]^*$.

(MC6) $x[x, x] = x$.

(MC7) $\langle x, y \rangle z = x[y, z]$.

One calls this data an *equivalence bimodule* for S and T. The inverse semigroups S and T are said to be *strongly Morita equivalent* if they possess an equivalence bimodule. In this paper, the inverse of an element s in an inverse semigroup is denoted s^*. In addition, we refer to a set on which an inverse semigroup acts as a *module* rather than the more usual term *act*.

Inverse semigroups have come to play an important role in the theory of topological groupoids and C^*-algebras. From this perspective, the above definition has important consequences: if S and T are strongly Morita equivalent then their associated étale groupoids, in the sense of [24], are Morita equivalent, and their universal and reduced C^*-algebras are strongly Morita equivalent [27]. These results make it important to obtain as much information as possible about strong Morita equivalence of inverse semigroups. The goal of this paper is to prove a number of different characterisations of this notion. Inverse semigroups inhabit at least three different worlds: the world of semigroup theory [13], the world of ordered groupoids [13], and the world of topos theory [4, 5, 17].[1] Accordingly, our characterisations of strong Morita equivalence will come from these three worlds. Before stating the main theorem of this paper, we shall provide some of the key definitions needed to understand it.

A semigroup S is *regular* if for each $s \in S$ there exists $t \in S$ such that $s = sts$ and $t = tst$. The element t is called an *inverse* of S. The set of inverses of s is denoted by $V(s)$. An important result about the behaviour of inverses in regular semigroups is the following: if $a' \in V(a)$ and $b' \in V(b)$ then $b'ha' \in V(ab)$ for some $h \in S$. See Theorem 2.5.4 of [6]. If each element of a regular semigroup has a unique inverse, then the semigroup is said to be *inverse*. Let S be a subsemigroup of the semigroup T. Then T is said to be an *enlargement* of S if $S = STS$ and $T = TST$. If R is an enlargement of subsemigroups S and T we say that R is a *joint enlargement* of S and T. If R is a regular semigroups we say that it is a *regular* joint enlargement. For undefined terms from inverse semigroup theory see [13].

Categories will be used both as structures on a par with monoids as well as the more usual categories of structures. It will be clear from the context which of these is meant. In addition to small categories, we shall also use *semigroupoids* which are categories without identities, but with objects. (Thus, a semigroup is a semigroupoid with one object.) Definitions from semigroup theory can be extended in an obvious way to semigroupoids. Our next definition is a version of the definition of a bipartite category given in [7] sharpened up in the light of the notion of 'bridge' discussed in [25]. Let $\mathbb{C}$ be a category. We say that $\mathbb{C} = [\mathbb{A}, \mathbb{B}]$ is *bipartite (with left part $\mathbb{A}$ and right part $\mathbb{B}$)* if it satisfies the following conditions:

[1] A pioneer paper in bridging these worlds was Loganathan's [18].

(B1) $\mathbb{C}$ has full disjoint subcategories $\mathbb{A}$ and $\mathbb{B}$ such that $C_0 = A_0 \cup B_0$.

(B2) For each identity $e \in A_0$ there exists an isomorphism x with domain e and codomain in B_0; for each identity $f \in B_0$ there exists an isomorphism y with domain f and codomain in A_0.

The category $\mathbb{C}$ is a disjoint union of four kinds of arrows: those in $\mathbb{A}$; those in $\mathbb{B}$; those starting in A_0 and ending in B_0; and those starting in B_0 and ending in A_0. Observe that if $\mathbb{A}$ and $\mathbb{B}$ are both strongly connected, then so too is $\mathbb{C}$. The crucial result [25] is that categories $\mathbb{A}$ and $\mathbb{B}$ are equivalent if and only if they form the left and right parts of a bipartite category $[\mathbb{A}, \mathbb{B}]$

If S is an inverse semigroup, then

$$C(S) = \{(e, s, f) \in E(S) \times S \times E(S) \colon \ esf = s\}$$

is a category called the *Cauchy completion* of S.

If S is an inverse semigroup, then

$$L(S) = \{(e, s) \in E(S) \times S \colon \ se = s\}$$

is a left cancellative category associated with S. Its composition is given by $(e, s)(f, t) = (e, st)$, provided $s^* s = f$.

An inverse semigroup S can also be regarded as an inductive groupoid $G(S)$. Inductive groupoids are special kinds of ordered groupoids. This approach to inverse semigroups is described in [13]. The theory can be extended to inverse semigroupoids and so with every inverse semigroupoid there is an underlying ordered groupoid. Let S and T be inverse semigroups with associated inductive groupoids $G(S)$ and $G(T)$. A *bipartite ordered groupoid enlargement* of $G(S)$ and $G(T)$ is an ordered groupoid $[G(S), G(T)]$ such that the set of identities of $[G(S), G(T)]$ is the disjoint union of the set of identities of $G(S)$ and $G(T)$ and for each $e \in G(S)_0$ there exists an arrow x such that $\mathbf{d}(x) = e$ and $\mathbf{r}(x) \in G(T)_0$ and dually.

In [28, 29, 30], Talwar introduced a notion of Morita equivalence for a class of semigroups that includes inverse semigroups. Let S be an inverse semigroup. A left S-module X is said to be *unitary* if $SX = X$. The category S-**Fact** is the category of unitary left S-acts of the inverse semigroup S.[2] The inverse semigroups S and T are *Morita equivalent* if the categories S-**Fact** and T-**Fact** are equivalent.

If S is an inverse semigroup then $E(S)$ denotes its semilattice of idempotents. The inverse semigroup S acts on $E(S)$ on the left when we define $s \cdot e = ses^*$. We call this the *Munn module*. A left S-module X paired with an S-homomorphism $X \xrightarrow{p} E(S)$ to the Munn module, such that $p(x) \cdot x = x$, is what we call *an étale left S-module* [5]. We denote the category of étale left S-modules by **Étale**.

[2] The definition of this category is actually more complicated than this, but reduces to this one in the case of inverse semigroups. See [28] for details.

Étale can be taken as the definition of *the classifying topos* of S, denoted $\mathscr{B}(S)$.[3] **Étale** (or $\mathscr{B}(S)$) is equivalent to the category $PSh(L(S))$ of presheaves on $L(S)$, a result essentially due to Loganathan [18] and used in [4, 5, 17]. (How **Fact** and **Étale** are related is studied further in §2.3.)

In this paper, we shall often refer simply to *modules*, always assumed unitary, and *étale modules*. An étale module is specified by giving the map $p\colon X \longrightarrow E(S)$. Morphisms between étale modules must preserve the corresponding maps. The relationship between these two kinds of inverse semigroup action is discussed in more detail in Section 2.3.

We are now ready to state the main theorem of this paper.

Theorem 1.1 *Let S and T be inverse semigroups. Then the following are equivalent.*

(i) *S and T are strongly Morita equivalent.*

(ii) *The classifying toposes of S and T are equivalent.*

(iii) *The inductive groupoids S and T have an ordered groupoid joint enlargement, which can be chosen to be bipartite.*

(iv) *The categories $C(S)$ and $C(T)$ are equivalent*

(v) *S and T have a regular semigroup joint enlargement.*

(vi) *S and T are Morita equivalent.*

Condition (v) raises a question: is it true that two inverse semigroups which are Morita equivalent have a joint *inverse* enlargement? We suspect this is not true, although we do not have a counterexample. In the light of Proposition 5.9 [27], however, we make the following conjecture. Let S be an inverse semigroup. We say that S is *directed* if for each pair of idempotents $e, f \in S$ there is an idempotent i such that $e, f \leq i$. This is equivalent to the condition that each subset of the form eSf is a subset of some local submonoid iSi. Semigroups with this property are studied in [22, 23]. We conjecture that if S and T are both directed then they are Morita equivalent if and only if they have an inverse semigroup joint enlargement.

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[3]The term 'classifying topos' and its $\mathscr{B}$ notation more generally refer to the topos associated with an étale, or even localic, groupoid [19]. An ordered groupoid is étale in this sense. It is not difficult to see that the definition $\mathscr{B}(S) = \mathscr{B}(G(S))$ ultimately amounts to the category of étale left S-modules.

2 Proofs

In Section 2.1, we prove the equivalence of (i), (ii), (iii) and (iv) of Theorem 1.1 using methods mainly from topos theory and ordered groupoids. Some of our results are proved for ordered groupoids more general than inductive. Combined with results from [13] on the role played by ordered groupoids within inverse semigroup theory, such as in the P-theorem, this suggests that a Morita theory of certain kinds of ordered groupoids would be worth developing. In Section 2.2, we prove the equivalence of (i), (iii), (iv), (v) and (vi) using mainly methods of semigroup theory. Section 2.3 is different. It addresses the question raised by our work on the relationship between unitary actions of an inverse semigroup and étale actions and is a mixture of semigroup and category theory.

2.1 Ordered groupoids and toposes

We begin with some categorical preliminaries. One approach to Morita theory for categories involves what are called essential points of a topos [3], whereas another uses what are called profunctors or bimodules [25]. It is the second approach we shall use in common with Section 2.2.

Let $\mathbb{C}$ and $\mathbb{D}$ be (small) categories. $PSh(\mathbb{C})$ denotes the category of presheaves on $\mathbb{C}$. A *profunctor* $\mathbb{C} \quad > \mathbb{D}$ is by definition a functor

$$\mathbb{C} \longrightarrow PSh(\mathbb{D}) \ .$$

A profunctor $U : \mathbb{C} \quad > \mathbb{D}$ may be equivalently given as a colimit preserving functor

$$U : PSh(\mathbb{C}) \longrightarrow PSh(\mathbb{D}) \ . \qquad (1)$$

Categories, profunctors, and natural transformations form a bicategory. For any $\mathbb{C}$, the identity profunctor $\mathbb{C} \quad > \mathbb{C}$ is Yoneda $\mathbb{C} \longrightarrow PSh(\mathbb{C})$. It is convenient to denote a profunctor $\mathbb{C} \quad > \mathbb{D}$, the actual functor $\mathbb{C} \longrightarrow PSh(\mathbb{D})$, and the corresponding colimit-preserving functor (1) by one and the same symbol.

We say that a profunctor has a right adjoint if it has a right adjoint in the usual bicategorical sense. It follows that a profunctor $\mathbb{C} \quad > \mathbb{D}$ has a right adjoint if and only if the corresponding colimit-preserving functor (1) has a colimit-preserving right adjoint (it always has a right-adjoint, but the right adjoint may not preserve colimits). The proof we give of the following probably well-known fact about profunctors is basically the same as the proof of the analogous fact about essential points [3], Prop. 4.2.

Lemma 2.1 *Suppose that a profunctor $U : \mathbb{C} \quad > \mathbb{D}$ has a right adjoint. Then for every object c of $\mathbb{C}$, $U(c)$ is a retract of a representable in $PSh(\mathbb{D})$. Moreover, if idempotents split in $\mathbb{D}$, then every $U(c)$ is isomorphic to a representable.*

Proof. We may cover $U(c)$

$$\coprod_A d \twoheadrightarrow U(c) \qquad (2)$$

by its elements: A is the set of pairs (d, x), where $d \xrightarrow{x} U(c)$ is an element of $U(c)$. The covering map is canonical: at an object c', it sends a pair $(c' \xrightarrow{\alpha} d, d \xrightarrow{x} U(c))$ to the composite $x\alpha$. Let $V : PSh(\mathbb{D}) \longrightarrow PSh(\mathbb{C})$ denote the right adjoint of U. Apply V to (2), using that V preserves colimits, hence coproducts and epimorphisms.

$$\coprod_A V(d) \twoheadrightarrow VU(c) \tag{3}$$

Evaluate the natural transformation (3) at c:

$$\coprod_A V(d)(c) \twoheadrightarrow VU(c)(c) \ .$$

The set $VU(c)(c)$ is isomorphic to the set $PSh(\mathbb{D})(U(c), U(c))$. Since the map above is onto, there is an element $d \xrightarrow{x} U(c)$ of A and a morphism $\xi : c \longrightarrow V(d)$, equivalently one $\hat{\xi} : U(c) \longrightarrow d$, making the following diagram commute.

$$\begin{tikzcd}
U(c) \arrow[r, "\hat{\xi}"] \arrow[dr, "1"'] & d \arrow[r, "y"] \arrow[d, "x"] & d' \arrow[d, "z"'] \arrow[dr, "1"] & \\
& U(c) \arrow[r, "\hat{\xi}"] & d \arrow[r, "y"'] & d'
\end{tikzcd}$$

This says that $U(c)$ is a retract of a representable. Notice that $\hat{\xi}x$ is an idempotent of $\mathbb{D}$, so that if idempotents split in $\mathbb{D}$, then we may split $\hat{\xi}x$ in $\mathbb{D}$ (depicted zy in the diagram). Then $U(c)$ is isomorphic to a representable since a little diagram chasing shows that $y\hat{\xi}$ is an isomorphism with inverse xz. □

Proposition 2.2 *A presheaf is connected and projective iff it is a retract of a representable. If idempotents split in the small category, then a presheaf is projective and connected iff it is isomorphic to a representable.*

Proof. Let P be a presheaf on a small category $\mathbb{D}$. We may cover P

$$\coprod_A d \twoheadrightarrow P$$

by its elements. If P is projective then this epimorphism must split, and if P is connected, then the splitting must factor through a unique section $d \xrightarrow{s} P$. It follows that P is a retract of the representable d. If idempotents split in $\mathbb{D}$, then as in Prop. 2.1 it follows that P is isomorphic to a representable.

The converse is easily seen to hold, first for represenable presheaves, and then for retracts of representables. □

A functor is *a weak equivalence* if it is full, faithful, and essentially surjective on objects. For example, an inverse subsemigroup $S \subseteq T$ is an enlargement if and only if its corresponding functor $L(S) \longrightarrow L(T)$ is a weak equivalence (Lemma 2.6). An equivalence profunctor is a profunctor that is an equivalence in the bicategory of profunctors.

Proposition 2.3 *Suppose that idempotents split in both $\mathbb{C}$ and $\mathbb{D}$. Then an equivalence profunctor is equivalently given by a 'Morita context'*

$$\begin{tikzcd} \mathbb{C} \arrow[dr, "F"'] & & \mathbb{D} \arrow[dl, "G"] \\ & \mathbb{U} & \end{tikzcd}$$

by which we mean a pair of weak equivalences (we may even assume that $\mathbb{U} = [\mathbb{C}, \mathbb{D}]$).

Proof. An equivalence profunctor between $\mathbb{C}$ and $\mathbb{D}$ is given by an equivalence of presheaf categories:

$$U : PSh(\mathbb{C}) \simeq PSh(\mathbb{D}) .$$

Let $\mathbb{U}$ denote the full subcategory of $PSh(\mathbb{D})$ on the representables d and objects $U(c)$. The functor F is Yoneda for $\mathbb{C}$ followed by U. The functor G is Yoneda for $\mathbb{D}$. F and G are full and faithful. To see that F is essentially surjective on objects let V denote the pseudo-inverse of U. Then $V \dashv U \dashv V$, and of course both functors preserve colimits. For any d, $V(d)$ is isomorphic to a representable c (Lemma 2.1). Then $d \cong UV(d) \cong U(c)$, showing that F is essentially surjective on objects. In the same way, G is essentially surjective on objects. In fact, we have $\mathbb{U} = [\mathbb{C}, \mathbb{D}]$.

On the other hand, given a Morita context, then we have equivalence functors

$$PSh(\mathbb{C}) \simeq PSh(\mathbb{U}) \simeq PSh(\mathbb{D}) ,$$

which gives an equivalence profunctor between $\mathbb{C}$ and $\mathbb{D}$. □

Proof of the equivalence of (i) and (ii).

Let S and T be inverse semigroups, and assume that the toposes $\mathscr{B}(S)$ and $\mathscr{B}(T)$ are equivalent. We use Proposition 2.3. In this case, $\mathbb{C} = L(S)$ and $\mathbb{D} = L(T)$ are left cancellative categories, so the identities are their only (split) idempotents. By Proposition 2.3, there is an equivalence $U : \mathscr{B}(S) \simeq \mathscr{B}(T)$ if and only if there is a Morita context

$$\begin{tikzcd} L(S) \arrow[dr, "P"'] & & L(T) \arrow[dl, "Q"] \\ & \mathbb{U} & \end{tikzcd}$$

where $\mathbb{U}$ is the (left-cancellative) category whose objects are the idempotents of S and T (disjoint collection). $\mathbb{U} = [L(S), L(T)]$ has three kinds of morphisms:

(i) those of $L(S)$,

(ii) those of $L(T)$, and

(iii) the connecting ones between $d \in E(S)$ and $e \in E(T)$, which are understood as natural transformations between presheaves $U(d)$ and e in $\mathscr{B}(T)$.

We may reorganize this data into a Morita context. Let X denote the set of connecting isomorphisms from an idempotent of T to an idempotent of S; that is, the morphisms of type (iii) above, but only the isomorphisms and only in the direction from T to S.

The action by S is composition on the right, which we write as a left action. Let $e \xrightarrow{x} d$ be an element of X: this is an isomorphism $x : e \cong U(d)$ in $\mathscr{B}(T)$. Let $s \in S$. If $s^*s = d$, then sx is the composite isomorphism $e \cong U(d) \cong U(ss^*)$. This defines a partial action by S, which we can make total with the help of the following lemma.

Lemma 2.4 *Let $U : \mathscr{B}(G) \simeq \mathscr{B}(H)$ be an equivalence of classifying toposes of ordered groupoids G and H. Let $b \leq d$ in G_0 and $x : e \cong U(d)$ be an isomorphism of $\mathscr{B}(H)$. Then there is a unique idempotent $a \leq e$ in H_0, and a unique isomorphism $bx : a \cong U(b)$ such that*

$$\begin{tikzcd} a \arrow[r, "bx"] \arrow[d] & U(b) \arrow[d] \\ e \arrow[r, "x"] & U(d) \end{tikzcd}$$

is a pullback in $\mathscr{B}(H)$.

Proof. By Lemma 2.1, there is $c \in H_0$ and an isomorphism $y : c \cong U(b)$. Consider the composite

$$c \cong U(b) \longrightarrow U(d) \cong e$$

in $\mathscr{B}(H)$, where the last isomorphism is x^{-1}. By Yoneda, this comes from a unique morphism $c \xrightarrow{t} e$ in $L(H)$. Let $a = \mathbf{r}(t) \leq e$, and $bx = yt^{-1}$.

Such an a is unique because a subobject (which is an isomorphism class of monomorphisms) of a representable e corresponds uniquely to a downclosed subset of elements of H_0 under e, and a principal one corresponds uniquely to an element of H_0 under e. If a and a' both make the square a pullback, then they are in the same isomorphism class of monomorphisms into e, hence they represent the same subobject, hence $a = a'$. The isomorphism bx is also unique because $U(b) \longrightarrow U(d) \cong e$ is a monomorphism. □

Returning to inverse semigroups, we see how to make the action total: let $b = ds^*s \leq d$, and let $sx = sd \cdot bx$.

The inner product $\langle\ ,\ \rangle : X \times X \longrightarrow S$ is defined as follows. If two isomorphisms $x : e \cong U(d)$ and $y : e \cong U(c)$ have the same domain, then $\langle x, y \rangle = yx^{-1}$. This is an isomorphism of $\mathscr{B}(T)$ between $U(d)$ and $U(c)$, but that amounts to an isomorphism of $L(S)$, which in turn is precisely an element of S. In general, the inner product of $x : f \cong U(d)$ and $y : e \cong U(c)$ is defined by using variations of Lemma 2.4.

$$\begin{tikzcd} U(a) \arrow[r] \arrow[d] & ef \arrow[d] \\ U(d) \arrow[r, "x^{-1}"] & f \end{tikzcd} \qquad \begin{tikzcd} ef \arrow[r] \arrow[d] & U(b) \arrow[d] \\ e \arrow[r, "y"] & U(c) \end{tikzcd}$$

These "variations" can be established in the same way as in Lemma 2.4, or they can be deduced from Lemma 2.4 by transposing under the pseudo-inverse V. For example, the right-hand square above can be obtained by applying Lemma 2.4 (with V instead of U) to the transpose of y^{-1}, as in the following diagram.

$$\begin{tikzcd} b \arrow[r] \arrow[d] & V(ef) \arrow[d] \\ c \arrow[r, "\widehat{y^{-1}}"] & V(e) \end{tikzcd}$$

The right action by T and the inner product $[\ ,\ ] : X \times X \longrightarrow T$ are entirely analogous. The axioms of a Morita context for semigroups are easily verified. For example, for any $x : f \cong U(d)$, the rule $\langle x, x\rangle x = x$ is the fact that the composite $xx^{-1}x$ is equal to x (in $\mathbb{U}$):

$$f \cong U(d) \cong f \cong U(d) \ \ ; \ \langle x, x\rangle x = xx^{-1}x = x \ .$$

We have therefore proved that (ii) implies (i).

The fact that (i) implies (ii), was proved by Steinberg [27]. However, it is of interest to see how to build a Morita context in the category sense from a Morita context X in the semigroup sense.

$$\begin{tikzcd} L(S) \arrow[dr, "P"'] & & L(T) \arrow[dl, "Q"] \\ & \mathbb{U} & \end{tikzcd}$$

By definition, the objects of the bipartite category $\mathbb{U} = [L(S), L(T)]$ are disjointly the objects of $L(S)$ and $L(T)$, which are the idempotents of S and of T. A morphism of $\mathbb{U}$ is either:

(i) one of $L(S)$,

(ii) one of $L(T)$,

(iii) one of the form $(x, d) \in X \times E(S)$, such that $\langle x, x\rangle \leq d$, where the domain of this morphism is $[x, x] \in E(T)$, and its codomain is d, or

(iv) one of the form $(x, e) \in X \times E(T)$, such that $[x, x] \leq e$, where the domain of this morphism is $\langle x, x\rangle \in E(S)$, and its codomain is e.

We compose the various kinds of morphisms in $\mathbb{U}$ by using the inner products and actions in X by S and T. For example, by definition

$$\begin{tikzcd} s^*s \arrow[r, "s"] \arrow[dr, "s^*x"'] & d \arrow[d, "x"] \\ & e \end{tikzcd}$$

commutes in $\mathbb{U}$, where $s \in S$, $d \in E(S)$, $x \in X$, $d = \langle x, x\rangle$, $s = ds$, $e \in E(T)$ and $[x, x] \leq e$. In other words, we define $(x, e)(s, d) = (s^*x, e)$. The pair

(s^*x, e) is indeed a legitimate morphism of $\mathbb{U}$ because the idempotent product $[x, x][s^*x, s^*x]$ is equal to

$$[x, \langle x, s^*x\rangle s^*x] = [x, \langle x, x\rangle ss^*x] = [x, dss^*x] = [x, ss^*x] = [s^*x, s^*x] \ .$$

Therefore, $[s^*x, s^*x] \leq [x, x] \leq e$. The domain of (s^*x, e) is

$$\langle s^*x, s^*x\rangle = s^*\langle x, x\rangle s = s^*ds = s^*s \ ,$$

which is the domain of (s, d) as it should be. For another example,

$$\begin{tikzcd} \langle x,x\rangle \arrow[r, "x"] \arrow[dr, "\langle y,x\rangle"'] & {[y,y]} \arrow[d, "y"] \\ & e \end{tikzcd}$$

commutes, where $[x, x] \leq [y, y]$. The domain of the composite $\langle y, x\rangle$ is

$$\langle y, x\rangle^*\langle y, x\rangle = \langle x, y\rangle\langle y, x\rangle = \langle x[y, y], x\rangle = \langle x, x\rangle \ ,$$

since $x = x[x, x] = x[x, x][y, y] = x[y, y]$. It follows that $\mathbb{U}$ is a category, that $\mathbb{U} = [L(S), L(T)]$, and that the obvious functors P, Q are weak equivalences.

Corollary 2.5 *The category $\mathbb{U}$ constructed from a Morita context X is left cancellative.*

Proof. This is true because $\mathbb{U}$ is weakly equivalent to a left cancellative category. However, the following calculations give more information. For example, if

$$\begin{tikzcd} s^*s \arrow[r, "s"] \arrow[dr, "y"'] & d \arrow[d, "x"] \\ & e \end{tikzcd}$$

commutes in $\mathbb{U}$, where $d = \langle x, x\rangle$ and $[x, x] \leq e$, then $y = s^*x$ (by definition) and

$$s = ds = \langle x, x\rangle s = \langle x, s^*x\rangle = \langle x, y\rangle \ .$$

Thus, s is uniquely determined by x and y. The other possibility, but keeping (x, e), is

$$\begin{tikzcd} {[y,y]} \arrow[r, "y"] \arrow[dr, "t={[x,y]}"'] & d \arrow[d, "x"] \\ & e \end{tikzcd}$$

where $\langle y, y\rangle \leq d$. Then y is determined by x and t since

$$y = \langle y, y\rangle y = \langle x, x\rangle\langle y, y\rangle y = \langle x, x\rangle y = x[x, y] = xt \ .$$

It follows that (x, e) is a monomorphism. $\square$

Proof of the equivalence of (ii) and (iii).

An ordered functor $\theta\colon G \longrightarrow H$ is said to be *a local isomorphism* if it satisfies the following two conditions.

(LI1) θ is full, faithful and essentially surjective.

(LI2) $\theta_0 : G_0 \longrightarrow H_0$ is a discrete fibration (in the same sense that the domain map of an ordered groupoid is one).

An enlargement is a local isomorphism.

Lemma 2.6 *An ordered functor $\theta : G \longrightarrow H$ is a local isomorphism if and only if $L(\theta) : L(G) \longrightarrow L(H)$ is a weak equivalence* (in the category sense).

Proof. Clearly $L(\theta)$ is essentially surjective if θ is. $L(\theta)$ is full: let $\theta(d) \xrightarrow{t} \theta(e)$ be a morphism of $L(H)$. Consider the unique lifting $c \leq e$ of $\mathbf{r}(t) \leq \theta(e)$, so that $\theta(c) = \mathbf{r}(t)$. Since θ is full there is $d \xrightarrow{s} e$ (in G) such that $\theta(s) = t$. Thus, $L(\theta)(s) = t$. $L(\theta)$ is faithful: suppose that $L(\theta)(s) = L(\theta)(t)$, where $s, t : d \longrightarrow e$ in $L(G)$. Let $c = \theta(\mathbf{r}(s)) = \theta(\mathbf{r}(t))$. The two inequalities $\mathbf{r}(s) \leq e$ and $\mathbf{r}(t) \leq e$ both lie above $c \leq \theta(e)$, so they must be equal by the uniqueness of liftings along θ_0. Thus, if θ is faithful, then $s = t$.

For the converse, if $L(\theta)$ is a weak equivalence, then we see easily that θ is full, faithful, and essentially surjective. Condition (LI2) can be seen to hold as follows. We have a commuting square of toposes

$$\begin{tikzcd} PSh(G_0) \arrow[r] \arrow[d] & PSh(H_0) \arrow[d] \\ \mathscr{B}(G) \arrow[r] & \mathscr{B}(H) \end{tikzcd}$$

where the bottom horizontal is an equivalence (associated with the weak equivalence $L(\theta)$). Since the verticals are étale, so is the other horizontal. Therefore, $G_0 \longrightarrow H_0$ is a discrete fibration. $\square$

Theorem 2.7 *The following are equivalent for ordered groupoids G and H:*

(i) *the classifying toposes of G and H are equivalent;*

(ii) *G and H have a joint bipartite enlargement $[G, H]$;*

(iii) *there is an ordered groupoid K and local isomorphisms $G \longrightarrow K \longleftarrow H$.*

Proof. $1 \Rightarrow 2$ Given an equivalence $U : \mathscr{B}(G) \simeq \mathscr{B}(H)$, consider the groupoid K such that $K_0 = G_0 + H_0$ and $K_1 = G_1 + H_1 + Y$, where Y is set of isomorphisms of $\mathscr{B}(H)$ between objects $U(d)$ and e. K_1 is partially ordered: for $i : U(d) \cong e$

and $j : U(a) \cong b$, we declare $i \leq j$ when $d \leq a$ in G_0 and $e \leq b$ in H_0 and the square of natural transformations

$$\begin{tikzcd} U(d) \arrow[r, "i"] \arrow[d] & e \arrow[d] \\ U(a) \arrow[r, "j"] & b \end{tikzcd}$$

commutes in $\mathscr{B}(H)$. The definition of $\leq$ for isomorphisms in the other direction is similar. By Lemma 2.4, the domain map $K_1 \longrightarrow K_0$ is a discrete fibration.

$2 \Rightarrow 3$ holds because an enlargement is a local isomorphism. $3 \Rightarrow 1$ holds because given such local isomorphisms, then $\mathscr{B}(G)$ and $\mathscr{B}(H)$ are equivalent by Lemma 2.6 since the geometric morphism associated with a weak equivalence of categories is an equivalence. □

Having proved the equivalence of (i), (ii) and (iii), we ought to be able to obtain from a given Morita context X between inverse semigroups S and T a common ordered groupoid enlargement of $G(S)$ and $G(T)$ directly. We do this in Proposition 2.21, (4), where it is denoted $G(S, T; X)$. This can also be done using the Schützenberger object

$$\mathbf{S}(e) = \begin{cases} \{s \in S \mid s^*s = e\} + \{x \in X \mid \langle x, x\rangle = e\}\ , & e \in E(S) \\ \{t \in T \mid t^*t = e\} + \{x \in X \mid [x, x] = e\}\ , & e \in E(T) \end{cases}$$

in the étendue $PSh(\mathbb{U})$, where $\mathbb{U}$ denotes the left cancellative category built from X (as in Cor. 2.5).

Lemma 2.8 $\mathbf{S}$ *is a torsion-free generator of the étendue* $PSh(\mathbb{U})$.

Proof. The category of elements of $\mathbf{S}$ is a preorder since $\mathbb{U}$ is left cancellative (Cor. 2.5). Therefore, $\mathbf{S}$ is torsion-free. $\mathbf{S}$ has global support so it is a generator. □

Let $\mathbb{S}_0 \longrightarrow \mathbb{U}$ denote the discrete fibration corresponding to the presheaf $\mathbf{S}$. $\mathbb{S}_0$ is the category of elements of $\mathbf{S}$, whose objects are 'elements' $e \xrightarrow{u} \mathbf{S}$. By Lemma 2.8, $\mathbb{S}_0$ is a preorder, and the category pullback

$$\begin{tikzcd} \mathbb{S}_1 \arrow[r] \arrow[d] & \mathbb{S}_0 \arrow[d] \\ \mathbb{S}_0 \arrow[r] & \mathbb{U} \end{tikzcd}$$

defines a *pre*ordered groupoid $(\mathbb{S}_0, \mathbb{S}_1)$. It follows that the ordered groupoid $G(S, T; X)$ is order-isomorphic to the posetal collapse of $(\mathbb{S}_0, \mathbb{S}_1)$: the object-poset of $G(S, T; X)$ equals the posetal collapse of $\mathbb{S}_0$, which may be identified with the map

$$\mathbb{S}_0 \twoheadrightarrow E(S) + E(T)$$

such that an element

$$e \xrightarrow{u} \mathbf{S} \mapsto \begin{cases} uu^* & u \in S \text{ or } u \in T \\ \langle u,u\rangle & u \in X \text{ and } e = [u,u] \\ [u,u] & u \in X \text{ and } e = \langle u,u\rangle \end{cases} .$$

Likewise, the morphism-poset of $G(S,T;X)$ equals the posetal collapse of $\mathbb{S}_1$. Moreover, the underlying groupoid of $G(S,T;X)$, where we ignore its order structure, equals the isomorphism subcategory of $\mathbb{U}$.

Proof of the equivalence of (ii) and (iv).

An ordered groupoid G is said to be *principally inductive* if for each identity e the poset $e^{\downarrow} = \{f \in G_0 \colon\ f \leq e\}$ is a meet semilattice under the induced order [11]. It is worth noting that if G is an ordered groupoid, then it is principally inductive precisely when the left cancellative category $L(G)$ has pullbacks. It is routine to verify that ordered groupoid enlargements of principally inductive groupoids are also principally inductive.

Principally inductive groupoids have a Cauchy completion, which we denoted $C(S)$ in the inverse case. Let G be such a groupoid. Define

$$C(G) = \{(e,x,f) \in G_0 \times G \times G_0 \colon\ \mathbf{d}(x) \leq f, \mathbf{r}(x) \leq e\}$$

and define a partial binary operation by $(e,x,f)(f,y,i) = (e, x \otimes y, i)$. Observe that $x \otimes y$ is defined because $\mathbf{d}(x), \mathbf{r}(y) \leq f$ and the fact that G is assumed to be principally inductive. Furthermore $C(G)$ is always an inverse category. However, it is not always strongly connected as in the inverse case.

Lemma 2.9 *Let G and H be principally inductive. An ordered functor $\theta : G \longrightarrow H$ is a local isomorphism if and only if $C(\theta) : C(G) \longrightarrow C(H)$ is a weak equivalence.*

Proof. The forward implication is similar to the proof of Lemma 2.6. On the other hand, if $C(\theta)$ is a weak equivalence, then so is $L(\theta)$ so we may use Lemma 2.6. □

Proposition 2.10 *Assume (AC). Let G and H be principally inductive ordered groupoids. Then the categories $C(G)$ and $C(H)$ are equivalent if and only if $\mathscr{B}(G) \simeq \mathscr{B}(H)$.*

Proof. If $C(G) \simeq C(H)$, then $L(G) \simeq L(H)$ since $L(G)$ equals the subcategory of $C(G)$ consisting of those morphisms with retracts. Hence, $\mathscr{B}(G) \simeq \mathscr{B}(H)$. Conversely, an equivalence of classifying toposes gives weak equivalences $L(G) \longrightarrow \mathbb{U} \longleftarrow L(H)$, and hence (by AC) an equivalence $L(G) \simeq L(H)$. Therefore, $C(G) \simeq C(H)$ because $C(G)$ is canonically equivalent to $\mathrm{Span}(L(G))$, where the Span of a category with pullbacks is given by the same objects, but whose morphisms are spans $\cdot \longleftarrow \cdot \longrightarrow \cdot$ in the given category. (This aspect is further explained following Prop. 2.23.) □

2.2 Inverse semigroups

We shall prove first that (v) is equivalent to strong Morita equivalence.

We begin with some preliminaries taken from [16] where proofs of all unproved statements can be found. A category $\mathbb{C} = (C_0, C_1)$ is said to be *strongly connected* if for each pair of identities e and f there is an arrow from e to f. All categories in this part will be strongly connected. Let $\mathbb{C}$ be a strongly connected category. A *consolidation* for $\mathbb{C}$ is a function $p\colon C_0 \times C_0 \longrightarrow C_1$, $p(e,f) = p_{e,f}$, where $p_{e,f}$ is an arrow from f to e and $p_{e,e} = e$. Given a category $\mathbb{C}$ equipped with a consolidation p we can define a binary operation $\circ$ on $\mathbb{C}$ by $x \circ y = xp_{e,f}y$ where x has domain e and y has codomain f. It is easily checked that this converts $\mathbb{C}$ into a semigroup, denoted $\mathbb{C}^p$. If we omit $\circ$, then we mean the category product. A category $\mathbb{C}$ is said to be *regular* if for each morphism a there exists another one b such that $a = aba$.

Lemma 2.11 *Let $\mathbb{C}$ be a strongly connected regular category, and let p be a consolidation on $\mathbb{C}$. Then $\mathbb{C}^p$ is regular.*

A consolidation r of a bipartite category $\mathbb{C} = [\mathbb{A}, \mathbb{B}]$ induces consolidations on the full subcategories $\mathbb{A}$ and $\mathbb{B}$. Thus $\mathbb{A}^r$ and $\mathbb{B}^r$ are subsemigroups of $\mathbb{C}^r$.

Lemma 2.12 *Let $\mathbb{C} = [\mathbb{A}, \mathbb{B}]$ be a bipartite category and let r be a consolidation defined on $\mathbb{C}$. Then $\mathbb{C}^r$ is an enlargement of both $\mathbb{A}^r$ and $\mathbb{B}^r$.*

Lemma 2.13 *Let $\mathbb{C} = [\mathbb{A}, \mathbb{B}]$ be a bipartite category. If $\mathbb{A}$ and $\mathbb{B}$ are both regular, then we can assume that $\mathbb{C}$ is also regular.*

Let $\mathbb{C} = [\mathbb{A}, \mathbb{B}]$ be a bipartite category, let p be a consolidation on $\mathbb{A}$, and q a consolidation on $\mathbb{B}$. We define a consolidation r on $\mathbb{C}$ as a whole as follows. Choose an identity $i_0 \in A_0$ and an isomorphism ξ with domain i_0 and codomain $j_0 \in B_0$. Define the consolidation r on $\mathbb{C}$ as follows:

$$r_{e,f} = \begin{cases} p_{e,f} & \text{if } e, f \in A_0 \\ q_{e,f} & \text{if } e, f \in B_0 \\ q_{e,j_0}\xi p_{i_0,f} & \text{if } e \in B_0, f \in A_0 \\ p_{e,i_0}\xi^{-1}q_{j_0,f} & \text{if } e \in A_0, f \in B_0 \ . \end{cases}$$

In other words, r agrees with p and q on $\mathbb{A}$ and $\mathbb{B}$ respectively, and then uses ξ to do the simplest possible thing to define it on the whole of $\mathbb{C}$ using the isomorphism ξ. We say that r is a *natural extension of p and q to $\mathbb{C}$.*

Proposition 2.14 *We assume the above setup. Let π_1 be a congruence on $\mathbb{A}^p$, and let π_2 be a congruence on $\mathbb{B}^q$. Let π be the congruence on $\mathbb{C}^q$ generated by $\pi_1 \cup \pi_2$.*

1. *$\pi \cap (\mathbb{A}^p \times \mathbb{A}^p) = \pi_1$ if the following three conditions hold:*

   (i) $(a, a') \in \pi_1 \Rightarrow (\xi^{-1} \circ a, \xi^{-1} \circ a') \in \pi_1$.

(ii) $(a, a') \in \pi_1 \Rightarrow (a \circ \xi, a' \circ \xi) \in \pi_1$.

(iii) $(b, b') \in \pi_2 \Rightarrow (\alpha \circ b \circ \beta, \alpha \circ b' \circ \beta) \in \pi_1$ *for all isomorphisms* α *and* β *where* α *has domain in* B_0 *and codomain in* A_0 *and* β *has domain in* A_0 *and codomain in* B_0.

2. $\pi \cap (\mathbb{B}^q \times \mathbb{B}^q) = \pi_2$ *if the following three conditions hold:*

(i) $(b, b') \in \pi_2 \Rightarrow (\xi \circ b, \xi \circ b') \in \pi_2$

(ii) $(b, b') \in \pi_2 \Rightarrow (b \circ \xi^{-1}, b' \circ \xi^{-1}) \in \pi_2$.

(iii) $(a, a') \in \pi_1 \Rightarrow (\alpha \circ a \circ \beta, \alpha \circ a'\beta) \in \pi_2$ *for all isomorphisms* α *and* β *where* α *maps* A_0 *to* B_0 *and* β *maps* B_0 *to* A_0.

Proposition 2.15 *Let* S *and* T *be inverse semigroups. Then the following are equivalent:*

(i) *S and T are strongly Morita equivalent;*

(ii) *The categories $C(S)$ and $C(T)$ are equivalent;*

(iii) *There is a regular semigroup which is a joint enlargement of S and T.*

Proof. We have already proved that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Let $C(S)$ and $C(T)$ be equivalent categories. By [25], we can find a bipartite category $\mathbb{C} = [C(S), C(T)]$. Both $C(S)$ and $C(T)$ are regular, so we can assume that $\mathbb{C}$ is regular by Lemma 2.13. Moreover, both $C(S)$ and $C(T)$ are strongly connected, so $\mathbb{C}$ is strongly connected. We now make the following definitions.

- The identities of $C(S)$ are of the form (e, e, e) where e is an idempotent of S. We abbreviate (e, e, e) by $\overline{e}$. On $C(S)$ we define the consolidation $p_{\overline{e},\overline{f}} = (e, ef, f)$. The function $\pi_1^\natural \colon C(S)^p \longrightarrow S$ given by $(e, s, f) \mapsto s$ is a surjective homomorphism.

- The identities of $C(T)$ are of the form (i, i, i) where i is an idempotent of T. We abbreviate (i, i, i) by $\mathbf{i}$. On $C(T)$ we define the consolidation $q_{\mathbf{i},\mathbf{j}} = (i, ij, j)$. The function $\pi_2^\natural \colon C(T)^q \longrightarrow T$ given by $(i, t, j) \mapsto t$ is a surjective homomorphism.

Let $\overline{e_0}$ be any identity in $C(S)$. Since $\mathbb{C}$ is bipartite, there is an isomorphism $\xi \in \mathbb{C}$ with domain $\overline{e_0}$ and codomain $\mathbf{f_0}$ for some identity in $C(T)$. Let r be a natural extension of p and q to $\mathbb{C}$ defined using this ξ.

We now verify that the conditions of Proposition 2.14 (1) hold; that those of (2) also hold follows by symmetry.

Condition (i). Let $(e, s, f) \, \pi_1 \, (e', s, f')$. Then simple calculations show that $\xi^{-1} \circ (e, s, f) = (e_0, e_0 s, f)$ and $\xi^{-1} \circ (e', s, f') = (e_0, e_0 s, f')$. Hence $\xi^{-1} \circ (e, s, f) \, \pi_1 \, \xi^{-1} \circ (e', s, f')$.

Condition (ii). Let $(e,s,f)\,\pi_1\,(e',s,f')$. Then simple calculations show that $(e,s,f)\circ\xi = (e,se_0,e_0)$ and $(e',s,f')\circ\xi = (e,se_0,e_0)$. Hence $(e,s,f)\circ\xi\,\pi_1\,(e',s,f')\circ\xi$.

Condition (iii). Let $(i,t,j)\,\pi_2\,(i',t,j')$. Let $\overline{f} \xrightarrow{\alpha} \mathbf{e}$ and $\mathbf{e}' \xrightarrow{\beta} \overline{f'}$ be isomorphisms in $\mathbb{C}$. Then simple calculations show that $\alpha\circ(i,t,j)\circ\beta = \alpha(f,ftf',f')\beta$ and $\alpha\circ(i',t,j')\circ\beta = \alpha(f,ftf',f')\beta$. Thus these two elements are actually equal and so clearly π_1-related.

By Lemma 2.11, the semigroup $\mathbb{C}^r$ is regular, and by Lemma 2.12 it is an enlargement of both $C(S)^p$ and $C(T)^q$. Thus $R = \mathbb{C}^r/\pi$ is a regular semigroup that contains (isomorphic copies of) S and T as regular subsemigroups. But enlargements are preserved under homomorphisms by Proposition 2.9 of [7]. Thus R is an enlargement of both S and T, as required.

(iii) $\Rightarrow$ (i). Let the regular semigroup R be a joint enlargement of inverse subsemigroups S and T. Let $x\in SRT$. Then $x = srt$. Let s^* be the unique inverse of s in S and let t^* be the unique inverse of t in T. Then x has an inverse of the form $t^*r's^*\in TRS$ where $r'\in R$ is some element. Put

$$X = \{(x,x')\colon\; x\in SRT \text{ and } x'\in V(x)\cap TRS\}.$$

Observe that

$$xx'\in(SRT)(TRS) = S(RTTR)S\subseteq S$$

and

$$x'x\in(TRS)(SRT) = T(RSSR)T\subseteq T.$$

Thus we may define a left action of S on X by $s(x,x') = (sx, x's^*)$ and a right action of T on X by $(x,x')t = (xt, t^*x')$. Thus X is an (S,T)-bimodule. Define $\langle (x,x'),(y,y')\rangle = xy'$ and $[(x,x'),(y,y')] = x'y$. We need to show that these maps are surjections. We prove that the first is surjective; the proof that the second is surjective follows by symmetry. Let $s\in S$. Then $s = bta'$ where $aa' = s^*s$ and $bb' = ss^*$ and $a\in V(a)$ and $b\in V(b)$. That this is possible is proved in [12]. Let $t\in V(t)$ such that $t't = a'a$ and $tt' = b'b$. Then $(b,b'),(at',ta')\in X$ and $\langle (b,b'),(at',ta')\rangle = bta' = s$, as required. It is now routine to verify that axioms (MC1)–(MC7) hold and that we have therefore defined a Morita context. $\square$

Our proof of Theorem 1.1 is concluded by the following result which connects strong Morita equivalence to Morita equivalence.

Proposition 2.16 *Let S and T be inverse semigroups. Then S and T are Morita equivalent if and only if they are strongly Morita equivalent.*

Proof. Let S and T be strongly Morita. Then by Theorem 1.1 there is a regular semigroup R which is an enlargement of both S and T. Thus from [16, 12, 14], S is Morita equivalent to each of S and T and so S and T are Morita equivalent to each other.

We prove the converse directly.[4] Let S be an inverse semigroup and let S-**Fact** be its category of unitary left S-actions. We write left S-homomorphisms on the right of their arguments. This category has arbitrary coproducts: disjoint unions of unitary left S-modules are unitary left S-modules. It can be proved, using essentially the same argument as that in [1], that in this category epimorphisms are precisely the surjections. The left S-modules Se where e is an idempotent are clearly unitary and it can be directly verified that they are indecomposable projectives. By the same argument as in Proposition II.14.3 [21], coproducts of projectives are projectives. Let X be an arbitrary unitary left S-act and let $x \in X$. Since $SX = X$, by assumption, there exists $s \in S$ and $y \in X$ such that $sy = x$. But then $ss^*x = ss^*sy = sy = x$. Thus for each $x \in X$, there exists an idempotent $e_x \in S$ such that $e_x x = x$. Form the coproduct $\coprod_{x \in X} Se_x$. This is projective and unitary and there is an obvious surjection from it onto X. It follows that the category S-**Fact** has enough projectives. By the same argument as in Proposition II.14.2 of [21], every surjection onto a projective is a retraction. Let X be an arbitrary indecomposable projective. Then there is a surjection $\pi\colon \coprod_{x \in X} Se_x \longrightarrow X$ given by $\pi(se_x) = sx$. By the above, this map is a retraction and so there is an injective left S-homomorphism $\sigma\colon X \longrightarrow \coprod_{x \in X} Se_x$ such that $\sigma\pi$ is the identity on X. Now $X\sigma$ is a submodule of $\coprod_{x \in X} Se_x$ and indecomposable thus it must be contained inside Se_y for some y. It follows that $\sigma\colon X \longrightarrow Se_y$ defines an injective left S-homomorphism. But using the fact that $\sigma\pi = 1_X$ we find that $X = (Se_y)\pi$. Now Se_y is a cyclic left S-module and so X is a cyclic left S-module. We may therefore assume that X is a projective cyclic left S-module where $X = Sx$ for some $x \in X$. Now X is unitary and so there is an idempotent $e \in S$ such that $ex = x$. Define $\varphi\colon Se \longrightarrow X$ by $(s)\varphi = sx$. Then φ is a surjection. But Sx is projective and so there exists a map $\psi\colon X \longrightarrow Se$ such that $\psi\varphi = 1_P$. We therefore have an injective map $\psi\colon X = Sx \longrightarrow Se$. Put $f = (x)\psi$. Then $f = (x)\psi = (ex)\psi = e(x)\psi = ef$, and since $x \in Sf$ we have that $fe = f$. Observe that $f^2 = fefe = fe = f$ and so f is an idempotent and $f \leq e$. It follows that ψ induces an isomorphism between $X = Sx$ and Se, as required.

We have proved that each indecomposable projective in the category S-**Fact** is isomorphic to one of the form Se where e is an idempotent. The full subcategory of S-**Fact** whose objects are the left S-modules of the form Se as e varies over the idempotents is isomorphic to the category $C(S)$ and equivalent to the full subcategory of S-**Fact** whose objects are all indecomposable projectives. It follows that if S is Morita equivalent to T then $C(S)$ is equivalent to $C(T)$. □

Although the two definitions of Morita equivalence turn out to be the same, there are great advantages to working with strong Morita equivalence and equivalence bimodules as defined in [27] when working with inverse semigroups.

[4] This result is in principle derivable from [28, 29]. However, Neklyudova [22] points out that Talwar's argument uses monoid results which are not directly applicable in the case of semigroups with local units. For this reason, we reproved his results carefully in [16]. The argument given there simplifies in the case of inverse semigroups and so it makes sense to give the proof in the inverse case directly.

To conclude, we describe an application of strong Morita equivalence to the theory of E-unitary inverse semigroups. With each E-unitary inverse semigroup S we can associate a triple (G, X, Y), called a *McAlister triple*, where G is a group, X a poset, and Y a subposet of X which is a semilattice for the induced order [13]. This triple is required to satisfy certain conditions, one of which is that G acts on X by order automorphisms. If (G, X) and (G', X') each consist of a group acting by order automorphisms on a poset, then we say they are *equivalent* if G and G' are isomorphic, X and X' are order-isomorphic, and the actions under these isomorphisms are the same.

Proposition 2.17 *Let S and T be E-unitary inverse semigroups with associated McAlister triples (G, X, Y) and (G', X', Y'). Then S and T are Morita equivalent if and only if (G, X) is equivalent to (G', X').*

Proof. Let S and T be such that (G, X) is equivalent to (G', X'). Then after making appropriate identifications, we have from the classical theory of E-unitary inverse semigroups [13] that the ordered groupoid $G \ltimes X$, being the Grothendieck or semidirect product construction, is a common enlargement of the inductive groupoids $G(S)$ and $G(T)$.

Conversely, suppose that S and T are strongly Morita equivalent. Then the toposes $\mathscr{B}(S)$ and $\mathscr{B}(T)$ are equivalent. The topos explanation of the P-theorem is simply an interpretation of X, Y, and G in topos terms: X comes from the (connected) universal covering morphism of the classifying topos and so must be the same for S and T, and G is the fundamental group of the classifying topos and so again necessarily the same for S and T. An explicit description of an equivalence of (G, X) and (G', X') derived directly from and in terms of a given Morita context ought to be readily available, but we leave this excercise for the reader. $\square$

Let us say that an inverse semigroup S is *locally E-unitary* if the local submonoid eSe is E-unitary for every idempotent e. An E-unitary inverse semigroup is locally E-unitary.

Lemma 2.18 *S is locally E-unitary if and only if $L(S)$ is right-cancellative.*

Proof. Suppose that $L(S)$ is right-cancellative. Let $s = ese$ and suppose that $d \leq s$, where d is an idempotent. Then the diagram $d \leq s^*s \xrightarrow{s, s^*s} e$ in $L(S)$ commutes. Therefore, $s = s^*s$ so that s is an idempotent.

Conversely, suppose that S is locally E-unitary. Suppose that $d \xrightarrow{t} e \xrightarrow{s,r} f$ commutes in $L(S)$. Then $rs^* \in fSf$. Also $rtt^*s^* = rt(st)^* = st(st)^*$ is idempotent, and we have $rtt^*s^* \leq rs^*$. Therefore, $rs^* = b$ is an idempotent by locally E-unitary. Hence, $r = rr^*r = re = rs^*s = bs$, so that $r \leq s$. Similarly, $s \leq r$ so that $s = r$. $\square$

We take the opportunity to improve [5], Cor. 4.3.

Corollary 2.19 *$\mathscr{B}(S)$ is locally decidable* (as it is called) *if and only if S is locally E-unitary.*

Proof. This follows from Lemma 2.18 and the well-known fact that the topos of presheaves on a small category is locally decidable if and only if the category is right-cancellative [5]. □

Corollary 2.20 *If two inverse semigroups are strongly Morita equivalent and one of them is locally E-unitary, then so is the other one.*

We conclude this section by giving a direct proof of the equivalence of (i) and (iii) of Theorem 1.1 by different means.

Proposition 2.21 *Two inverse semigroups are strongly Morita equivalent if and only if their associated inductive groupoids have a bipartite ordered groupoid enlargement.*

Proof. Let $(S, T, X, \langle -, - \rangle, [-, -])$ be an equivalence bimodule. Put $I = \{1, 2\}$, and regard $I \times I$ as a groupoid in the usual way, $S' = \{1\} \times S \times \{1\}$ and $T' = \{2\} \times T \times \{2\}$ and

$$\mathcal{R} = \mathcal{R}(S, T; X) = S' \cup T' \cup (\{1\} \times X \times \{2\}) \cup (\{2\} \times X \times \{1\})$$

We shall define a partial binary operation on $\mathcal{R}$. The product of (i, α, j) and (k, β, l) will be defined if and only if $j = k$ in which case the product will be of the form (i, γ, l). Specifically, we define products as follows

- $(1, s, 1)(1, s', 1) = (1, ss', 1)$.
- $(2, t, 2)(2, t', 2) = (2, tt', 2)$.
- $(1, s, 1)(1, x, 2) = (1, sx, 2)$.
- $(1, x, 2)(2, t, 2) = (1, xt, 2)$.
- $(2, t, 2)(2, x, 1) = (2, xt^*, 1)$.
- $(2, x, 1)(1, s, 1) = (2, s^*x, 1)$.
- $(2, x, 1)(1, y, 2) = (2, [x, y], 2)$.
- $(1, x, 2)(2, y, 1) = (1, \langle x, y \rangle, 1)$.

This operation is associative whenever it is defined. To prove this, one essentially checks all possible cases of triples of elements, however the restrictions on what elements can be multiplied reduces the number of cases that need to be checked. Within this list of possibilities, associativity of multiplication in the inverse semigroups S and T combined with the 'associativity' of left, right and bimodule actions reduces the number of cases still further. One then uses the definition of an equivalence bimodule, and particularly Proposition 2.3 of [27], to check all the remaining cases. Thus $\mathcal{R}$ is a semigroupoid. Observe that $(1, x, 2)(2, x, 1) = (1, \langle x, x\rangle, 1)$ and that $(2, x, 1)(1, x, 2) = (2, [x, x], 2)$. Thus

$$(1, x, 2)(2, x, 1)(1, x, 2) = (1, \langle x, x\rangle x, 2) = (1, x, 2)$$

by (MC3). Similarly

$$(2, x, 1)(1, x, 2)(2, x, 1) = (2, [x, x], 2)(2, x, 1) = (2, x[x, x], 1) = (2, x, 1)$$

by (MC6). Thus $\mathcal{R}$ is a regular semigroupoid. But the only idempotents in $\mathcal{R}$ are those coming from S' and T' and so the idempotents commute whenever the products of two idempotents is defined. It follows that $\mathcal{R}$ is an inverse semigroupoid. Clearly $S' = S'\mathcal{R}S'$ and $T' = T'\mathcal{R}T'$ and it is easy to check that $\mathcal{R} = \mathcal{R}S'\mathcal{R}$ and $\mathcal{R} = \mathcal{R}T'\mathcal{R}$. Every inverse semigroupoid gives rise to an ordered groupoid in a way that directly generalises the way in which inverse semigroups give rise to ordered groupoids. We denote this ordered groupoid by

$$G(S, T; X)\ . \tag{4}$$

We see that $G(S, T; X)$ is an enlargement of both $G(S')$ and $G(T')$. This proves the result.

Conversely, let S and T be inductive groupoids which are ordered subgroupoids of the ordered groupoid G and where G is an enlargement of them both. Let X be the set of all the arrow of G that have domains in T and codomains in S. We define a left action of S on X by $sx = s \otimes x$, and a right action of T on X by $xt = x \otimes t$. Define $\langle x, y\rangle = x \otimes y^{-1}$ and $[x, y] = x^{-1} \otimes y$. Here $\otimes$ is the pseudoproduct in the ordered groupoid G; the pseudoproduct $a \otimes b$ is defined whenever $a^{-1}a$ and bb^{-1} have a meet in the partially ordered set of identities of G. If this meet is e then $a \otimes b = (a \mid e)(e \mid b)$. The pseudoproduct is associative whenever this makes sense. The theory behind pseudoproducts is explained in [13]. It is routine using this theory to check that in this way we have defined a Morita context. □

2.3 Actions and étale actions

In this paper, we have studied two kinds of categories of actions of an inverse semigroup S: the unitary and the étale. We denote the former by **Fact** and the latter by **Étale** since in this section the inverse semigroup S will be fixed. Recall that when the categories of unitary actions of two inverse semigroups are equivalent we say that the inverse semigroups are Morita equivalent, whereas when the categories of étale actions are equivalent the semigroups are said to be strongly Morita equivalent. We proved in Proposition 2.16 that these two notions of Morita equivalence are the same; it is the goal of this section to better understand this equivalence.

Étale may be taken as the definition of the topos $\mathscr{B}(S)$: an object is a set X equipped with a left action by S and a map $X \xrightarrow{p} E$ (the étale structure) such that $p(sx) = sp(x)s^*$ and $p(x)x = x$. Maps in **Étale** commute with the actions and with the projections to E. Thus, **Étale** is the full subcategory of $\mathbf{Fact}/E$ on those objects $X \xrightarrow{p} E$ satisfying $p(x)x = x$, whose inclusion has a right adjoint denoted V in (7).

Under the equivalence of **Étale** with presheaves on $L(S)$, the representable presheaves correspond to the étale actions $Se \longrightarrow E$, $s \mapsto ss^* = \mathbf{r}(s)$ (the reader will easily verify that this map is indeed étale), and the Yoneda embedding $L(S) \longrightarrow PSh(L(S))$ is identified with the functor $L(S) \longrightarrow$ **Etale** carrying e to $Se \longrightarrow E$. A morphism $d \xrightarrow{s} e$ goes to the map $\alpha_s : Sd \longrightarrow Se$ (over E) such that $\alpha_s(t) = ts^*$. For instance, $\alpha_s(d) = ds^* = (sd)^* = s^*$, so $s = \alpha_s(d)^*$. Any étale map $\alpha : Sd \longrightarrow Se$ is uniquely determined this way by the morphism $x = \alpha(d)^* : d \longrightarrow e$ of $L(S)$, since $\alpha(t) = \alpha(td) = t\alpha(d) = \alpha_x(t)$. We have thus proved the Yoneda Lemma, which asserts in this case that there is a functorial bijection between the étale morphisms $Sd \longrightarrow Se$ and $L(S)(d, e)$.

We proved (also in Proposition 2.16) that in **Fact** the left actions $Se = U(Se \longrightarrow E)$ are precisely the projective indecomposables up to isomorphism. The functor $e \mapsto Se$ of $C(S)$ into **Fact** is full and faithful (Prop. 2.26), so that $C(S)$ is therefore equivalent to the full subcategory of **Fact** on the projective decomposables. When this functor is restricted to the subcategory $L(S)$, the following diagram of functors commutes.

$$\begin{tikzcd} L(S) \arrow[d, "\text{Yoneda}"'] \arrow[dr] & \\ \mathbf{Etale} \arrow[r, "U"] & \mathbf{Fact} \end{tikzcd} \tag{5}$$

The functor $U(X \longrightarrow E) = X$ that forgets étale structure is faithful.

Lemma 2.22 *Let S be an inverse semigroup.*

(i) *A morphism of* **Étale** *is a monomorphism if and only if it is injective. In particular, an étale morphism $Se \longrightarrow Sf$ is injective.*

(ii) *A morphism of* **Étale** *is an epimorphism if and only if it is a surjection.*

Proof. The presheaf on $L(S)$ that corresponds to $X \xrightarrow{p} E$ is the 'fiber map' $e \mapsto p^{-1}(e)$. A morphism of étale actions is an epimorphism iff its corresponding natural transformation of presheaves is an epimorphism iff its component maps are surjections iff the given map of étale actions is a surjection. Likewise for monomorphisms and injections.

A map between representables is injective because such a map must come from a morphism of $L(S)$ (Yoneda). But every morphism of $L(S)$ is a monomorphism, and the natural transformation corresponding to a monomorphism (under Yoneda) must have injective component maps. □

Proposition 2.23 *An étale action $X \longrightarrow E$ is isomorphic to a representable one $Se \longrightarrow E$ if and only if it is projective and indecomposable. The Yoneda functor* (explained above) *gives an equivalence between $L(S)$ and the full subcategory of* **Étale** *on the projective indecomposable objects.*

Proof. This is a consequence of Prop. 2.2, although in Prop. 2.2 we use the term "connected" instead of "indecomposable." $\square$

In the proof of Prop. 2.10 we encountered the fact that $C(S)$ is equivalent to $\mathrm{Span}(L(S))$. Indeed, two functors

$$\begin{tikzcd} C(S) \arrow[r, bend left] & \mathrm{Span}(L(S)) \arrow[l, bend left] \end{tikzcd}$$

giving the equivalence are $(e,s,f) \mapsto ((f,s^*),(e,ss^*))$, and $((f,t),(e,s)) \mapsto (e,st^*,f)$. We emphasizes this equivalence in terms of projective indecomposable module and étale actions. For instance, a module map $Sf \xrightarrow{\theta} Se$ corresponds to the morphism (e,x,f) of $C(S)$, where $x = \theta(f)^*$. The functor above carries this to the span $((f,x^*),(e,xx^*))$ of morphisms in $L(S)$, which in turn corresponds to a span of étale maps

$$\begin{tikzcd} & Sxx^* \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] & \\ Sf & & Se \end{tikzcd}$$

defined as follows: $\theta_1(s) = sx$, and $\theta_2(s) = se$. Observe that θ_2 is subset inclusion since $xx^* \leq e$. Spans are composed in an obvious manner by pullback.

We return to the faithful functor U (5) that forgets étale structure.

Proposition 2.24 *U has a right adjoint R:*

$$R(X) = \coprod_E eX \longrightarrow E \ ; \ (e,x) \mapsto e \ ,$$

where

$$eX = \{x \in X \mid ex = x\} = \{ex \mid x \in X\} \cong \mathbf{Fact}(Se, X)$$

for an idempotent e.

Proof. We denote a typical member of the coproduct $\coprod_E eX$ by (e,x). The action by S that $\coprod_E eX$ carries is defined by:

$$s(e,x) = (ses^*, sx) \ .$$

Since idempotents commute in S, if e fixes x, then ses^* fixes sx: $ses^*sx = ss^*sex = sx$. The projection to E is easily to be étale. The unit of $U \dashv R$ at $X \xrightarrow{p} E$ is the étale map

$$\begin{tikzcd} X \arrow[rr, "x \mapsto (p(x),x)"] \arrow[dr, "p"'] & & \coprod_E eX \arrow[dl] \\ & E & \end{tikzcd} \quad . \tag{6}$$

The counit $UR(X) \longrightarrow X$ of $U \dashv R$ at a unitary action X is the map $\coprod_E eX \longrightarrow X$, $(e,x) \mapsto x$. $\square$

R may also be described as the equalizer:

$$
\begin{tikzcd}
R(X) \arrow[r, tail] & E \times X \arrow[r, bend left, "ex"] \arrow[r, bend right, "x"'] & X
\end{tikzcd} .
$$

Evidently, R is the composite

$$
\begin{tikzcd}
\mathbf{Fact} \arrow[d, "E^*"'] \arrow[dr, "R"] & \\
\mathbf{Fact}/E \arrow[r, "V"] & \mathbf{Etale}
\end{tikzcd} \tag{7}
$$

of two right adjoints, where $E^*(X) = E \times X \longrightarrow E$, and

$$V(X \xrightarrow{p} E) = \{x \mid p(x)x = x\} \longrightarrow E \ ,$$

which is right adjoint to inclusion. Because idempotents commute in S, the action of S in X restricts to $\{x \mid p(x)x = x\}$:

$$p(sx)sx = sp(x)s^*sx = ss^*sp(x)x = sx \ .$$

R is defined for any S-action, not just the unitary ones.

Lemma 2.25 *An S-action X is unitary ($SX = X$) if and only if the counit of $U \dashv R$ at X is an epimorphism. In particular, the right adjoint R is faithful when restricted to unitary actions.*

Proof. We have seen that the unitary condition $SX = X$ is equivalent to the condition

$$\forall x \in X \exists e \in E, ex = x \ ,$$

which holds if and only if $\coprod_E eX \longrightarrow X$ is onto. A simple diagram chase shows that if the counit maps of any adjoint pair are epimorphisms, then the right adjoint is faithful. $\square$

Proposition 2.26 *A unitary action is projective and indecomposable if and only if it is isomorphic to Se, for some idempotent e. The set $\{Se \mid e \in E\}$ of unitary actions separates maps in* **Fact**. *The functor $C(S) \longrightarrow$* **Fact**, *$e \mapsto Se$, is full and faithful.*

Proof. The first statement is already proved in Proposition 2.16. The representable étale actions $Se \longrightarrow E$ separate maps in the topos **Étale**. Therefore, the Se do the same in **Fact** because U has a faithful right adjoint. The functor $e \mapsto Se$ is full and faithful because an action preserving map $\alpha : Sd \longrightarrow Se$ is uniquely determined by $\alpha(d)$: this calculation is the same as for $L(S)$ except that we can only conclude $\alpha(d) \in dSe$, so that $x = \alpha(d)^* \in C(S)(d, e)$, and $\alpha(t) = tx^* = \alpha_x(t)$. $\square$

The theory of monads is adequately explained in the literature [2], but briefly a monad in a category is an endofunctor M of the category equipped with an associative multiplication $M^2 \longrightarrow M$ and a unit $\mathrm{id} \longrightarrow M$. The (Eilenberg-Moore) algebras for a monad form a category that maps to the given category by forgetting an algebra's M structure. A functor is said to be monadic if it is equivalent to such a forgetful functor from the category of algebras for a monad. We will use the following well-known sufficient conditions: if a functor has a left adjoint, reflects isomorphisms, coequalizers exist and the functor preserves them, then it is monadic. A comonad is a monad in the opposite category. All topos terminology and facts that we use are part of the basic theory [19].

An example of a monad (ultimately explained in Theorem 2.32) is the one in **Étale** associated with the adjoint pair $U \dashv R$: its endofunctor $M = RU$ carries an étale action $X \xrightarrow{p} E$ to $\coprod_E eX \longrightarrow E$. We shall show that the category of Eilenberg-Moore algebras for this monad is equivalent to $PSh(C(S))$, identifying the forgetful functor with the inverse image functor I^*. We begin by explaining this functor.

Restriction of presheaves along the inclusion functor $I : L(S) \longrightarrow C(S)$ is denoted

$$I^* : PSh(C(S)) \longrightarrow PSh(L(S)) \ .$$

Under the equivalence of $PSh(L(S))$ and **Étale**, if P is a presheaf on $C(S)$, then $I^*(P)$ is the étale action

$$\coprod_E P(e) \longrightarrow E \ ,$$

where $s(e, x) = (ses^*, P(es^*)(x))$. I^* is the inverse image functor of a geometric morphism of toposes

$$I^* \dashv I_* : \mathbf{Etale} \longrightarrow PSh(C(S)) \ .$$

The right adjoint I_* is given by 'taking sections,' whose explicit description we omit. The above geometric morphism is termed a surjection because its inverse image functor I^* reflects isomorphisms. Thus, in a geometric sense, $C(S)$ is a quotient of $L(S)$. By our sufficient criteria (previous paragraph), I^* is comonadic by a finite limit preserving comonad. (A well-known fact from topos theory is that a functor is equivalent to the inverse image functor of a surjective geometric morphism iff it is comonadic by a finite limit preserving comonad.)

I^* also has a left adjoint $I_!$ (calculated in Lemma 2.29): by definition, if $X \xrightarrow{p} E$ is étale, and e is an idempotent, then

$$I_!(p)(e) = \varinjlim_{\mathbb{X}} \ (\ x \mapsto C(S)(e, p(x)) \ ) \ , \tag{8}$$

where $\mathbb{X}$ is the category with objects X, and morphisms $x \xrightarrow{s} y$, such that $p(x) \xrightarrow{s} p(y)$ is a morphism of $L(S)$ satisfying $s^*y = x$. I^* is also monadic (again by the same criteria) by a monad that preserves all colimits, explained further in Theorem 2.32.

Consider the following commuting diagrams of functors.

$$
\begin{tikzcd}
 & C(S) \arrow[dl, "e\mapsto Se"'] \arrow[dr, "\text{Yoneda}"] & \\
\mathbf{Fact} \arrow[rr, "\Sigma^*"] \arrow[d, "E^*"'] \arrow[drr, "R"] & & PSh(C(S)) \arrow[d, "I^*"] \\
\mathbf{Fact}/E \arrow[rr, "V"] & & \mathbf{Etale}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C(S) \arrow[dl, "e\mapsto Se"'] \arrow[dr, "\text{Yoneda}"] & \\
\mathbf{Fact} & & PSh(C(S)) \arrow[ll, "\Sigma"'] \\
\mathbf{Fact}/E \arrow[u] & & \mathbf{Etale} \arrow[ll] \arrow[u, "I_!"'] \arrow[ull, "U"']
\end{tikzcd}
$$

We have $\Sigma^*(X)(e) = eX \cong \mathbf{Fact}(Se, X)$. Σ^* is faithful since R is. I^* and E^* are also faithful. Of course the corresponding diagram of left adjoints commutes (above, right). Only the left adjoint Σ of Σ^* deserves more explanation; its existence depends on the existence of colimits in **Fact**.

Lemma 2.27 ***Fact*** *has all small colimits, created in the category of sets. All small limits also exist in* **Fact** (but they are not created in sets).

Proof. A small coproduct $\coprod_A X_a$ of unitary actions is an S-set in the obvious way, which is easily seen to be unitary. The set coequalizer

$$
\begin{tikzcd}
X \arrow[r, bend left] \arrow[r, bend right] & Y \arrow[r, two heads] & Z
\end{tikzcd}
$$

of two S-maps also has an action by S in an obvious way (just use the universal property of Z), which again is unitary.

Limits are slightly more complicated than colimits. For example, a finite product $X \times Y$ has underlying set $\{(x, y) \mid \exists e \in E, ex = x, ey = y\}$. Arbitrary products follow the same pattern. Equalizers are, like coequalizers, created in sets. $\square$

Proposition 2.28 Σ^* *has a left adjoint* Σ *given by* (colimit extension)*:*

$$
\Sigma(P) = \varinjlim_{\mathbb{P}} \left(\mathbb{P} \longrightarrow C(S) \xrightarrow{e\mapsto Se} \mathbf{Fact} \right) ,
$$

where $\mathbb{P} \longrightarrow C(S)$ *is the discrete fibration corresponding to a presheaf* P*. We have* $\Sigma I_! \cong U$*, and* Σ *commutes with Yoneda.*

Lemma 2.29 *We have* $I_! \cong \Sigma^* U$*: for any étale* $X \xrightarrow{p} E$ *and any* $e \in E$*,* $I_!(p)(e) \cong eX$*.*

Proof. We argue this fact by direct calculation. Let $X \xrightarrow{p} E$ be an étale action. We claim that the unit map $I_!(p) \longrightarrow \Sigma^*\Sigma I_!(p) \cong \Sigma^* U(p)$ is a natural isomorphism (of presheaves on $C(S)$). For any $e \in E$, the component map at e of this unit is

$$
I_!(p)(e) = \coprod_{x\in X} C(S)(e, p(x))/\sim\ \longrightarrow eX \text{ ; equiv. class of } (x, e \xrightarrow{s} p(x)) \mapsto s^*x ,
$$

where the left-hand side is the colimit (8), calculated as a coproduct factored by an equivalence relation. This map has inverse $x \mapsto (x, e \xrightarrow{p(x)} p(x))$, where $e \xrightarrow{p(x)} p(x)$ is the inequality $p(x) \leq e$ understood as a map in $C(S)$, which holds because $ex = x$, hence $ep(x) = p(x)$. Furthermore, given any $(x, e \xrightarrow{s} p(x))$, the map $s^*x \xrightarrow{s} x$ in the category $\mathbb{X}$ (from 8) witnesses that $(x, e \xrightarrow{s} p(x))$ is equivalent in the colimit to $(s^*x, e \xrightarrow{p(s^*x)} p(s^*x))$, noting $p(s^*x) = s^*p(x)s = s^*s \leq e$. ◻

Proposition 2.30 *U reflects isomorphisms, U has a right adjoint, and* **Étale** *has all equalizers and U preserves them. U is therefore comonadic.*

Proof. U preserves equalizers because they are created in both categories by their underlying sets. ◻

Proposition 2.31 *$I_!$ reflects isomorphisms, $I_!$ has a right adjoint, and* **Étale** *has all equalizers and $I_!$ preserves them. $I_!$ is therefore comonadic.*

Proof. $I_!$ reflects isomorphisms because U does and $\Sigma I_! \cong U$. By Lemma 2.29, $I_!$ preserves any limit U does, such as an equalizer, because Σ^* preserves all limits. ◻

Summing up, I^*, $I_!$ and U are all comonadic, but we wish to emphasize the following fact.

Theorem 2.32 *The monads in* **Étale** *associated with the adjoint pairs $U \dashv R$ and $I_! \dashv I^*$ coincide. The endofunctor of this monad carries $X \xrightarrow{p} E$ to $\coprod_E eX \longrightarrow E$, as in (6). It preserves all colimits. The category of Eilenberg-Moore algebras for this monad is equivalent to $PSh(C(S))$, and the comparison functor associated with $U \dashv R$ is Σ^*.*

Proof. We have already mentioned that I^* is monadic. By Lemma 2.29, we have $I^*I_! \cong I^*\Sigma^*U \cong RU$. ◻

3 Atlases

This section has a different goal from the rest of the paper and depends only on the definition of equivalence bimodule.

Inverse semigroups originated in differential geometry as pseudogroups of transformations. One of the founders of the field, V. V. Wagner[5], was one of the few who continued to seek inspiration from this source. In differential geometry, pseudogroups are usually not studied on their own but in combination with the notion of an atlas. Just as Wagner defined inverse semigroups to be the

[5]This name is usually transliterated as 'Vagner' in the literature, but we understand that 'Wagner' was his preferred transliteration.

algebraic versions of pseudogroups, so too he defined a class of structures, called *generalized heaps*, to be the algebraic versions of atlases. Unlike semigroups, which are equipped with a binary operation, heaps are defined in terms of a ternary operation [31, 32, 33]. A convenient place to find an axiomatization is Boris Schein's paper [26]. This work by Wagner and his school did not become well known outside of Eastern Europe for a variety of reasons: mathematically, heaps are unusual in being based on a ternary operation rather than a binary one; more substantively, the theory of heaps appears to be tangential to the main theory of inverse semigroups, perhaps nothing more than a generalization for generalization's sake; finally, the theory was developed at a time when political tensions between East and West impeded the dissemination of ideas. Whatever the reasons, although Wagner is one of the founding fathers of the field the details of this particular aspect of his work have been largely forgotten.

It was while the authors were working on this paper, that they began to sense that the notion of equivalence bimodule might be connected in some way to generalized heaps. Our first calculation, which is now Proposition 3.1 below, showed that from an equivalence bimodule we could construct a generalized heap, a structure satisfying the axioms in Schein's paper. Having gone in one direction, it was natural to wonder if we could go back. It turned out that we could: we proved there is a bijective correspondence between equivalence bimodules and generalized heaps. In other words, equivalence bimodules are to atlases as inverse semigroups are to pseudogroups. Far from being tangential to inverse semigroup theory or a generalization for generalization's sake, generalized heaps are the mathematical devices which witness a Morita equivalence between two inverse semigroups. Because the term 'generalized heap' does not sound good in English, we have preferred to call them atlases.

What follows is an exposition of the exact correspondence between equivalence bimodules and atlases. We found it convenient to prove this correspondence in terms of Kock's notion of a 'pregroupoid' [9, 10] which is a categorical formulation of the differential geometric notion of atlas and which enables one to envisage what is going on quite clearly.

We should stress that we make no claim to originality in what follows: these are Wagner's ideas in modern dress. We hope that it will provide a new lease of life for Wagner's pioneering work.

Motivation

Given two spaces X and Y, a *concrete atlas* A from X to Y is a set of partial bijections such that the union of their domains is X and the union of their images is Y. The set $T = A^{-1}A$ is a collection of partial bijections defined on X, and $S = AA^{-1}$ is a collection of partial bijections defined on Y. For example, if X and $Y = \mathbb{R}^n$ are topological spaces, A consists of homeomorphisms, and S is the pseudogroup of all smooth maps defined between open subsets of Y then the atlas A defines the structure of a differential manifold on X. Other such local structures can be defined in a similar way.

This concrete notion of an atlas can be made algebraic. Observe that if $x, y, z \in A$, an atlas, then also $xy^{-1}z \in A$, as long as A is sufficiently large.

Define now $\{xyz\} = xy^{-1}z$. We may therefore regard the set A as an algebraic structure equipped with the ternary operation $(x, y, z) \mapsto \{xyz\}$. One may seek to axiomatise the properties of this structure and relate it back, in the spirit of Cayley's theorem, to the original concrete notion of an atlas. This idea formed the basis of a number of papers by Wagner and his students, but whereas Wagner's ideas on inverse semigroups entered the mainstream, his work on abstract atlases has been largely neglected. In this section, we shall repair that neglect.

The definition of atlas we shall use runs as follows. An *atlas* is a set X equipped with a ternary operation $(x, y, z) \mapsto \{xyz\}$ that satisfies the following axioms; they are not independent and we refer the reader to [26] for further information.

(A1) $\{xxx\} = x$.

(A2) $\{\{x_1x_2x_3\}x_4x_5\} = \{x_1\{x_4x_3x_2\}x_5\} = \{x_1x_2\{x_3x_4x_5\}\}$.

(A3) $\{xyx\} = x$ and $\{yxy\} = y$ implies that $x = y$.

(A4) $\{xx\{yyz\}\} = \{yy\{xxz\}\}$.

(A5) $\{\{zxx\}yy\} = \{\{zyy\}xx\}$.

From equivalence bimodules to atlases

We begin by proving the easy direction.

Proposition 3.1 *Let $(S, T, X, \langle -, - \rangle, [-, -])$ be an equivalence bimodule. On the set X define a ternary operation*

$$\{xyz\} = \langle x, y \rangle z.$$

Then $(X, \{\})$ is an atlas.

Proof. (A1) holds. We have that $\{xxx\} = \langle x, x \rangle x = x$ by (MC3).

(A2) holds. By definition, we have that

$$\{\{x_1x_2x_3\}x_4x_5\} = \langle \langle x_1, x_2 \rangle x_3, x_4 \rangle x_5 = \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle x_5$$

by (MC1);

$$\{x_1, \{x_4x_3x_2\}x_5\} = \langle x_1, \langle x_4, x_3 \rangle x_2 \rangle x_5 = \langle x_1, x_2 \rangle \langle x_4, x_3 \rangle^* x_5 = \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle x_5$$

by (MC1) and (MC2);

$$\{x_1x_2\{x_3x_4x_5\}\} = \langle x_1, x_2 \rangle \langle x_3, x_4 \rangle x_5$$

where we have used the associativity of the action.

(A3) holds. Suppose that $\{xyx\} = x$ and $\{yxy\} = y$. Then $\langle x, y \rangle x = x$ and $\langle y, x \rangle y = y$. Observe that

$$\langle x, x \rangle = \langle \langle x, y \rangle x, x \rangle = \langle x, y \rangle \langle x, x \rangle.$$

Thus $\langle x, x\rangle \leq \langle x, y\rangle$ since $\langle x, x\rangle$ is an idempotent by Proposition 2.3(7) of [27]. It follows that $\langle x, x\rangle \leq \langle y, x\rangle$ also. Now

$$x = \langle x, x\rangle x = \langle x, x\rangle\langle y, x\rangle x = \langle x, x\rangle y[x, x]$$

by (MC7). However, this implies that $x \leq y$ using Proposition 3.6 of [27] and the order defined in Proposition 3.2. A dual argument shows that $y \leq x$ and so $x = y$, as required.

(A4) holds. We have that

$$\{xx\{yyz\}\} = \langle x, x\rangle\langle y, y\rangle z$$

whereas

$$\{yy\{xxz\}\} = \langle y, y\rangle\langle x, x\rangle z.$$

These two elements are equal because $\langle x, x\rangle$ and $\langle y, y\rangle$ are idempotents and so commute.

(A5) holds. We have that

$$\{\{zxx\}yy\} = z[x, x][y, y]$$

whereas

$$\{\{zyy\}xx\} = z[y, y][x, x]$$

using (MC1) and (MC7). These two elements are equal because $[x, x]$ and $[y, y]$ are idempotents and so commute. □

From atlases to equivalence bimodules

This direction is more complex and will be carried out in a series of steps. Our first goal is to show that from each atlas we can construct a pregroupoid in the sense of Kock [9, 10]. Here is the definition. Let X be a set equipped with a *partially defined* ternary operation $\{\}$, and surjections $p\colon X \longrightarrow E$ and $q\colon X \longrightarrow F$ such that $\{xyz\}$ is defined if and only if $q(x) = q(y)$ and $p(y) = p(z)$ and such that the following axioms hold:

(PG1) $p(\{xyz\}) = p(x)$ and $q(\{xyz\}) = q(z)$.

(PG2) $\{xxz\} = z$ and $\{yxx\} = y$.

(PG3) $\{vy\{yxz\}\} = \{vxz\}$ and $\{\{yxz\}zw\} = \{yxw\}$.

Then we call $(X, \{\}, p, q)$ a *pregroupoid*.

Recall that a *band* is a semigroup in which every element is an idempotent. A band is *left normal* if it satisfies the law $xyz = xzy$, and it is *right normal* if it satisfies the law $xyz = yxz$. A commutative band is just a *semilattice*.

Lemma 3.2 *Let X be an atlas.*

(i) *Define the binary operation* $\circ$ *on* X *by* $x \circ y = \{xxy\}$. *Then* $(X, \circ)$ *is a right normal band. The minimum semilattice congruence on* X° *is given by the* $\mathcal{R}$*-relation. Put* $E = X^{\circ}/\mathcal{R}$ *and denote the natural map from* X *to* E *by* p.

(ii) *Define the binary operation* $\bullet$ *on* X *by* $x \bullet y = \{xyy\}$. *Then* $(X, \bullet)$ *is a left normal band. The minimum semilattice congruence on* X° *is given by the* $\mathcal{L}$*-relation. Put* $F = X^{\bullet}/\mathcal{L}$ *and denote the natural map from* X *to* F *by* q.

Proof. We prove (i); the proof of (ii) follows by symmetry. The fact that every element is an idempotent follows by (A1). We prove associativity. We have that $(x \circ y) \circ z = \{\{xxy\}\{xxy\}z\}$ and $x \circ (y \circ z) = \{xx\{yyz\}\}$. But

$$(x \circ y) \circ z = \{\{\{xxy\}yx\}xz\}$$

by (A2). By (A2) and (A4) we have that

$$\{\{\{xxy\}yx\}xz\} = \{\{xx\{yyx\}\}xz\} = \{\{yy\{xxx\}\}xz\}$$

but by (A1) this is equal to

$$\{\{yyx\}xz\}.$$

Finally we use (A2) and (A4) to get

$$\{yy\{xxz\}\} = \{xx\{yyz\}\},$$

as required. We have thefore proved that we have a band. To show that we have a right normal band observe that

$$x \circ y \circ z = \{xx\{yyz\}\} = \{yy\{xxz\}\} = y \circ x \circ z$$

using (A4). □

By the above

$$p(x) = p(y) \Leftrightarrow x = y \circ x \text{ and } y = x \circ y$$

and

$$q(x) = q(y) \Leftrightarrow x = x \bullet y \text{ and } y = y \bullet x.$$

The elements of the atlas X should be regarded as arrows

$$p(x) \xleftarrow{x} q(x).$$

If X is an atlas, then we define the *restricted product* on X to be the ternary operation restricted to those triples (x, y, z) where $q(x) = q(y)$ and $p(y) = p(z)$

Proposition 3.3 *Let* X *be an atlas and let* $p\colon X \longrightarrow E$ *and* $q\colon X \longrightarrow F$ *be defined as in Lemma 3.2. Then with respect to the restricted product* $(X, \{\}, p, q)$ *is a pregroupoid.*

Proof. (PG1) Suppose that $q(x) = q(y)$ and $p(y) = p(z)$. Thus $x = \{xyy\}$, $y = \{yxx\}$ and $y = \{zzy\}$ and $z = \{yyz\}$. We have that

$$\{xx\{xyz\}\} = \{\{xxx\}yz\} = \{xyz\}$$

and

$$\{\{xyz\}\{xyz\}x\} = \{\{xyz\}z\{yxx\}\} = \{\{xyz\}zy\} = \{xy\{zzy\}\} = \{xyy\} = x.$$

Thus $p(\{xyz\}) = p(x)$.

We also have that

$$\{\{xyz\}zz\} = \{xy\{zzz\}\} = \{xyz\}$$

and

$$\{z\{xyz\}\{xyz\}\} = \{\{zzy\}x\{xyz\}\} = \{yx\{xyz\}\} = \{\{yxx\}yz\} = \{yyz\} = z.$$

Thus $q(\{xyz\}) = q(z)$.

(PG2) Both of these follow immediately from the definitions

(PG3) We have that

$$\{vy\{yxx\}\} = \{\{vyy\}xz\} = \{vxz\}.$$

Similarly, we have that

$$\{\{yxz\}zw\} = \{yx\{zzw\}\} = \{yxw\}.$$

□

We now follow Kock [9, 10] and use this pregroupoid structure to construct two groupoids that we denote by XX^{-1} and $X^{-1}X$. We define $X^{-1}X$; the definition of XX^{-1} is obtained dually. Let

$$XpX = \{(x, y) \in X \times X \colon\ p(x) = p(y)\}.$$

We identify a pair of elements (x, y) and (u, v) of this set if and only if $q(x) = q(u)$ and $q(y) = q(v)$ and $y = \{xuv\}$. This is an equivalence relation. We denote the equivalence class containing the pair (x, y) by $x^{-1}y$ and the set of equivalence classes by $X^{-1}X$. The element $x^{-1}y$ should be regarded as an arrow from $q(y)$ to $q(x)$. Define a partial binary operation on $X^{-1}X$ by

$$x^{-1}y \cdot u^{-1}v = x^{-1}\{yuv\}$$

if and only if $q(y) = q(u)$. With respect to this operation $X^{-1}X$ is a groupoid. We check that the identities of the groupoid $X^{-1}X$ can be identified with the semilattice F. Identities have the form $x^{-1}x$. We map $x^{-1}x$ to $q(x)$. This is well-defined and surjective. Suppose that $x^{-1}x$ and $y^{-1}y$ are such that $q(x) = q(y)$. From our results above we have that $x = \{xyy\}$ and so in fact $x^{-1}x = y^{-1}y$.

Lemma 3.4 *With the above notation, make the following definitions.*

(i) *Define* $\langle -,-\rangle \colon X \times X \longrightarrow XX^{-1}$ *by*

$$\langle x,y\rangle = \{xyy\}\{yxx\}^{-1}$$

a surjective map.

(ii) *Define* $[-,-] \colon X \times X \longrightarrow X^{-1}X$ *by*

$$[x,y] = \{yyx\}^{-1}\{xxy\}$$

a surjective map.

(iii) *Define* $XX^{-1} \times X \longrightarrow X$ *by* $xy^{-1} \cdot z = \{xyz\}$.

(iv) *Define* $X \times X^{-1}X \longrightarrow X$ *by* $x \cdot y^{-1}z = \{xyz\}$.

(v) *Axioms (MC2),(MC3),(MC5),(MC6) and (MC7) hold.*

Proof. (i). We prove that $q(\{xyy\}) = q(\{yxx\})$. We calculate one part of the proof

$$\{\{xyy\}\{yxx\}\{yxx\}\} = \{\{\{xyy\}xx\}y\{yxx\}\} = \{\{xyy\}y\{yxx\}\}$$

this is equal to

$$\{x\{yyy\}\{yxx\}\} = \{xy\{yxx\}\} = \{\{xyy\}xx\} = \{xxx\}yy\} = \{xyy\}$$

It remains to show that this map is surjective. Let $xy^{-1} \in XX^{-1}$. Then $q(x) = q(y)$. Thus $x = \{xyy\}$ and $y = \{yxx\}$. It follows that $\langle x,y\rangle = \{xyy\}\{yxx\}^{-1} = xy^{-1}$, as required.

(ii). We prove that $p(\{yyx\}) = p(\{xxy\})$. We calculate one part of the proof.

$$\{\{yyx\}\{yyx\}\{xxy\} = \{\{\{yyx\}x\}\{xxy\}\} = \{\{\{yyx\}xy\}y\{xxy\}\}$$

this is equal to

$$\{\{yyx\}x\{yy\{xxy\}\}\} = \{\{yyx\}x\{xxy\}\} = \{yy\{xxy\}\} = \{xxy\}.$$

It remains to show that this map is surjective. Let $x^{-1}y \in X^{-1}X$. Then by assumption $p(x) = p(y)$. Thus $x = \{yyx\}$ and $y = \{xxy\}$. It follows that $[x,y] = \{yyx\}^{-1}\{xxy\} = x^{-1}y$, as required.

(iii) We have to show that this operation is well-defined; this is similar to the proof of (iv) below.

(iv). We have to show that this operation is well-defined. Let $y^{-1}z = u^{-1}v$. We have that

$$x \cdot y^{-1}z = \{xyz\}$$

and

$$x \cdot u^{-1}v = \{xuv\}.$$

By assumption $z = \{yuv\}$. Thus

$$\{xyz\} = \{xy\{yuv\}\} = \{x\{uyy\}v\}.$$

But $q(y) = q(u)$ and so $u = \{uyy\}$. Thus $\{xyz\} = \{xuv\}$, as required.

(v). (M2) By construction $\langle x, y\rangle$ and $\langle y, x\rangle$ are groupoid inverses of each other.

(M3) By definition $\langle x, x\rangle x = \{xxx\} = x$.

(M5) By construction $[x, y]$ and $[y, x]$ are groupoid inverses of each other.

(M6) By definition $x[x, x] = \{xxx\} = x$.

(M7) By definition

$$\langle x, y\rangle z = \{\{xyy\}\{yxx\}z\}$$

which quickly simplifies to $\{xyz\}$. By definition

$$x[y, z] = \{x\{zzy\}\{yyz\}\}$$

which quickly simplifies to $\{xyz\}$. □

We next show that $X^{-1}X$ and XX^{-1} are in fact inverse semigroups by using the theory of ordered groupoids [13].

Proposition 3.5

(i) *Define a relation $\leq$ on $X^{-1}X$ by*

$$x^{-1}y \leq u^{-1}v \Leftrightarrow x = x \bullet u \textit{ and } y = \{xuv\}.$$

This relation is well-defined and a partial order. With respect to this order $X^{-1}X$ is an inductive groupoid with pseudoproduct given by

$$x^{-1}y \otimes u^{-1}v = \{\{yuu\}yx\}^{-1}\{yuv\}.$$

(ii) *Define a relation $\leq$ on XX^{-1} by*

$$xy^{-1} \leq uv^{-1} \Leftrightarrow y = y \circ v \textit{ and } x = \{uvy\}.$$

This relation is well-defined and a partial order. With respect to this order XX^{-1} is an inductive groupoid with pseudoproduct given by

$$xy^{-1} \otimes uv^{-1} = \{xyu\}\{vu\{yyu\}\}^{-1}.$$

Proof. We prove (i); the proof of (ii) is obtained dually.

We show first that the relation is well-defined. Suppose that $x_1^{-1}y_1 = x^{-1}y$ and $u_1^{-1}v_1 = u^{-1}v$. We have that $q(x_1) = q(x)$ and $q(u_1) = q(u)$. Thus in $X^\bullet$

we have that $x_1 \mathcal{L} x$ and $u_1 \mathcal{L} u$. It follows that $X \bullet x_1 \subseteq X \bullet u_1$ and so $x_1 = x_1 \bullet u_1$. We now calculate

$$\{x_1u_1v_1\} = \{x_1u_1\{u_1uv\}\} = \{\{x_1u_1u_1\}uv\} = \{x_1uv\} = \{\{x_1xx\}uv\}$$

which is equal to

$$\{x_1x\{xuv\}\} = \{x_1xy\} = y_1.$$

Next we check that this is a partial order. Let $x^{-1}y \in X^{-1}X$. Then $x = x \bullet x$ and $y = \{xxy\}$ since $p(x) = p(y)$. Thus $x^{-1}y \leq x^{-1}y$.

Suppose that $x^{-1}y \leq u^{-1}v$ and $u^{-1}v \leq x^{-1}y$. We have that $q(x) = q(u)$ and $y = \{xuv\}$. We prove that $q(y) = q(v)$. We calculate

$$\{yvv\} = \{\{xuv\}vv\} = \{xu\{vvv\}\} = \{xuv\} = y$$

and

$$\{vyy\} = \{\{uxy\}yy\} = \{ux\{yyy\}\} = \{uxy\} = v.$$

Finally, suppose that $x^{-1}y \leq u^{-1}v \leq w^{-1}z$. We have that

$$X \bullet x = X \bullet x \bullet u \subseteq X \bullet u = X \bullet u \bullet w \subseteq X \bullet w.$$

Thus $x = x \bullet w$. We now calculate

$$y = \{xuv\} = \{xu\{uwz\}\} = \{\{xuu\}wz\} = \{xwz\}.$$

Observe that on the set of identities $x^{-1}x \leq u^{-1}u$ if and only if $x = x \bullet u$. Thus the order we have defined induces a semilattice ordering on the set of identities. We shall now prove that with respect to this order $X^{-1}X$ is an ordered groupoid and so by the observation above an inductive groupoid.

Suppose that $x^{-1}y \leq u^{-1}v$. We prove that $y^{-1}x \leq v^{-1}u$. We have that

$$y \bullet v = \{yvv\} = \{\{xuv\}vv\} = \{xu\{vvv\}\} = \{xuv\} = y,$$

and

$$\{yvu\} = \{\{xuv\}vu\} = \{xuu\} = x.$$

Suppose that $x_1^{-1}y_1 \leq u_1^{-1}v_1$ and $x_2^{-1}y_2 \leq u_2^{-1}v_2$. We prove that $x_1^{-1}y_1x_2^{-1}y_2 \leq u_1^{-1}v_1u_2^{-1}v_2$ where the products are groupoid products. We have that $x_1 = x_1 \bullet u_1$ and $y_1 = \{x_1u_1v_1\}$ and $x_2 = x_2 \bullet u_2$ and $y_2 = \{x_2u_2v_2\}$. We shall prove that $x_1^{-1}\{y_1x_2y_2\} \leq u_1^{-1}\{v_1u_2v_2\}$. We have $x_1 = x_1 \bullet u_1$. Now $q(v_1) = q(u_2)$ and so $v_1 = \{v_1u_2u_2\}$. Thus

$$\{y_1u_2u_2\} = \{\{x_1u_1v_1\}u_2u_2\} = \{x_1u_1\{v_1u_2u_2\}\} = \{x_1u_1v_1\} = y_1.$$

We now calculate

$$\{x_1u_1\{v_1u_2v_2\}\} = \{\{x_1u_1v_1\}u_2v_2\} = \{y_1u_2v_2\} = \{y_1u_2\{u_2x_2y_2\}\} = \{\{y_1u_2u_2\}x_2y_2\}$$

which is equal to $\{y_1x_2y_2\}$ as required.

We now construct corestrictions, which is sufficient to prove that $X^{-1}X$ is an ordered groupoid [13]. Let $z^{-1}z \leq x^{-1}x$. Define

$$(z^{-1}z \mid x^{-1}y) = z^{-1}\{zxy\}.$$

We prove that this is a corestriction.

It is easy to check that $z^{-1}\{zxy\} \leq x^{-1}y$. Let $u^{-1}v \leq x^{-1}y$ where $u^{-1}u = z^{-1}z$. Then

$$\{uz\{zxy\}\} = \{\{uzz\}xy\} = \{uxy\} = v.$$

We have therefore proved uniqueness.

Using the restriction and corestriction operations we can now calculate the pseudoproduct. We get

$$x^{-1}y \otimes u^{-1}v = \{\{yuu\}yx\}^{-1}\{yuu\} \cdot \{yuu\}^{-1}\{\{yu\}uv\}.$$

This quickly simplifies to

$$\{\{yuu\}yx\}^{-1}\{yuv\}.$$

□

From now on we shall denote the pseudoproducts by concatenation.

Proposition 3.6 *X is a $(XX^{-1}, X^{-1}X)$-bimodule, and MC1) and (MC4) hold.*

Proof. We show that X is a left XX^{-1}-module. We have that

$$(xy^{-1}uv^{-1}) \cdot z = \{\{xyu\}\{vu\{yyu\}\}z\}$$

whereas

$$xy^{-1} \cdot (uv^{-1} \cdot z) = \{xy\{uvz\}\}.$$

But

$$\{\{xyu\}\{vu\{yyu\}\}z\} = \{\{xy\{u\{yyu\}u\}\}vz\} = \{\{\{xy\{yyu\}\}vz\}$$

which is equal to

$$\{\{xyy\}yu\} = \{\{xyu\}vz\} = \{xy\{uvz\}\}.$$

Thus X is a left XX^{-1}-module. A dual argument shows that X is a right $X^{-1}X$-module.

To show that it is a bimodule we calculate $(xy^{-1} \cdot z) \cdot u^{-1}v$ and $xy^{-1} \cdot (z \cdot u^{-1}v)$. But these are equal by (A2).

(MC1) holds. We calculate $\langle xy^{-1} \cdot u, v \rangle$ and $xy^{-1}\langle u, v \rangle$. Now

$$\langle xy^{-1} \cdot u, v \rangle = \langle \{xyu\}, v \rangle = \{\{xyu\}vv\}\{v\{xyu\}\{xyu\}\}^{-1}.$$

At this point we introduce something and apply (A5); the reason for doing this will become clear

$$\{v\{xyu\}\{xyu\}\} = \{\{vvv\}\{xyu\}\{xyu\}\} = \{\{v\{xyu\}\{xyu\}\}vv\}.$$

But

$$\{v\{xyu\}\{xyu\}\} = \{vu\{yx\{xyu\}\}\} = \{vu\{\{yxx\}yu\}\} = \{vu\{yyu\}\} = \{\{vuy\}yu\}.$$

Thus we have shown that

$$\langle xy^{-1} \cdot u, v\rangle = \{\{xyu\}vv\}\{\{\{vuy\}yu\}vv\}^{-1}.$$

But

$$xy^{-1}\langle u, v\rangle = xy^{-1} \cdot \{uvv\}\{vuu\}^{-1} = \{xy\{uvv\}\}\{\{vuu\}\{uvv\}\{yy\{uvv\}\}\}^{-1}.$$

We now simplify the second component

$$\{\{vuu\}\{uvv\}\{yy\{uvv\}\}\} = \{\{\{vuu\}vv\}u\{yy\{uvv\}\}\} = \{\{vuu\}u\{yy\{uvv\}\}\}$$

this is equal to

$$\{\{vu\{yy\{uu\{uvv\}\}\}\} = \{vu\{yy\{\{uuu\}vv\}\}\} = \{vu\{yy\{uvv\}\}\} = \{\{vuy\}y\{uvv\}\}$$

which is just

$$\{\{\{vuy\}yu\}vv\}.$$

We have shown that

$$xy^{-1}\langle u, v\rangle = \{\{xyu\}vv\}\{\{\{vuy\}yu\}vv\}^{-1}.$$

We have therefore shown that (MC1) holds.

The fact that (MC4) holds follows by a dual argument. □

Combining the above results we get the following.

Theorem 3.7 *With each atlas $(X, \{\})$ we can associate an equivalence bimodule*

$$(XX^{-1}, X^{-1}X, X, \langle -, -\rangle, [-, -]).$$

Back and forth

It remains to show that the two constructions we have described are essentially inverses of each other. If we start with an atlas, construct the corresponding equivalence bimodule, and then construct the atlas from that then we arrive back where we started. Thus we need only prove the following.

Proposition 3.8 *Let $(S, T, X, \langle -, -\rangle, [-, -])$ be an equivalence bimodule and let $(XX^{-1}, X^{-1}X, X, \langle -, -\rangle_1, [-, -]_1)$. be the equivalence bimodule that arises after successively applying our two constructions. Then the two equivalence bimodules are isomorphic.*

Proof. We show first that $X^{-1}X$ is isomorphic to T.

Define a map $X^{-1}X \longrightarrow S$ by $x^{-1}y \mapsto [x,y]$. This map is well-defined, for suppose that $x^{-1}y = u^{-1}y$. Then $y = \{xuv\} = \langle x,u\rangle v$. We calculate

$$[x,y] = [x, \langle x,u\rangle v] = [\langle u,x\rangle x, v] = [u[x,x],v] = [u,v]$$

since $q(x) = q(u)$ and so $x = \{xuu\} = \langle x,u\rangle u = x[u,u]$. Next we show that this map is injective. Suppose that $[x,y] = [u,v]$. Then

$$y = \langle y,y\rangle y = \langle x,x\rangle y = x[x,y] = x[u,v] = \langle x,u\rangle v.$$

Thus $y = \{xuv\}$. Next we show that $q(x) = q(u)$ and $q(y) = q(v)$. We have that

$$x = \langle x,x\rangle x = \langle y,y\rangle x = y[y,x] = y[v,u].$$

But $[v,u][u,u] = [v,u]$. Thus $x[u,u] = x$. Thus $x = \{xuu\}$. We may similarly show that $u = \{uxx\}$. Thus $q(x) = q(u)$. A similar argument shows that $q(y) = q(v)$. Thus the map is injective. We now show that the map is surjective. Let $s \in S$. Then by assumption there exists $(x,y) \in X \times X$ such that $[x,y] = s$. Consider the element $yyx^{-1}\{xxy\} \in X^{-1}X$. Then

$$[\{yyx\}, \{xxy\}] = [\langle y,y\rangle x, \langle x,x\rangle y] = [x,y] = s$$

using Proposition 2.3 of [27]. It remains to show that this function is a homomorphism. Again using Proposition 2.3 of [27], one quickly shows that the image of $x^{-1}y \cdot u^{-1}v$ is equal to $[x,y][u,v]$.

We have therefore shown that $\alpha\colon\ X^{-1}X \longrightarrow S$ given by $\alpha(x^{-1}y) = [x,y]$ is an isomorphism of semigroups. A dual argument shows that $\beta\colon\ XX^{-1} \longrightarrow T$ given by $\beta(xy^{-1}) = \langle x,y\rangle$ is an isomorphism of semigroups.

We now show that the actions are isomorphic. By definition

$$x \cdot y^{-1}z = \{xyz\} = \langle x,y\rangle z = x[y,z] = x\alpha(y^{-1}z).$$

A dual argument holds for the action of XX^{-1} on X.

Finally, we compare $[-,-]_1$ and $[-,-]$. By definition

$$[x,y]_1 = \{yyx\}^{-1}\{xxy\} = [\{yyx\},\{xxy\}] = [\langle y,y\rangle x, \langle x,x\rangle y] = [x,y]$$

using Proposition 2.3 of [27]. The dual argument compares $\langle -,-\rangle_1$ and $\langle -,-\rangle$. $\square$

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