# A monoid associated with a self-similar group action

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**Abstract**

We prove that there is a correspondence between self-similar group actions and the class of left cancellative right hereditary monoids satisfying the dedekind height property. The monoids in question turn out to be coextensive with the Zappa-Szép products of free monoids and groups, and the ideal structure of the monoid reflects properties of the group action. These monoids can also be viewed as 'tensor monoids' of covering bimodules, and also arise naturally from a double category associated with the action. There is also a correspondence between self-similar group actions and a class of inverse monoids, which are congruence-free when the actions are faithful; these inverse monoids arise naturally in the construction of the Cuntz-Pimsner algebras associated with the actions, and generalise the polycyclic monoids from which the Cuntz algebras are constructed. Finally, these results have the effect of correcting an error in a paper of Nivat and Perrot.

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## 1 Self-similar group actions

The results of this paper are closely related to a general theory originating in the pioneering papers of Rees [22] and Clifford [5], and subsequently developed by a number of authors [23, 17, 18, 16]. However the special case we consider is of independent interest, is developed from scratch, and has the added effect of shedding a new light on this general theory. The paper arose in the first instance from trying to write out a proof of Proposition 6 of [20], and failing: this is amplified in the remark following Proposition 4.8.

In this section, we shall define what we mean by the term ‘self-similar group action’ in this paper: it is used in a slightly more general sense than in [19]. We begin by recalling some definitions from [1]. Let $X^\omega$ denote the set of all (right) infinite strings over the alphabet $X$. An action of $G$ on $X^\omega$, denoted $\circ$, is said to be ‘self-similar’ if for each $g \in G$ and $x \in X$ there exists $h \in G$ and $y \in X$ such that

$$g \circ (xw) = y(h \circ w)$$

for all $w \in X^\omega$. The group element $h$ depends on $g$ and $x$ but is not assumed to be unique. Applying the above definition a number of times we deduce the following: for each string $u$ of length $n$ and group element $g$ there exists a string $v$ of length $n$ and a group element $k$ such that

$$g \circ (uw) = v(k \circ w)$$

for all $w \in X^\omega$. Now define a function from $G \times X^*$ to $X^*$ by $g \cdot u = v$. This defines an action of $G$ on the free monoid $X^*$ on $X$. This action is *length-preserving* in the sense that $|g \cdot x| = |x|$ for all $x \in X^*$, and *prefix-preserving* in the sense that $x = yz$ implies that $g \cdot x = (g \cdot y)z'$ for some string $z'$. Suppose now that $G$ acts faithfully on $X^\omega$. Then the element $h$ is uniquely determined by $g$ and $x$; we will denote it by $g|_x$. Under the assumption that the action is faithful, it is easy to check that the following properties hold. Observe that we use 1 to denote both the identity of the group $G$ and the empty string of $X^*$.

(SS1) $1 \cdot x = x$.

(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$.

(SS3) $g \cdot 1 = 1$.

(SS4) $1|_x = 1$.

(SS5) $g|_1 = g$.

(SS6) $g|_{xy} = (g|_x)|_y$.

(SS7) $(gh)|_x = g|_{h \cdot x} h|_x$.

(SS8) $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

Observe that (SS1) and (SS2) simply restate that we have an action of $G$ on $X^*$. Property (SS3) follows from the fact that the action is length-preserving. Property (SS4) follows from the fact that $1 \cdot (xy) = xy$ for all $x$ and $y$. Property (SS5) follows from the fact that $g \cdot (1x) = g \cdot x$ for all $x$. Property (SS6) follows from the fact that $g \cdot ((xy)z) = g \cdot (x(yz))$ for all $x$, $y$ and $z$. Property (SS7) follows from the fact that $(gh) \cdot (xy) = g \cdot (h \cdot (xy))$ for all $x$ and $y$. Property (SS8) is just a restatement of the definition.

More generally, let $G$ be a group, $X$ a set, $G \times X^* \to X^*$ an operation, called the *action*, denoted by $(g,x) \mapsto g \cdot x$, and $G \times X^* \to G$ an operation, called the *restriction*, denoted by $(g,x) \mapsto g|_x$, such that the above eight axioms hold. Then we say that the action of $G$ on $X^*$ is *self-similar*. Our definition of self-similar action contains the faithful self-similar actions in the sense of [1] and [19], but is more specialised than the general definition because the uniqueness of the restriction operation and its properties are inbuilt.

**Lemma 1.1** *Let $G$ act on $X^*$ in such a way that the axioms (SS1)–(SS8) hold. Then the action is length-preserving and prefix-preserving.*

**Proof** Prefix-preserving follows from (SS8). We now prove that the action is length-preserving. Observe first that by (SS3), if $x$ is the empty string so too is $g \cdot x$. Conversely, if $g \cdot x = 1$ then $x = g^{-1} \cdot 1 = 1$ by (SS3). Thus $g \cdot x$ is the empty string iff $x$ is. Let $x \in X$. Suppose that $g \cdot x = yz$ where $y$ is a letter and $z$ is a string, possibly empty. Then by (SS8), we have that

$$x = (g^{-1} \cdot y)(g^{-1}|_y \cdot z).$$

We know that $g^{-1} \cdot y$ cannot be empty and so has length at least one. Since the leftthand side has length one, and lengths add we deduce that $(g^{-1}|_y \cdot z)$ has length zero. Thus $z$ is the empty string. It follows that letters are mapped to letters. The result now follows by (SS8) and induction. ∎

## 2 A class of left cancellative monoids

In this section, we describe the class of monoids we shall associate with self-similar group actions.

An *$S$-act* or *act* $(X,S)$ is an action of a monoid $S$ on a set $X$ on the right. If $S$ is a monoid then $(S,S)$ is an act by right multiplication. If $Y \subseteq X$ is a subset such that $YS \subseteq Y$ then we say that $Y$ is an *$S$-subact* or just a *subact*. Right ideals of $S$ are subacts under right multiplication. If $X$ and $Y$ are acts then a function $\theta$ from $X$ to $Y$ is an *$S$-homomorphism* or just a *homomorphism* if $\theta(xs) = \theta(x)s$ for all $x \in X$ and $s \in S$. For a fixed $S$, we can form the category consisting of $S$-acts and the homomorphisms between them. The usual definitions from module theory can be adapted to the theory of acts. In particular, we can define when an act is *projective*. A monoid $S$ is said to be *right PP* if all its principal right ideals are projective as right $S$-acts, and *right hereditary* if all its right ideals are projective as right $S$-acts. The following was proved by Dorofeeva [7].

**Theorem 2.1** *A monoid $S$ is right hereditary iff it is right PP, incomparable principal right ideals are disjoint, and $S$ has the ascending chain condition for principal right ideals.* ∎

We do not need the general characterisation of right PP monoids for this paper; it is enough to know that the right PP monoids with a single idempotent are precisely the left cancellative monoids.

**Remarks**

1. We shall often use the fact that (ACC) on principal right ideals is equivalent to the condition that every non-empty set of principal right ideals has a maximal element.

2. From now on, 'ideal' will always mean 'principal right ideal' unless otherwise stated, and 'maximal ideal' will always mean 'maximal proper principal right ideal'.

3. If two maximal ideals intersect in a left cancellative right hereditary monoid then they are equal; this is because they must be comparable, but both are maximal.

4. We denote the group of units of a monoid $S$ by $G(S)$ or just $G$.

5. In a left cancellative monoid $S$ we have that $aS = bS$ iff $a = bg$ for some unit $g$; we say that $a$ and $b$ are *associates*.

6. In a left cancellative monoid $S$ we have that $aS = S$ iff $a$ is invertible.

7. Generators of maximal ideals will be called *irreducible elements*.

8. Let $S$ be a monoid and $a \in S$. A *left factor* of $a$ is an element $b \in S$ such that $a \in bS$.

We shall study left cancellative right hereditary monoids satisfying a further finiteness condition [3]. Let $S$ be a left cancellative right hereditary monoid and $a \in S$. Then the set of all principal right ideals containing $a$ need not be finite, but if it is we say that $S$ has the *dedekind height property*.

Let $aS$ and $bS$ be two principal right ideals. A *chain of length $n$ from $aS$ to $bS$* is a sequence

$$aS = a_0S \subset a_1S \subset a_2S \subset \ldots \subset a_nS = bS.$$

**Lemma 2.2** *Let $S$ be a left cancellative right hereditary monoid. Then the following are equivalent.*

(i) *$S$ has the dedekind height property.*

(ii) *For each $a \in S$ there exists a unique chain of maximum finite length starting at $aS$ and concluding at $S$.*

**Proof** (i) $\Rightarrow$ (ii). The set of all principal right ideals containing $a$ is finite. Thus there is a bound on the length of chains starting at $aS$ and ending at $S$. Given two such chains of maximum length they must be equal. To prove this, we show that the two chains must agree term by term. We use the fact that if two ideals have a non-empty intersection, then they must be comparable. Let

$$aS = a_0 S \subset a_1 S \subset \ldots \subset a_m S = S$$

and

$$aS = b_0 S \subset b_1 S \subset \ldots \subset b_n S = S$$

be two such chains. We claim that $a_1 S = b_1 S$. To see why observe that they are comparable because both contain $aS$. Thus either $a_1 S \subset b_1 S$ or vice-versa. If the former we could refine the second chain, if the latter we could refine the first chain. But neither refinement is possible since each chain is of maximum length. Thus $a_1 S = b_1 S$. This process continues. If $m > n$ then we could use the first chain to refine the second. If $n > m$ then we could use the second chain to refine the first. So the two chains must have the same length and the same terms.

(ii) $\Rightarrow$ (i). All the distinct principal right ideals containing $aS$ must be comparable so they will form a totally ordered set from $aS$ to $S$. This will be a chain of maximum length and so equal to the unique such chain assumed to exist. Thus the set of all principal right ideals containing $aS$ must be finite. ■

The next lemma provides us with a class of examples of monoids satisfying the dedekind height property.

**Lemma 2.3** *Let $S$ be a left cancellative right hereditary monoid equipped with a monoid homomorphism $\lambda \colon S \to \mathbb{N}$ such that $\lambda^{-1}(0) = G(S)$. Then $S$ satisfies the dedekind height property.*

**Proof** Let $aS \subseteq bS$. Then $a = bs$ and so $\lambda(a) = \lambda(b) + \lambda(s)$. Thus, in particular, $\lambda(a) \geq \lambda(b)$. Suppose, in addition, that $\lambda(a) = \lambda(b)$. Then $\lambda(s) = 0$ and so $s$ is a unit. It follows that in this case, $aS = bS$. We deduce that if $aS \subset bS$ then $\lambda(a) > \lambda(b)$. Thus the length of any chain of principal right ideals starting at $aS$ is bounded by $\lambda(a)$. ■

We define a *length function* on an arbitrary monoid $S$ to be a homomorphism $\lambda \colon S \to \mathbb{N}$ such that $\lambda^{-1}(0) = G(S)$.

Let $S$ be left cancellative and right hereditary satisfying the dedekind height property. Let $aS = S_0 \subset S_1 \subset S_2 \ldots \subset S_n = S$ be a chain of principal right ideals of maximum length. We define $\lambda(a) = n$.

**Lemma 2.4** *Let $S$ be a left cancellative right hereditary monoid satisfying the dedekind height property. Let*

$$bS = b_0 S \subset b_1 S \subset b_2 S \subset \ldots \subset b_{\lambda(b)} S = S$$

*be a chain of maximum length joining $bS$ to $S$. Then*

$$abS = ab_0 S \subset ab_1 S \subset ab_2 S \subset \ldots \subset ab_{\lambda(b)} S = aS$$

*is a chain of maximum length joining $abS$ to $aS$.*

**Proof** We show first that the inclusions really are distinct. Suppose that $ab_iS = ab_{i+1}S$ for some $i$. Then $ab_i = ab_{i+1}g$ for some unit $g$. By left cancellation, $b_i = b_{i+1}g$ giving $b_iS = b_{i+1}S$, which contradicts our assumption. Next we show that the chain is of maximum length. Suppose not. Then we can interpolate a principal right ideal somewhere

$$ab_iS \subset cS \subset ab_{i+1}S.$$

Let $ab_i = cf$ for some $f$ and $c = ab_{i+1}d$ for some $d$. Thus by left cancellation, $b_i = b_{i+1}df$. We therefore have

$$b_iS \subseteq b_{i+1}dS \subseteq b_{i+1}S.$$

Suppose that $b_iS = b_{i+1}dS$. Then $b_i = b_{i+1}dg$ for some unit $g$. By left cancellation, it follows that $g = f$ and is a unit. Thuis $ab_iS = cS$, which is contradiction. Suppose that $b_{i+1}dS = b_{i+1}S$. Then $b_{i+1}d = b_{i+1}h$ for some unit $h$. By left cancellation, $d = h$ and so $cS = ab_{i+1}S$, which is a contradiction. However, we now have

$$b_sS \subset b_{i+1}dS \subset b_{i+1}S$$

which contradicts the fact that our original chain was of maximum length. It follows that our new chain is of maximum length. ■

**Lemma 2.5** *Let $S$ be a left cancellative right hereditary monoid satisfying the dedekind height property. Then the function $\lambda$ defined before Lemma 2.4 is a length function.*

**Proof** By Lemma 2.4, if

$$bS = b_0S \subset b_1S \subset b_2S \subset \ldots \subset b_{\lambda(b)}S = S$$

is a chain of maximum length joining $bS$ to $S$, then

$$abS = ab_0S \subset ab_1S \subset ab_2S \subset \ldots \subset ab_{\lambda(b)}S = aS$$

is a chain of maximum length joining $abS$ to $aS$. Now glue this to a chain

$$aS = a_0 \subset a_1S \subset \ldots \subset a_{\lambda(a)}S = S$$

of maximum length. The resulting chain links $abS$ to $S$ and has maximum length, and this length is $\lambda(a) + \lambda(b)$. Thus $\lambda$ is a homomorphism. Those elements $a$ of length 0 are precisely those where $aS = S$, which are just the invertible elements. ■

We combine Lemmas 2.2,2.3 and 2.5 in the following theorem.

**Theorem 2.6** *Let $S$ be a left cancellative, right hereditary monoid. Then the following are equivalent.*

(i) *$S$ satisfies the dedekind height property.*

(ii) *For each $a \in S$ there exists a unique chain of maximum finite length starting at $aS$ and concluding at $S$.*

(iii) *The monoid $S$ is equipped with a length function.* ■

**Remark** We shall see in Section 3, that left cancellative right hereditary monoids satisfying the dedekind height property possess length functions which are surjective and are induced by the usual length function on a free submonoid.

An arbitrary monoid $M$ is said to be *equidivisible* if for all $a, b, c, d \in M$ the fact that $ab = cd$ implies that either $a = cu, ub = d$ for some $u \in M$ or $c = av, b = vd$ for some $v \in M$.

**Lemma 2.7** *Let $S$ be a left cancellative monoid. Then the following are equivalent*

(i) *Incomparable principal right ideals are disjoint.*

(ii) *$S$ is equidivisible.*

*If either holds, then incomparable principal left ideals are disjoint.*

**Proof** (i)⇒(ii). Suppose that $ab = cd$. Then $aS \cap cS \neq \emptyset$. Thus $aS \subseteq cS$ or $cS \subseteq aS$. Suppose the former. Then $a = cu$ for some $u \in S$. But $ab = cd$ and so $cub = cd$. By left cancellation, $ub = d$. Suppose the latter. Then $c = av$ for some $v \in S$. But $ab = cd$ and so $ab = avd$. By left cancellation, $b = vd$. Thus $S$ is equidivisible.

(ii)⇒(i). This is immediate.

To prove the last assertion, suppose that $Sb \cap Sd \neq \emptyset$. Then $ab = cd$ for some $b, c \in S$. The result now follows by equidivisibility. ■

The following is immediate from Theorem 2.6 and Lemma 2.7 and Corollary 5.1.6 of [11] and the fact that free monoids are left cancellative, right hereditary and their length functions really are length functions in our sense.

**Corollary 2.8** *Let $S$ be a left cancellative right hereditary monoid satisfying the dedekind height property. Then $S$ is a free monoid if and only if the group of units is trivial.* ■

**Remark** The above corollary tells us that left cancellative right hereditary monoid satisfying the dedekind height property are natural generalisations of free monoids.

The class of left cancellative right hereditary monoids satisfying the dedekind height property is a proper subclass of the class of all left cancellative monoids.

We shall now show how closely these two classes are related. We shall use the theory of Rhodes-expansions described in [3] adapted to our situation.

Let $S$ be a left cancellative monoid. We shall be interested in finite sequences of elements of $S$

$$\mathbf{x} = (x_1, \ldots, x_n)$$

where $x_{i+1} \in x_i S$ but $x_{i+1}S \neq x_n S$ and where $x_1$ is a unit. We denote by $\hat{S}$ the set of all such sequences. We shall now define a product on such sequences. Let

$$\mathbf{x} = (x_1, \ldots, x_m) \text{ and } \mathbf{y} = (y_1, \ldots, y_n).$$

Consider the sequence

$$x_1, \ldots, x_{m-1}, x_m, x_m y_1, \ldots, x_m y_n.$$

Because $y_1$ is a unit, we have that $x_m S = x_m y_1 S$. Clearly, $x_m y_1 S \subset x_{m-1}S$. Also from $y_{i+1}S \subset y_i S$ we get $x_m y_{i+1} S \subseteq x_m y_i S$. Observe that if $x_m y_{i+1} S = x_m y_i S$ then $x_m y_{i+1} = x_m y_i g$ for some unit $g$. Thus by left cancellation, $y_{i+1} = y_i g$ implying that $y_{i+1}S = y_i S$, contradicting our assumption. It follows that

$$\mathbf{xy} = (x_1, \ldots, x_{m-1}, x_m y_1, \ldots, x_m y_n)$$

is a well-defined element of $\hat{S}$. This defines a binary operation on $\hat{S}$. The fact that this is a semigroup follows from the general theory in [3]. It is easy to check that it is a monoid with identity (1), and that left cancellation in $S$ is inherited by $\hat{S}$.

**Proposition 2.9** *For each left cancellative monoid $S$, the monoid $\hat{S}$ is left cancellative, right hereditary and equipped with a dedekind height function. There is a surjective homomorphism from $\hat{S}$ onto $S$.*

**Proof** We first characterise the left factors of an element of $\hat{S}$. Suppose that $\mathbf{x} \in \mathbf{y}\hat{S}$. Then

$$(x_1, \ldots, x_m) = (y_1, \ldots, y_n)(z_1, \ldots, z_p).$$

Thus $m \geq n$, $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ and $y_n S = x_n S$. Conversely, suppose that $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ are such that $m \geq n$, $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ and $y_n S = x_n S$. For $0 \leq i \leq m-n$ define $z_{i+1}$ by $x_{n+i} = y_n z_{i+1}$. Observe that $z_1$ is a unit. It is easy to check that $\mathbf{z} = (z_1, \ldots, z_p)$ is a well-defined element of $\hat{S}$ and that $\mathbf{x} = \mathbf{yz}$.

We can now show that $\hat{S}$ is right hereditary and satisfies the dedekind height property. Suppose that $\mathbf{x}\hat{S} \cap \mathbf{y}\hat{S} \neq \emptyset$. Then there is a $\mathbf{z}$ which has both $\mathbf{x}$ and $\mathbf{y}$ as left factors. Let $\mathbf{z} = (z_1, \ldots, z_p)$, $\mathbf{x} = (x_1, \ldots, x_m)$, and $\mathbf{y} = (y_1, \ldots, y_n)$. Then $p \geq m, n$ and $x_1 = z_1, \ldots, x_{m-1} = z_{m-1}, z_m S = x_m S$ and $y_1 = z_1, \ldots, y_{n-1} = z_{n-1}, z_n S = y_n S$. Without loss of generality, suppose that $m \leq n$. Then $x_1 = y_1, \ldots, x_{m-1} = y_{m-1}$ and $x_m S = z_m S = y_m S$. Thus $\mathbf{y} \in \mathbf{x}\hat{S}$.

From the above we can easily derive the criterion for $\mathbf{y}\hat{S} = \mathbf{x}\hat{S}$: $\mathbf{x}$ and $\mathbf{y}$ have the same length, all the components are the same except the rightmost ones which are associate.

It follows from the above two characterisations that the dedekind height property is satisfied. Define $\eta_S: \hat{S} \to S$ by $(x_1, \ldots, x_n) \mapsto x_n$. Then this is a surjective homomorphism. Observe that restricted to the $\mathcal{R}$-classes of $\hat{S}$, this homomorphism is injective. ■

We conclude this section by looking at the *cancellative* right hereditary monoids satisfying the dedekind height property. A group $G$ is said to be *indicable* if there is a surjective homomorphism $\theta: G \to \mathbb{Z}$. We shall call the full inverse image under $\theta$ of the natural numbers (and zero!) the *positive cone* of $G$. In [22], Rees considers the structure of left cancellative monoids in which the principal right ideals form a descending chain order-isomorphic to the natural numbers under the reverse of the usual ordering. For convenience, I shall call such monoids as *left cancellative $\omega$-monoids*. They are clearly right hereditary and satisfy the dedekind height property. The following is due to Stuart Margolis (private communication).

**Proposition 2.10** *The positive cones of indicable groups are precisely the cancellative $\omega$-monoids.*

**Proof** Let $G$ be an indicable group and $\theta: G \to \mathbb{Z}$ its surjective homomorphism. Put $S = \theta^{-1}(\mathbb{N})$, the positive cone of $G$. It is easy to check that the group of units of $S$ is precisely the group $\theta^{-1}(0)$, and that for all $s, t \in S$, we have that $sS = tS$ if and only if $\theta(s) = \theta(t)$. Let $a$ be a fixed element of $S$ such that $\theta(a) = 1$, which exists by surjectivity. Now given $s, t \in S$, suppose that $\theta(s) \leq \theta(t)$. Then $\theta(sa^{\theta(t)-\theta(s)}) = \theta(t)$, and so $tS = sa^{\theta(t)-\theta(s)}S \subseteq sS$. It follows that $S$ is a cancellative $\omega$-monoid.

Conversely assume that $S$ is a cancellative $\omega$-monoid. From [22], there is a surjective homomorphism $\theta: S \to \mathbb{N}$ such that $\theta^{-1}(0)$ is the group of units of $S$. Then $S$ satisfies the Ore conditions and thus has a group of fractions $G$ such that each element of $G$ is of the form $st^{-1}$ where $s, t \in S$. It is easy to see from the structure of groups of fractions that $\theta$ extends uniquely to a surjective function $G$ to $\mathbb{Z}$, proving that $G$ is indicable and that $S$ is the positive cone.■

For each monoid $S$ there is a group $U(S)$ and a homomorphism $\iota: S \to U(S)$ such that for each homomorphism $\phi: S \to G$ to a group there is a unique homomorphism $\bar{\phi}: S \to G$ such that $\phi = \bar{\phi}\iota$. The group $U(S)$ is called the *universal group* of $S$. The monoid $S$ can be embedded in a group iff $\iota$ is injective. It can be deduced from the results of Section 0.5 of [6] that every cancellative monoid in which any two principal right ideals are either disjoint or comparable can be embedded in a group. It follows that the cancellative right hereditary monoids satisfying the dedekind height property can be embedded in their universal groups. If $S$ is a cancellative right hereditary monoids satisfying the dedekind height property, then there is a homomorphism from $S$ onto $\mathbb{N}$ and so a homomorphism from $S$ to $\mathbb{Z}$. It follows that the universal group of $S$ admits a homomorphism to $\mathbb{Z}$. Since the image of this homomorphism contains $\mathbb{N}$ it is in fact the whole of $\mathbb{Z}$ and so surjective. We have therefore proved the following.

**Proposition 2.11** *The universal group of a cancellative right hereditary monoid satisfying the dedekind height property is indicable.* ■

## 3 The correspondence

In this section, we set up a correspondence between the self-similar group actions of Section 1 and the left cancellative right hereditary monoids satisfying the dedekind height property of Section 2.

**Proposition 3.1** *Let $S$ be a left cancellative right hereditary monoid satisfying the dedekind height property. Let $X$ be a transversal of the generators of the maximal proper principal right ideals, and denote by $X^*$ the submonoid generated by the set $X$. Then*

(i) $S = X^* G(S)$.

(ii) *Each element of $S$ can be written uniquely as a product of an element of $X^*$ and an element of $G(S)$.*

(iii) *The monoid $X^*$ is free.*

**Proof** (i) Let $s \in S \setminus G(S)$. Consider the set of all proper ideals that contain $s$. This set contains a maximal element $x_1 S$, which is necessarily a maximal ideal, and $x_1 \in X$. Thus $s = x_1 s_1$. If $s_1$ is a unit or irreducible the process stops. Otherwise, repeat this process with $s_1$ to get $s_1 = x_2 s_2$ and so on. Thus we can write $s = x_1 \ldots x_i s_i$. To show that this process terminates observe that

$$sS \subset x_1 \ldots x_i S \subset \ldots \subset x_1 S.$$

Thus termination follows from the dedekind height property. It follows that we can write $s = x_1 \ldots x_n g$ where $g$ is a unit.

(ii) Let $s = xu = yv$ where $x, y \in X^*$ and $u, v \in G(S)$. Then $xuS = xS$ and $yvS = yS$ so that $xS = yS$. Let $x = x_1 \ldots x_m$ and $y = y_1 \ldots y_n$ where the $x_i, y_j \in X$. Now $xS = yS \subseteq x_1 S, y_1 S$. It follows that $x_1 S = y_1 S$ and so $x_1 = y_1$. By left cancellation, $x_2 \ldots x_m = y_2 \ldots y_n a$. Assume $m < n$. Then repeating the above argument we eventually get $1 = y_{m+1} \ldots y_n a$. Thus $y_{m+1} \ldots y_n$ is invertible. But then $S = y_{m+1} S$ meaning $y_{m+1}$ is invertible, which is a contradiction. We also get a contradiction if $m > n$. Thus in fact, we must have that $m = n$, and so $x_i = y_i$ and this gives $a = 1$. By left cancellation again we get that $u = v$.

(iii) Observe that $X = X^* \setminus (X^*)^2$. The result now follows by Proposition 5.1.3 of [11] and (ii) above. ■

**Remark** The above proposition says that $S$ is a 'Zappa-Szép product' of a free monoid by a group, a point we shall return to later.

The following is a consequence of the theory of Zappa-Szép products, but we sketch a proof anyway.

**Proposition 3.2** *With each left cancellative right hereditary monoid satisfying the dedekind height property we can associate a self-similar group action.*

**Proof** From Proposition 3.1, if $g \in G(S)$ and $x \in X^*$ then we can write

$$gx = x'g'$$

for unique elements $x' \in X$ and $g' \in G(S)$. We write these unique elements as follows

$$gx = (g \cdot x)g|_x.$$

Thus we have defined a function $G(S) \times X^* \to X^*$ given by $(g, x) \mapsto g \cdot x$ and a function $G(S) \times X^* \to G(S)$ given by $(g, x) \mapsto g|_x$. It is now straightforward to check that these operations satisfy axioms (SS1)–(SS8): (SS1) and (SS4) follow from the fact that $x = 1x$; (SS2) and (SS7) follow from the fact that $(gh)x = g(hx)$; (SS6) and (SS8) follow from the fact that $g(xy) = (gx)y$; (SS3) and (SS5) follow from the fact that $g1 = g$. ■

We shall now look at the converse of the above result. Let $G$ be an arbitrary group, and $M$ an arbitrary left cancellative monoid (not necessarily free) equipped with a function $G \times M \to M$, denoted by $(g, m) \mapsto g \cdot m$, and a function $G \times M \to G$, denoted by $(g, m) \mapsto g|_m$, satisfying the obvious generalisations of (SS1)–(SS8). On the set $M \times G$ define the binary operation by

$$(x, g)(y, h) = (x(g \cdot y), g|_y h).$$

The following is part of the general theory of Zappa-Szép products, but we prove it anyway.

**Proposition 3.3** *With the above product, $M \times G$ is a left cancellative monoid containing copies of $M$ and $G$ such that $M \times G$ can be written as a unique product of these copies.*

**Proof** We begin by proving associativity. We calculate first

$$[(x, g)(y, h)](z, k).$$

By (SS2), (SS7), and (SS6) we get

$$(x(g \cdot y)g|_y \cdot (h \cdot z), g|_{y(h \cdot z)} h|_z k).$$

We now calculate

$$(x, g)[(y, h)(z, k)].$$

Using (SS8), we get the same result.

We now show that $(1, 1)$ is the identity. We calculate

$$(1, 1)(x, g) = (1(1 \cdot 1), 1|_x g) = (x, g)$$

using (SS1) and (SS4). We calculate

$$(x, g)(1, 1) = (x(g \cdot 1), g|_1 1) = (x, g)$$

using (SS3) and (SS5). We have now used all the axioms (SS1)–(SS8).

Next we show that $M \bowtie G$ is left cancellative. Suppose that

$$(x, g)(y, h) = (x, g)(z, k).$$

Then

$$(x(g \cdot y), g|_y h) = (x(g \cdot z), g|_z k).$$

Left cancellation in $M$ gives us $g \cdot y = g \cdot z$ and so because this is an action $y = z$. Hence $h = k$.

We now have to show that $M$ and $G$ are each embedded in $M \bowtie G$. Define $\iota_M\colon M \to M \bowtie G$ by $x \mapsto (x, 1)$. This is an injective homomorphism by (SS1) and (SS4). Denote its image by $M'$. Define $\iota_G\colon G \to M \bowtie G$ by $g \mapsto (1, g)$. This is an injective homomorphism by (SS3) and (SS5). Denote its image by $G'$. Observe that $(x, g) = (x, 1)(1, g)$. Thus $M \bowtie G = M'G'$. This decomposition is evidently unique. ■

The monoid constructed in Proposition 3.3 is called the *Zappa-Szép product of $M$ and $G$* and is denoted $M \bowtie G$.

**Proposition 3.4** *Let $S$ be a monoid. Suppose that $S = MG$ uniquely where $M$ is a left cancellative monoid and $G$ is a group. Then $S$ is a left cancellative monoid whose ideal structure is order isomorphic with the ideal structure of $M$. In particular, when $M$ is a free monoid, the monoid $S$ is right hereditary and equipped with a length function.*

**Proof** Observe that $\{1\} = G \cap M$. To see why if $g \in G \cap M$ then $g = 1g = g1$ and so we would lose uniqueness. We use the notation $gx = (g \cdot x)g|_x$. We prove first that $S$ is left cancellative. Let $ab = ac$ where $a = mg$, $b = nh$, and $c = pk$. Then $mgnh = mgpk$. Thus $m(g \cdot n)g|_n h = m(g \cdot p)g|_p k$. In the monoid $M$ we have that $m(g \cdot n) = m(g \cdot p)$, and in the group $G$ we have that $g|_n h = g|_p k$. By left cancellation in $M$ and properties of the group action we get $n = p$ and so $h = k$. Hence $b = c$, as required.

We now show that the ideal structures of $M$ and $S$ are order-isomorphic. If $a \in S$ then $a = xg$ and so $aS = xS$. We prove that $xS \subseteq yS$ iff $xM \subseteq yM$. Suppose that $x = yb$ for some $b \in S$. Let $b = zu$ where $z \in M$ and $u \in G$. Then $x = (yz)u$. By uniqueness, $u = 1$ and so $x \in yM$. The converse is clear.

Finally, when $M$ is a free monoid, the monoid $S$ will be right hereditary and satisfy the dedekind height property because $S$ and $M$ have isomorphic ideal structures. The fact that $S$ has a length function follows from Theorem 2.6. ■

**Remark** By the result above, the length functions of left cancellative right hereditary monoids satisfying the dedekind height function can be assumed to extend the length function on a submonoid that is free; in particular, they can be assumed surjective.

Combining Propositions 3.1,3.2,3.3 and 3.4, we obtain the following.

**Theorem 3.5** *A monoid is left cancellative, right hereditary and satisfies the dedekind height property if and only if it isomorphic to a Zappa-Szép product of a free monoid by a group. Furthermore, Zappa-Szép products of free monoids by groups determine, and are determined by, self-similar group actions.* ■

We have therefore set up a correspondence between self-similar group actions and left cancellative right hereditary monoid satisfying the dedekind height property. Each determines the other upto isomorphism.

# 4 Special cases

With each self-similar group action of $G$ on $X^*$ we have the *associated monoid* $S = X^* \bowtie G$. In this section, we shall explore the connection between the structure of the group action and the structure of $S$.

Green's relation $\mathcal{J}$ is defined by $a\mathcal{J}b$ iff $SaS = SbS$. If $a = xg$, where $x \in X^*$ and $g \in G$ then $SaS = SxS$. Thus to study the principal ideals it is enough to study those of the form $SxS$ where $x \in X^*$.

**Lemma 4.1** *Let $x, y \in X^*$. Then $SxS \subseteq SyS$ implies $|x| \geq |y|$.*

**Proof** By assumption, $x = syt$ for some $s, t \in S$. Let $s = wg$ and $t = zh$ where $w, z \in X^*$ and $g, h \in G$. Then

$$x = wgyzh = w(g \cdot y)g|_y zh = w(g \cdot y)(g|_y \cdot z)g|_{yz}h.$$

By uniqueness, $g|_{yz}h = 1$ and so $x = w(g \cdot y)(g|_y \cdot z)$. Thus $|x| = |w| + |g \cdot y| + |g|_y \cdot z| \geq |g \cdot y| = |y|$, since the action is length-preserving by Lemma 1.1. Hence $|x| \geq |y|$. ■

**Lemma 4.2** *Let $x, y \in X^*$. Then the following are equivalent:*

(i) *$SxS \subseteq SyS$ and $|x| = |y|$.*

(ii) *$x = g \cdot y$ for some $g \in G$.*

(iii) *$SxS = SyS$.*

**Proof** (i)⇒(ii). Suppose that $SxS \subseteq SyS$ and $|x| = |y|$. Then $x = sxt$ where $s, t \in S$. Let $s = wg$ and $t = zh$ where $w, z \in X^*$ and $g, h \in G$. Then

$$x = wgyzh = w(g \cdot y)g|_y zh = w(g \cdot y)(g|_y \cdot z)g|_{yz}h.$$

By uniqueness, $g|_{yz}h = 1$ and so $x = w(g \cdot y)(g|_y \cdot z)$. Thus $|x| = |w| + |g \cdot y| + |g|_y \cdot z|$. Since the action is length-preserving by Lemma 1.1, we have that $|y| = |g \cdot y|$ and $|z| = |g|_y \cdot z|$. Thus $|x| = |w| + |y| + |z|$. By assumption, $|x| = |y|$. Thus $|w| = 0$ and $|z| = 0$, and so $w$ and $z$ are the empty strings. Hence $x = g \cdot y$, as required.

(ii)⇒(iii). Suppose that $x = g \cdot y$. Observe that $SgyS = S(g \cdot y)g|_y S = S(g \cdot y)S = SxS$. Thus $SxS = SyS$.

(iii)⇒(i). Suppose that $SxS = SyS$. By Lemma 4.1, we have that $|x| = |y|$. Thus (i) holds. ■

**Proposition 4.3** *The associated monoid has a maximum proper principal two-sided ideal if and only if the action of $G$ on $X$ is transitive.*

**Proof** Suppose that for some $x \in X^*$, the two-sided ideal $SxS$ is maximum and proper. We show first that $x \in X$. Let $x = yz$ where $y$ has length one. Thus $SxS \subseteq SyS$. Now $SxS$ is maximal proper and so either $SyS = S$, which would imply $y$ has length zero, or $SxS = SyS$ which implies $x$ has length one (and so $z$ is the empty string). Now let $y \in X$. Then $SyS \subseteq SxS$ and $|y| = |x|$. Thus by Lemma 4.2 we have that $y = g \cdot x$ for some $g$. Since $y$ was arbitrary with length one, it follows that the action of $G$ on $X$ is transitive.

Conversely, suppose that the action of $G$ on $X$ is transitive. Then for any two strings $x$ and $y$ of length one, we know that $x = g \cdot y$ and so by Lemma 4.3, we know that $SxS = SyS$. Let $SzS$ be any two-sided principal ideal where $|z| \geq 2$. Now $z = xy$ where $|x| = 1$. Thus $SzS \subseteq SxS$. Hence $SxS$ is a maximum proper principal two-sided ideal. ■

If $G$ acts on $X^*$ by length-preserving prefix-preserving transformations in such a way that the action of $G$ on $X^n$ is transitive for all $n$, then we say that the action is *level transitive*.

**Proposition 4.4** *The principal two-sided ideals form a descending chain in the associated monoid if and only if the action of $G$ on $X^*$ is level-transitive.*

**Proof** Suppose first that the action of $G$ on $X^*$ is level transitive. Let $I_n = SxS$ where $x \in X^n$ is a string of length $n$. By level transitivity, $I_n = SyS$ where $y$ is any string of length $n$. If $|x| = n + 1$ then $x = yz$ where $|y| = n$. Thus $SxS \subseteq SyS$. Hence $I_{n+1} \subseteq I_n$, and we have our descending chain.

Conversely, suppose that the principal two-sided ideals form a descending chain. We denote them by $I_n = Sx_nS$ for some $x_n \in X^*$ where $I_{n+1} \subseteq I_n$. Since $x_{n+1} \in Sx_nS$ we know that $|x_{n+1}| \geq |x_n|$. We also know that $S = I_0$.

Let us now consider $I_1$. We claim that $x_1$ has length one. Suppose that $|x_1| \geq 2$. Then $x_1 = xy$ where $x$ has length one. Thus $Sx_1S \subseteq SxS$. Now either $SxS = S$ which is impossible or $Sx_1S = SxS$, which implies that $x_1$ has length one by Lemma 4.2. Now we prove that $SxS = Sx_1S$ for any string $x$ of length one. Suppose $SxS = I_n$ where $n > 1$. Then $SxS$ is contained in $I_1$. Thus

$SxS \subseteq Sx_1S$ and $|x| = |x_1|$. Hence $x = g \cdot x_1$ by Lemma 4.2 and so $SxS = Sx_1$. In particular, $G$ is transitive on $X$.

Suppose now that for all $m \leq n$ we have proved that $Sx_mS$ is equal to $SxS$ where $x$ is any string of length $m$. We now prove the same for $I_{n+1}$. We prove first that $x_{n+1}$ has length $n+1$. It cannot have length $n$ or less thus we can write it as $x_{n+1} = xyz$ where $x$ has length $n$, and $y$ has length one. Then $Sx_{n+1}S \subseteq SxyS$. If they are not equal, then $Sxy$ has got to equal one of the earlier ideals in the chain. But that would mean $xy$ would have length at most $n$ which is a contradiction. Thus $Sx_{n+1}S = SxyS$. It follows that $x_{n+1}$ has length $n+1$. Now let $SxS$ be any ideal where $x$ has length $n+1$. It cannot be equal to any of the earlier ideals and so it is equal either to $I_{n+1}$ or to a later ideal in the chain. In any event, $SxS \subseteq Sx_{n+1}S$. But $x$ and $x_{n+1}$ have the same length and so by Lemma 4.2, we have that $x = g \cdot x_{n+1}$. It follows that $I_{n+1}$ is generated by any element of $X^n$ and that $G$ acts transtively on $X^n$.■

We may summarise as follows.

**Theorem 4.5** *Left cancellative right hereditary monoids satisfying the dedekind height property in which the principal two-sided ideals form a descending chain are in correspondence with level transitive self-similar group actions.* ■

For each $x \in X^*$ denote by $G_x$ the stabiliser of $x$ in $G$ under the action. Put $M' = \bigcap_{x \in X^*} G_x$, a subgroup of $G$. The action is *faithful* iff $M'$ is the identity. Define

$$M = \{g \in G\colon\, gs \in sG \text{ for all } s \in S\}.$$

The definition of $M$ is due to Rees [22].

**Lemma 4.6** *With the above definitions, $M = M'$.*

**Proof** Let $g \in M'$. Thus $g \cdot x = x$ for all $x \in X^*$. Let $s \in S$. Then $s = xh$. Thus $gs = gxh = (g \cdot x)g|_x h = xg|_x h = xh(h^{-1}g|_x h) = s(h^{-1}g|_x h)$. It follows that $g \in M$. To prove the reverse inclusion, let $g \in M$. Let $x \in X^*$. Then $gx = xh$ but $gx = (g \cdot x)g|_x$. Thus $x = g \cdot x$ by uniqueness. Hence $g \in M'$. ■

The subgroup $M$ is the greatest 'right normal divisor' of $S$. Monoids for which $M = \{1\}$ we shall call *fundamental*.

**Corollary 4.7** *A left cancellative right hereditary monoid satisfying the dedekind height property is fundamental if and only if its associated group action is faithful.* ■

Rees [22] shows that the fundamental monoids are the building blocks for the arbitrary ones.

**Remark** It follows by Corollary 4.7 that there is a correspondence between fundamental left cancellative right semihereditary monoid satisfying the dedekind height property and faithful self-similar group actions of [1, 19].

Green's relation $\mathcal{R}$ is defined by $a \,\mathcal{R}\, b$ iff $aS = bS$. In our case, this is equivalent to the existence of a unit $g$ such that $a = gb$. This relation is always a left congruence.

**Proposition 4.8** *The relation $\mathcal{R}$ is a right congruence, and so a congruence, if and only if the action of $G$ on $X^*$ is trivial.*

**Proof** Suppose that $\mathcal{R}$ is a right congruence. Let $g \in G$ and $x \in X$. Then $gS = S = 1S$ and so $g \,\mathcal{R}\, 1$. By assumption, $gx \,\mathcal{R}\, x$. Thus $gxS = xS$. There is therefore a unit $h$ such that $gx = xh$. But $gx = (g \cdot x)g|_x$. By uniqueness, $x = g \cdot x$, and so the action is trivial.

Suppose now that the action is trivial. Let $a \,\mathcal{R}\, b$. We prove that $ac \,\mathcal{R}\, bc$. Let $a = bg$ and $c = yh$. Then

$$ac = bgc = bgyh = b(g \cdot y)g|_y h = byg|_y h = byh(h^{-1}g|_y h) = bc(h^{-1}g|_y h)$$

and so $ac \,\mathcal{R}\, bc$, as required. ■

**Remark** Suppose that the action is trivial. Then the multiplication in $X^* \bowtie G$ is given by

$$(x, g)(y, h) = (xy, g|_y h).$$

Thus there is a surjective homomorphism from $X^* \bowtie G$ onto $X^*$. For each $x \in X^*$ define $\phi_x \colon G \to G$ by $\phi_x(g) = g|_x$. Then $\phi_1$ is the identity function, $\phi_{xy} = \phi_y \phi_x$, and $\phi_x(gh) = \phi_x(g)\phi_x(h)$. It follows that the multiplication is determined by $\phi$. This is essentially the struture theorem described in Proposition 6 of [20]. However, there it is claimed to give the structure of all left cancellative right hereditary monoids satisfying the dedekind height property: this is wrong. It gives the structure only of those monoids in the class where $\mathcal{R}$ is a congruence. As a special case, it gives the structure of left cancellative $\omega$-monoids (see Section 2), since in that case the action is automatically trivial. That the action need not be trivial is immediate from the many examples to be found in, say. [19]. But here is a different example. Centre the Sierpinski gasket at the origin, and consider the monoid $S$ of all similarities of the plane that map the gasket into itself. I shall now pick out certain important elements of $S$: a clockwise rotation by $\frac{2\pi}{3}$ denoted by $\rho$; a reflection in the vertical denoted by $\sigma$; and three similarities denoted $T$, $L$ and $R$ which halve the size of the gasket and then translate it to the top, left and right parts of the original gasket. It is not hard to see that the monoid generated by these similarities is $S$, and that the group of units is the symmetry group of the equilateral triangle. The submonoid of $S$ generated by $T$, $L$ and $R$ is the free monoid on three generators. Simple calculations show that

$$\rho T = R\rho, \quad \rho L = T\rho, \quad \rho R = L\rho$$

and

$$\sigma T = T\sigma, \quad \sigma L = R\sigma, \quad \sigma R = L\sigma.$$

Thus the monoid $S$ is a Zappa-Szép product and the group action is non-trivial.

Let $G$ and $X^*$ be a self-similar group action. If $x \in G$ then as usual $G_x$ is the stabiliser of $x$ in $G$ and so a subgroup of $G$. The proofs of the following are straightforward.

**Lemma 4.9**

(i) *The function $\phi_x \colon G_x \to G$ given by $g \mapsto g|_x$ is a homomorphism.*

(ii) *Let $y = g \cdot x$. Then $G_y = gG_xg^{-1}$ and*

$$\phi_y(h) = g|_y \phi_x(g^{-1}hg)(g|_x)^{-1}.$$

■

**Lemma 4.10**

(i) *If $\phi_x$ is injective then $\phi_{g \cdot x}$ is injective.*

(ii) *If $\phi_x$ is injective for all $x \in X$ then $\phi_y$ is injective for all $y \in X^*$.*

**Proof** (i) This is immediate by Lemma 4.9.

(ii) We prove the result by induction on the length of $y$. The result is true for strings of length one by assumption. We assume the result true for strings of length $n$. We now prove it for strings of length $n+1$. Let $y$ be of length $n+1$. Thus $y = zx$ where $z$ has length $n$ and $x$ has length one. We prove that $\phi_y$ is injective on $G_y$. Let $h, k \in G_y$. Thus $h \cdot y = y = k \cdot y$. It follows that $h \cdot z = z = k \cdot z$ and $h|_z \cdot x = x = k|_z \cdot x$. Suppose that $\phi_y(h) = \phi_y(k)$. Thus $h|_y = k|_y$. By axiom (SS6), we have that $(h|_z)|_x = (k|_z)|_x$. Now observe that $h|_z, k|_z \in G_x$. Thus by injectivity $h|_z = k|_z$. Now observe that $h, k \in G_z$. Thus by injectivity $h = k$, as required. ■

**Proposition 4.11** *The associated monoid $S$ is right cancellative (and so cancellative) iff all $\phi_x$ are injective.*

**Proof** We suppose first that all the $\phi_x$ are injective. Suppose that $ab = cb$ where $a = wg$, $b = yk$ and $c = zl$. We therefore get $w(g \cdot y) = z(l \cdot y)$ and $g|_y k = l|_y k$. Since the action is length-preserving and by uniqueness we have that $g \cdot y = l \cdot y$ and $g|_y = l|_y$. Our result will follow if we can show that $g = l$. Observe that $g^{-1}l \in G_y$. Thus $\phi_y(g^{-1}l)$ is defined and equals $(g^{-1}l)|_y$. But this quickly reduces to $(g^{-1}g)|_y = 1$. Thus $g = l$, as required.

Conversely, suppose that $S$ is right cancellative. We prove that $\phi_x$ is injective. Let $g, h \in G_x$ and suppose that $\phi_x(g) = \phi_x(h)$. Thus $g|_x = h|_x$. Now $gx = (g \cdot x)g|_x = xg|_x$, and $hx = (h \cdot x)h|_x = xh|_x$. Thus $gx = hx$. By right cancellation, $g = h$, and so $\phi_x$ is injective. ■

**Remark** If the action of $G$ on $X$ is transitive then we need only know that one of the maps $\phi_x$ for a one letter string $x$ is injective to know that the associated monoid is right cancellative. Also $|X| = |G : G_x|$. Thus if $|X|$ is finite the homomorphism $\phi_x \colon G_x \to G$ is a virtual endomorphism.

Following [19], we say that a self-similar group action is *recurrent* if $G$ is transitive on $X$, and $\phi_x$ is onto for any $x \in X$.

**Proposition 4.12** *The self-similar action is recurrent iff the associated monoid has a maximum proper principal left ideal and a maximum proper principal two-sided ideal.*

**Proof** Suppose that the action is recurrent. Since the action is therefore transitive the monoid has a maximum proper principal two-sided ideal. Let $xg$ and $yh$ be any two elements such that $x, y \in X$. Because the action is transitive there exists $k \in G$ such that $k \cdot x = y$. Thus $k(xg) = (k \cdot x)k|_x g$. By assumption, $\phi_{k \cdot x} \colon G_{k \cdot x} \to G$ is onto. Thus there exists $p \in G_{k \cdot x}$ such that $\phi_{k \cdot x}(p) = hg^{-1}(k|_x)^{-1}$. Hence $(pk)(xg) = yh$. In particular, $Sxg = Syh$. Put $L = Sxg$. Let $Sx_1 \ldots x_n g'$ be any principal left ideal. Then $Sx_1 \ldots x_n g' \subseteq Sx_n g' = L$. Thus $L$ is a maximum proper principal left ideal.

Conversely, suppose that the associated monoid has a maximum proper principal left ideal and a maximum proper principal two-sided ideal. By Proposition 4.3, the action is transitive. Let $L = Sxg$ be the maximum proper principal left ideal. We claim that for any $y \in X$ and $h \in G$ we have that $L = Syh$. Clearly $Syh \subset L$. Thus $yh = a(xg)$. Let $a = zk$. Then $yh = z(k \cdot x)k|_x g$. Thus $z$ is the empty string (comparing lengths). It follows that $Syh = Sxg = L$. It remains to show that $\phi_x$ is onto for any $x \in X$. Let $g \in G$. Then $Sxg^{-1}$ is a proper principal left ideal, as is $Sx$. By the result above $Sxg^{-1} = Sx$. Thus $x = h(xg^{-1})$ for some $h \in G$. Thus $x = (h \cdot x)h|_x g^{-1}$. It follows that $h \cdot x = x$ and $h|_x = g$. Hence $h \in G_x$ and $\phi_x(h) = g$. ■

# 5 Covering bimodules and tensor monoids

In Chapter 2 of [19], the algebraic properties of self-similar group actions are handled using 'covering bimodules'. In this section, we show how to construct the monoid associated with the self-similar group action from the covering bimodule.

Let $X$ be a set and $S$ and $T$ monoids. We say that $X$ is a $(S, T)$*-biact* if $X$ is a left $S$-act, a right $T$-act and if $(sx)t = s(xt)$ for all $s \in S$, $t \in T$ and $x \in X$.[1] If $X$ and $Y$ are $(S, T)$-biacts then a function $\theta \colon X \to Y$ is called a

[1] In [19], the term 'commuting' is used for the last condition, which is misleading.

*bihomomorphism* if $\theta(sxt) = s\theta(x)t$ for all $s \in S$, $t \in T$ and $x \in X$. We shall be interested in biacts where both acting monoids are the same and are groups.

Let $S$ be a monoid with group of units $G$. Then under left and right multiplication $S$ is also a $(G, G)$-biact.

**Lemma 5.1** *Let $S$ be a left cancellative right hereditary monoid satisfying the dedekind height property. Let $M$ be the set of generators of the maximal proper principal right ideals of $S$. Then $M$ is a $(G, G)$-biact under left and right multiplication by $G$, and the right $G$-action is free.*

**Proof** Let $x$ be a generator of a maximal proper principal right ideal. Then $xS = xgS$ and so $xg$ is a generator of a maximal proper principal right ideal. Consider now $gx$. We prove that $gxS$ is a maximal proper principal ideal. If it is not maximal then there is a maximal proper principal right ideal $yS$ such that $gxS \subseteq yS$. Thus $xS \subseteq g^{-1}yS$. Now $xS$ is maximal and so either $g^{-1}yS = S$ or $gxS = yS$. The former cannot occur because $y$ is not invertible. Thus $gxS = yS$. Thus $gx$ is also a generator of a maximal proper principal right ideal. Observe that by left cancellation, the right $G$-action is free. ∎

**Remark** Let $S = X^* \bowtie G$. In this case, the set $M$ is $M = X \times G$. Observe that

$$(1, h)(x, g) = (h \cdot x, h|_x g) \text{ and } (x, g)(1, h) = (x, gh).$$

Thus if we define left and right actions by $G$ on $M$ as follows: $G \times M \to M$ is given by $h(x, g) = (h \cdot x, h|_x g)$, and $M \times G \to M$ is given by $(x, g)h = (x, gh)$ then we get a $(G, G)$-biact. In [19], biacts such as this are called 'covering bimodules'.

We define a *covering biact* to be a $(G, G)$-biact $M$ where the righthand action is free. In Lemma 5.1, we showed how to construct a covering biact $M$ from a left cancellative right hereditary monoid satisfying the dedekind height condition.

We shall now investigate the relationship between the original monoid $S$ and the covering biact $M$ constructed from it. It is convenient to assume that $S = X^*G$, uniquely. In this case, $M = XG$. Define $\iota$: $M \to S$ by $\iota(xg) = xg$. Recall that $S$ is a $(G, G)$-biact for left and right multiplication by $G$. The function $\iota$ is a $(G, G)$-bihomomorphism: this is simply because $M$ is a $(G, G)$-subact of $S$. The relationship between $M$ and $S$ is characterised by the following theorem.

**Theorem 5.2** *Let $S$ be a left cancellative right hereditary monoid satisfying the dedekind height property and with group of units $G$. Let $M$ be the covering biact associated with $S$. Let $T$ be a monoid with group of units $G$. Let $\alpha$: $M \to T$ be a $(G, G)$-bihomomorphism. Then there is a unique monoid homomorphism $\bar{\alpha}$: $S \to T$ such that $\alpha = \bar{\alpha}\iota$ and which is the identity on the group of units of $S$.*

**Proof** Define $\bar{\alpha}$ by

$$\bar{\alpha}(x_1 \ldots x_n g) = \alpha(x_1) \ldots \alpha(x_n) g.$$

It is clear that $\bar{\alpha}\iota = \alpha$. We need to prove that $\bar{\alpha}$ is a homomorphism. Let $\mathbf{x} = x_1 \ldots x_m g$ and $\mathbf{y} = y_1 \ldots y_n h$ be elements of $S$. Their product is

$$x_1 \ldots x_m (g \cdot y_1)(g|_{y_1} \cdot y_2) \ldots (g|_{y_1 \ldots y_i} \cdot y_{i+1}) \ldots (g|_{y_1 \ldots y_{n-1}} \cdot y_n) g|_{y_1 \ldots y_n} h.$$

We now calculate $\bar{\alpha}(\mathbf{xy})$. This is equal to

$$\alpha(x_1) \ldots \alpha(x_m)\alpha(g \cdot y_1)\alpha(g|_{y_1} \cdot y_2) \ldots \alpha(g|_{y_1 \ldots y_i} \cdot y_{i+1}) \ldots \alpha(g|_{y_1 \ldots y_{n-1}} \cdot y_n)\alpha(1, g|_{y_1 \ldots y_n} h).$$

We shall now use the fact that $\alpha$ is a bihomomorphism. Consider

$$\alpha(g \cdot y_1)\alpha(g|_{y_1} \cdot y_2).$$

We write this as

$$\alpha(g \cdot y_1) g|_{y_1} (g|_{y_1})^{-1} \alpha(g|_{y_1} \cdot y_2)$$

and now use the fact that $\alpha$ is a bihomomorphism and that

$$(g|_{y_1})^{-1}|_{g|_{y_1} \cdot y_2} = (g|_{y_1 y_2})^{-1}$$

by (SS1) and (SS7) to get

$$\alpha((g \cdot y_1) g|_{y_1})\alpha(y_2 (g|_{y_1 y_2})^{-1})$$

which is equal to

$$g\alpha(y_1)\alpha(y_2)(g|_{y_1 y_2})^{-1}.$$

We now consider the remaining product

$$(g|_{y_1 y_2})^{-1}\alpha(g|_{y_1 y_2} \cdot y_3) \ldots \alpha(g|_{y_1 \ldots y_i} \cdot y_{i+1}) \ldots \alpha(g|_{y_1 \ldots y_{n-1}} \cdot y_n)\alpha(g|_{y_1 \ldots y_n} h).$$

We now push the leftmost group element through the product using the fact that

$$(g|_{y_1 \ldots y_i})^{-1}|_{g|_{y_1 \ldots y_i} \cdot y_{i+1}} = (g|_{y_1 \ldots y_{i+1}})^{-1}.$$

The last term is

$$(g|_{y_1 \ldots y_n})^{-1}\alpha(g|_{y_1 \ldots y_n} h) = h.$$

It follows that

$$\bar{\alpha}(\mathbf{xy}) = \bar{\alpha}(\mathbf{x})\bar{\alpha}(\mathbf{y}).$$

It remains to prove uniqueness. Let $\alpha' \colon S \to T$ be another monoid homomorphism such that $\alpha'\iota = \alpha$. Then $\bar{\alpha}(xg) = \alpha'(xg)$ for all $x \in X$ and $g \in G$. By definition $\bar{\alpha}(x_1 \ldots x_n g) = \alpha(x_1) \ldots \alpha(x_n) g$. By assumption this is equal to $\alpha'(x_1) \ldots \alpha'(x_n) g$. But $\alpha'$ is a homomorphism and so this is equal to $\alpha'(x_1 \ldots x_n g)$, as required. ■

Let $M$ be an arbitrary covering $(G, G)$-biact. We may form the tensor product $M \otimes M$ whose elements we denote by $x \otimes y$. The bihomomorphism $\otimes \colon M \times M \to M \otimes M$ has the property that $xg \otimes y = x \otimes gy$; such maps are called *bimaps*. The tensor product is the universal such bimap. Observe that

$a \otimes b = c \otimes d$ iff $a = cg$ and $b = g^{-1}d$. The theory of tensor products of monoid acts is described in [10].

Define $M^{\otimes 0} = G$ and $M^{\otimes n} = M^{\otimes n-1} \otimes M$. For $p, q > 0$ there are isomorphisms $\phi_{p,q}$: $M^{\otimes p} \otimes M^{\otimes q} \to M^{\otimes p+q}$ which map $(u, v)$ to $u \otimes v$. Observe that all tensor products are free right $G$-acts. Put $S = \bigcup_{n=0}^{\infty} M^{\otimes n}$. There is the obvious embedding $\iota: M \to S$. The $(G, G)$-biact $S$ becomes a monoid under tensor products and left and right actions by $G$: we use the isomorphisms above to define the multiplication. We call $S$ the *tensor monoid* of the $(G, G)$-biact $M$ by analogy with the tensor algebra of a module [15].

More informally, the elements of $S$ can be regarded as the elements of $G$ together with all formal products $x_1 \otimes \ldots \otimes x_n$ where $x_i \in M$. The product of two formal products $u$ and $v$ is just the formal product $u \otimes v$ and the product of $g \in G$ and a formal product $x_1 \otimes \ldots \otimes x_n$ is given by $g(x_1 \otimes \ldots \otimes x_n) = gx_1 \otimes \ldots \otimes x_n$ and $(x_1 \otimes \ldots \otimes x_n)g = x_1 \otimes \ldots \otimes x_n g$.

**Lemma 5.3** *The tensor monoid of a covering $(G, G)$-biact is left cancellative, right hereditary and satisfies the dedekind height property.*

**Proof** The group of units of $S$ is $G$, and there is a surjective homomorphism from $S$ to $\mathbb{N}$, in which the inverse image of 0 is $G$. Thus $S$ is equipped with a length function. Because the action is free on the right, it is easy to check that if $\mathbf{x} \otimes \mathbf{y} = \mathbf{x} \otimes \mathbf{y}'$ in $S$ then $\mathbf{y} = \mathbf{y}'$ and so $S$ is left cancellative (this works because lengths match). Thus $S$ is left cancellative. We finish off by showing that $S$ is equidivisible (see Lemma 2.7) Suppose that $\mathbf{x} \otimes \mathbf{u} = \mathbf{y} \otimes \mathbf{v}$. There are three cases to consider depending on the relative lengths of $\mathbf{x}$ and $\mathbf{y}$. We shall just consider the case where the length of $\mathbf{x}$ is $m$, that of $\mathbf{y}$ is $n$ and where $m < n$. We therefore suppose that $\mathbf{u} = \mathbf{w} \otimes \mathbf{z}$ and that $\mathbf{x} \otimes \mathbf{w}$ has the same length as $\mathbf{y}$. Thus $\mathbf{x} \otimes \mathbf{w} = \mathbf{y}g$ and $\mathbf{z} = g^{-1}\mathbf{v}$. Thus $\mathbf{y} = \mathbf{x} \otimes (\mathbf{w}g^{-1})$. Once the argument is completed by the other two cases, it will follow that $S$ is equidivisible. ■

The following theorem can be proved using the universal properties of tensor products.

**Theorem 5.4** *Let $S$ be the tensor monoid of the covering $(G, G)$-biact $M$. Let $T$ be any monoid with group of units $G$, and let $\alpha$: $M \to T$ be a bihomomorphism. Then there is a unique monoid homomorphism $\bar{\alpha}$: $S \to T$ such that $\alpha = \bar{\alpha}\iota$ and which is the identity on the group of units.* ■

The results of this section can be placed in a categorical framework. We fix a group $G$ and consider the category whose objects are the left cancellative right hereditary monoids satisfying the dedekind height property with $G$ as their groups of units and whose morphisms are the monoid homomorphisms which are the identity on the groups of units and which map generators of maximal proper principal right ideals to generators of maximal proper principal right ideals. There is then a forgetful functor from this category to the category whose objects are the covering $(G, G)$-biacts and whose morphisms are the $(G, G)$-bihomomorphisms: associate with a monoid its covering biact. This functor has a left adjoint which associates with a covering biact its tensor monoid.

# 6 Categories and automata

In this section, we shall look at some further interpretations of self-similar group actions involving categories and automata. We first adapt to our setting some of the ideas to be found in [8] where full definitions can be found if required. Let $G$ be a group with a self-similar group action on $X^*$. We define a double category as follows. Its elements are squares of the form

$$\begin{tikzcd} \cdot \arrow[r, "g\cdot x"] \arrow[d, "g"'] & \cdot \arrow[d, "g|_x"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

We define *horizontal multiplication* as follows: let

$$\begin{tikzcd} \cdot \arrow[r, "h\cdot y"] \arrow[d, "h"'] & \cdot \arrow[d, "h|_y"] \\ \cdot \arrow[r, "y"'] & \cdot \end{tikzcd}$$

be another square such that $g|_x = h$. Then their product is

$$\begin{tikzcd} \cdot \arrow[r, "(g\cdot x)(h\cdot y)"] \arrow[d, "g"'] & \cdot \arrow[d, "h|_y"] \\ \cdot \arrow[r, "xy"'] & \cdot \end{tikzcd}$$

This is well-defined because $g \cdot (xy) = (g \cdot x)(g|_x \cdot y) = (g \cdot x)(h \cdot y)$, and $h|_y = (g|_x)|_y = g|_{xy}$. We define *vertical multiplication* as follows: we suppose now that $x = h \cdot y$. Then their product is

$$\begin{tikzcd} \cdot \arrow[r, "g\cdot x"] \arrow[d, "gh"'] & \cdot \arrow[d, "g|_x h|_y"] \\ \cdot \arrow[r, "y"'] & \cdot \end{tikzcd}$$

This is well-defined because $(gh) \cdot y = g \cdot (h \cdot y) = g \cdot x$, and $(gh)|_y = g|_{h\cdot y}h|_y = g|_x h|_y$. It is easy to check that the interchange law holds, so we have defined a double category from a self-similar group action. This double category has the vertical structure of a group and the horizontal structure of a free monoid. The following emulates Proposition 2.4 of [8].

**Proposition 6.1** *Let* **B** *be a double category in which the vertical structure is a group $G$, the horizontal structure is a free monoid $X^*$ such that the* star condition *holds: every pair*

$$\begin{tikzcd} \cdot \arrow[d, "g"'] & \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

*can be uniquely completed to a square*

$$\begin{tikzcd} \cdot \arrow[r, "g\cdot x"] \arrow[d, "g"'] & \cdot \arrow[d, "g|_x"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

*where $g \cdot x$ and $g|_x$ denote uniquely defined elements. Then there is a self-similar group action of $G$ on $X^*$.*

**Proof** From horizontal multiplication we get that (SS6) and (SS8) hold, from vertical multiplication we get that (SS2) and (SS7) hold. The remaining four axioms hold by considering the horizontal and vertical morphisms in the double category: squares of the form

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d, "1"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

are the horizontal morphisms and imply that axioms (SS1) and (SS4) hold, squares of the form

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "g"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

are the vertical morphisms and imply that axioms (SS3) and (SS5) hold. ■

With each double category can be associated a bisimplicial complex. The diagonal of this bisimplicial set is a simplicial set which is actually the nerve of a category. In our case, this category is a monoid: its elements are diagrams of the form

$$\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd}$$

and the product with

$$\begin{tikzcd} \cdot \arrow[r, "y"] & \cdot \arrow[d, "h"] \\ & \cdot \end{tikzcd}$$

is given by

$$
\begin{tikzcd}
{} \arrow[r, "x"] & {} \arrow[r, dash, dashed, "g\cdot y"] \arrow[d, "g"] & {} \arrow[d, dash, dashed, "g|_y"] \\
& {} \arrow[r, "y"] & {} \arrow[d, "h"] \\
& & {}
\end{tikzcd}
$$

using the star condition of Proposition 6.1 and so is just

$$
\begin{tikzcd}
{} \arrow[r, "x(g\cdot y)"] & {} \arrow[d, "g|_y h"] \\
& {}
\end{tikzcd}
$$

This monoid is just the monoid associated with the self-similar group action. The argument of Proposition 2.6 of [8] therefore yields the following result.

**Proposition 6.2** *Let a self-similar group action be given. Then the classifying space of the double category associated with the action is canonically homotopically equivalent to the classifying space of the monoid associated with the action.* ■

We now turn to automata. A *(non-initial) (Mealy) machine* $\mathbf{A} = (S, X, |, \cdot)$ consists of the following information: a set of states $S$, an input/output alphabet $X$, a state transition function $S \times X \to S$, denoted by $(s, x) \mapsto s|_x$, where $x \in X$ and an output function $S \times X \to X$, denoted by $(s, x) \mapsto s \cdot x$, where $x \in X$. Machines are defined to process input and output *letters*, but can easily be extended to process input and output strings in the following way. First, state transitions for strings are defined by

- $s|_1 = s$ for all states $s$.
- $s|_{ax} = (s|_a)|_x$ where $a$ is a letter and $x$ a string.

Second, outputs are defined for strings by

- $s \cdot 1 = 1$.
- $s \cdot (ax) = (s \cdot a)(s|_a \cdot x)$ where $a$ is a letter and $x$ a string.

Observe that these conditions are actually axioms (SS5), (SS6), (SS3), and (SS8).

A function $\theta$: $\mathbf{A} \to \mathbf{B}$ is a *homomorphism* of Mealy machines if $\theta$: $S \to T$, $\theta(s|_x) = \theta(s)|_x$, and $s \cdot x = \theta(s) \cdot x$. The composition of homomorphisms is a homomorphism and the identity function on a set of states is the identity function on the Mealy machine. An *isomorphism* is just a bijective homomorphism. We therefore have a category $\mathcal{A}$ whose objects are Mealy machines (over the same input/output alphabet $X$) and whose morphisms are the homomorphisms of Mealy machines. This category is endowed with extra structure which we now describe and exploit later. We denote by $\mathbf{I}$ the machine with one state, and which simply outputs the input. Given two machines $\mathbf{A}$, with set of states $S$, and $\mathbf{B}$, with set of states $T$, we define a new machine $\mathbf{A} * \mathbf{B}$ as follows: the set of states is $S \times T$; the input/output alphabet is $X$; the transition function is given by $(s,t)|_x = (s|_{t \cdot x}, t|_x)$; and the output function is given by $(s,t) \cdot x = s \cdot (t \cdot x)$. Intuitively, this machine is constructed by taking the output of $\mathbf{B}$ and using it as the input to $\mathbf{A}$. We call it the *cascade product of* $\mathbf{A}$ *and* $\mathbf{B}$. Observe that underlying the construction of $\mathbf{A} * \mathbf{B}$ is the product of sets, and that $\mathbf{I}$ has as underlying set the one-element set. Now the category of sets is a monoidal category with respect to products of sets and the one-element set as unit. Our homomorphisms are simply set functions satisfying certain algebraic conditions. We therefore have the following.

**Theorem 6.3** *The category $\mathcal{A}$ of Mealy machines over a fixed input/output alphabet $X$ is a monoidal category with respect to cascade product.* ■

For each state $s \in S$, there is an *initial Mealy machine* $\mathbf{A}_s$ where $s$ is the distinguished *initial state*. An initial Mealy machine $\mathbf{A}_s$ computes a function from $X^*$ to itself which maps $x \in X^*$ to $s \cdot x$. A homomorphism between initial Mealy machines is required to map initial states to initial states. It can be shown that if there is a homomorphism between two initial Mealy machines then they compute the same function. If $\mathbf{A}_s$ is an initial Mealy machine computing $f\colon X^* \to X^*$ and $\mathbb{B}_t$ is an initial Mealy machine computing $g\colon X^* \to X^*$ then $\mathbf{A}_s * \mathbf{B}_t$ computes $fg$, composing from right-to-left.

Suppose that we have a self-similar group action. Then we have a Mealy machine $\mathbf{A}(G) = (G, X, |, \cdot)$. Denote by $\mu\colon G \times G \to G$ the multiplication operation in $G$.

**Lemma 6.4** *The function $\mu$ is a homomorphism of Mealy machines $\mathbf{A} * \mathbf{A} \to \mathbf{A}$. In addition, there is a homomorphism $\eta\colon \mathbf{I} \to \mathbf{A}$ which maps the single state of $\mathbf{I}$ to the identity of $\mathbf{A}$.*

**Proof** Let $(g,h)$ be a state in $\mathbf{A} * \mathbf{A}$ Let $x$ be an input letter. Then by definition $(g,h)|_x = (g|_{h \cdot x}, h|_x)$. Thus $\mu((g,h)|_x) = (g|_{h \cdot x})(h|_x)$. On the other hand, $\mu(g,h)|_x = (gh)|_x$. These two are equal by axiom (SS7).

Let $(g,h)$ be a state and $x$ and input letter. Then $(g,h) \cdot x = g \cdot (h \cdot x)$ and $\mu(g,h) \cdot x = (gh) \cdot x$. These two are equal by axiom (SS2).

The fact that $\eta$ is a homomorphism follows by axioms (SS1) and (SS4). ■

**Remarks**

1. It follows from the fact that $\eta\colon \mathbf{I} \to \mathbf{A}$ is a homomorphism that $\mathbf{A}_1$ computes the identity function.

2. Lemma 6.4 shows that a self-similar group action gives rise to a *monoid in the monoidal category* $(\mathcal{A}, *, I)$.

3. The elements of the double category we constructed at the beginning of this section have an automata-theoretic interpretation: $(g, x)$ represents current state and new input whilst $(g|_x, g \cdot x)$ represents new state and output. Such square notation is used in [2].

# 7 An associated inverse monoid

The monoids associated with self-similar group actions can also be used to construct a class of inverse monoids. This is a well-known procedure so I shall simply sketch out the theory as it applies to our case. For more details see [13]. For all undefined terms from inverse semigroup theory see [12].

Let $S$ be a left cancellative, right hereditary monoid satisfying the dedekind height property. The inverse monoid $B(S)$ of all $S$-isomorphisms between the principal right ideals of $S$ together with the empty function is a 0-bisimple inverse monoid. There is a useful isomorphic representation of $B(S)$. Define an equivalence relation on the set of nonzero ordered pairs of elements of $S$ by $(a, b)$ is equivalent to $(au, bu)$ for all units $u \in S$. Denote by $[a, b]$ the equivalence class containing $(a, b)$. Consider now the set of all such equivalence classes together with a zero element. Define $[d, c][b, a]$ to be zero if $cS \cap bS$ is empty. If $cS \cap bS$ is not empty there are two possibilities. If $c = bs$ for some $s$ then we define the product to be $[d, as]$. If $b = cs$ for some $s$ then we define the product to be $[ds, a]$. It can be proved that the resulting structure is isomorphic to $B(S)$ and, from now on, we shall treat $B(S)$ in this way. The non-zero idempotents of $B(S)$ are the elements $[a, a]$. The natural partial order is given by $[a, b] \leq [c, d]$ iff $(a, b) = (c, d)p$ for some $p \in S$. The idempotent structure of $B(S)$ is isomorphic to the semilattice of principal right ideals of $S$ together with the empty set. It follows that if $e$ and $f$ are idempotents of $B(S)$ and $ef \neq 0$ then $e$ and $f$ are comparable with respect to the natural partial order. The identity of $B(S)$ is $[1, 1]$ and the $\mathcal{L}$-class of the identity consists of elements of the form $[a, 1]$ and forms a left cancellative monoid isomorphic to $S$.

It follows from the general theory of 0-bisimple inverse semigroups that there is a correspondence between the following two classes of monoids:

- Left cancellative, right hereditary monoids satisfying the dedekind height property.

- 0-bisimple inverse monoids with two properties: first, if $e$ and $f$ are idempotents and $ef \neq 0$ then $e$ and $f$ are comparable and second, there are only a finite number of idempotents above any non-zero idempotent.

Under this correspondence, fundamental monoids of the first class correspond to fundamental inverse monoids of the second. I shall call the inverse monoids that arise in this way the *associated inverse monoids*.

If $S$ is an inverse semigroup with zero, then $S^* = S \setminus \{0\}$. A *prehomomorphism* $\theta$ from an inverse semigroup $S$ to an inverse semigroup $T$ is a function $\theta$: $S^* \to T^*$ such that $ab \neq 0$ implies that $\theta(ab) = \theta(a)\theta(b)$. An inverse monoid is said to be *strongly* $E^*$*-unitary* if it admits a prehomomorphism to a group such that the inverse image of the identity consists only of idempotents. The associated inverse monoid is strongly $E^*$-unitary if and only if the associated monoid is cancellative [13].

The set of idempotents of an inverse semigroup is said to be 0-*disjunctive* if whenever $0 < e < f$ then there exists a nonzero idempotent $g$ such that $g \leq f$ and $ge = 0$.

**Lemma 7.1** *If $|X| > 1$, then the set of idempotents of an associated inverse monoid is 0-disjunctive.*

**Proof** Let $0 < [a, a] < [b, b]$ in $B(S)$. Then $a = bp$ in $S$. Let $a = xg$, $b = yh$ and $p = zk$. Then $xg = y(h \cdot z)h|_z k$. Thus by uniqueness $x = y(h \cdot z)$ and $g = h|_z k$. If $x = y$ then $h \cdot z$ is the identity and so $z$ would be the empty string. This would imply that $[a, a] = [b, b]$. It follows that $y$ is a proper prefix of $x$. Let $x = yw$ where $w$ has length at least one. Let $q \in X$ different from the first letter of $w$. Put $c = yq$. Then $0 < [c, c] \leq [b, b]$, and $[a, a][c, c] = 0$ by construction.■

Fundamental 0-bisimple inverse monoids with a 0-disjunctive set of indempotents are *congruence-free* (see page 181 of [21]). It follows by Corollary 4.7, that the inverse monoids associated with faithful self-similar group actions on free monoids with at least two letters are congruence-free.

It is possible to write the elements of $B(S)$ in a more straightforward form. If $x \in X^*$ then $x^{-1}$ denotes the *reverse string of* $x$. Observe that $(xy)^{-1} = y^{-1}x^{-1}$. If $z = xy$ then we define $x^{-1}z = y$. We can identity the nonzero elements of $B(S)$ with the formal products $xgy^{-1}$. The product of $xgy^{-1}$ and $whz^{-1}$ is then: zero if neither $y$ nor $w$ is a prefix of the other; $x(g \cdot p)g|_p hz^{-1}$ if $w = yp$; and $xg(h|_{h^{-1} \cdot p})(h^{-1} \cdot p)^{-1}z^{-1}$ if $y = wp$. Monoids of the form $B(X^*)$ are called the *polycyclic monoids* [20]. The form of the elements and product in $B(S)$ just described generalises the usual way in which polycylic monoids are described.

Finally, the inverse monoids defined here lead to Cuntz-Pimsner algebras in the same way that the polycyclic monoids lead to Cuntz algebras. See Section 13.2 of [1]

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