Constructing ordered groupoids

Mark V. Lawson
Department of Mathematics
University of Wales, Bangor
Dean Street
Bangor, Gwynedd LL57 1UT
United Kingdom

September 7, 2004

Abstract

We prove that every ordered groupoid is isomorphic to one constructed from a category acting in a suitable fashion on a groupoid arising from an equivalence relation. This construction can be used to provide greater understanding of Dehornoy's structural monoid associated with a balanced variety, and of the inverse semigroups used by Girard and Abramsky in their work on linear logic and reversible computation respectively.

2000 AMS Subject Classification: 20M18, 20L05.

1 Introduction

The theory of ordered groupoids was introduced by Ehresmann as a way of formalising the theory of pseudogroups of transformations, particularly in differential geometry [5].[1] They subsequently found their way into semigroup theory: first, in the original paper of Schein, translated as [22], and then, in Nambooripad's sophisticated description of regular semigroups and their sets of idempotents in [19]. Despite Nambooripad's work, most interest in ordered groupoids within semigroup theory has centred on their applications to studying inverse semigroups. This is probably due to the fact that inverse semigroups arise naturally in a wide variety of mathematical contexts. During the 1990's, the author, starting from Ehresmann's original papers, began a systematic investigation of the role of ordered groupoids in inverse semigroup theory. This work is summarised in [10] and has come to be termed 'the ordered groupoid approach to inverse semigroups'. This approach has been substantially advanced by Ben Steinberg [23], and his work has led in turn to a 'homotopy theory of inverse semigroups' that depends crucially for its development on ordered

[1] What we call 'ordered groupoids' were termed 'functorially ordered groupoids' by Ehresmann.

groupoids [12]. In a different direction, see Nick Gilbert's paper [6] and the references there for applications of ordered groupoids to the study of combinatorial inverse semigroup theory. Recently, Ben Steinberg and the author showed that ordered groupoids could be used to construct étendues [15].

We may summarise by saying that ordered groupoids are an important tool in studying inverse semigroups, and that inverse semigroups are turning out to be natural mathematical objects; the books [21], [20] and [10] provide many examples justifying this last claim.

The aim of this paper is to describe a way of constructing ordered groupoids. This may seem a strange ambition since a groupoid equipped with an order is a simple enough combination that might appear to defy further analysis. However, the construction I shall describe grew out of concrete examples: the clause inverse semigroup introduced by Girard in [7] for applications in linear logic; my construction of inverse semigroups from category actions in [11] that was motivated by an analysis of Girard's semigroup; and Dehornoy's construction of the structural monoid of an algebraic variety defined by a set of balanced equations [4]. In the remainder of this section, I shall describe the intuitive idea behind my construction.

Let G be a groupoid. I shall denote the right identity of $g \in G$ by $\mathbf{d}(g)$ and the left identity by $\mathbf{r}(g)$. I shall also denote g by an arrow $\mathbf{d}(g) \xrightarrow{g} \mathbf{r}(g)$. The partial product will be denoted by concatenation; note that the product gh is defined iff $\mathbf{d}(g) = \mathbf{r}(h)$. The set of identities of G is denoted G_o. A groupoid G is said to be *ordered* if it is equipped with a partial order $\leq$ in such a way that the following four axioms hold:

(OG1) $x \leq y$ implies $x^{-1} \leq y^{-1}$.

(OG2) If $x \leq y$ and $u \leq v$ and xu and yv are defined then $xu \leq yv$.

(OG3) Let $e \leq \mathbf{d}(x)$ where e is an identity. Then there exists a unique element $(x \mid e)$, called the *restriction of x to e*, such that $(x \mid e) \leq x$ and $\mathbf{d}(x \mid e) = e$.

(OG3)* Let $e \leq \mathbf{r}(x)$ where e is an identity. Then there exists a unique element $(e \mid x)$, called the *corestriction of x to e*, such that $(e \mid x) \leq x$ and $\mathbf{r}(e \mid x) = e$.

In fact, axiom (OG3)* is a consequence of the other axioms; see [10]. The homomorphisms between ordered groupoids are the *ordered functors*: those functors that are also order-preserving. An ordered functor $\alpha\colon G \to H$ is an *ordered embedding* if $g \leq h$ iff $\alpha(g) \leq \alpha(h)$.

In the class of groupoids, those that arise from equivalence relations deserve to be regarded as the simplest. They can be characterised as follows: if e and f are identities then there is at most one element g such that $e \xrightarrow{g} f$. We call such groupoids *combinatorial*. Our goal is to construct ordered groupoids from combinatorial groupoids plus some other data; what this other data is will have to involve the partial order, the other ingredient in an ordered groupoid. One way of constructing partial orders is by means of preorders: if $\preceq$ is a preorder on a set X then it defines an equivalence relation $\equiv$ on X by $x \equiv y$ iff $x \preceq y$ and $y \preceq x$, and a partial order on the set of equivalence classes $X/\equiv$. Combining these two ideas, we shall try to construct arbitrary ordered groupoids from combinatorial groupoids equipped with a preorder. But this raises the question of how the preorder should be described. At the moment, the preorder $\preceq$ is just a given on the combinatorial groupoid H, however preorders can be induced by actions. Monoid actions will induce preorders on a set, but we shall need the more general notion of a category action. We now have all the ingredients we need:

- A category C acts on a combinatorial groupoid H.
- It induces a preorder $\preceq$ on H whose associated equivalence relation is $\equiv$.
- The quotient structure $H/\equiv$ is a groupoid on which the preorder induces an order.
- The groupoid $H/\equiv$ is ordered and every ordered groupoid is isomorphic to one constructed in this way.

In Sections 2 and 3, I shall show that this construction can be realised. In Section 4, I prove a representation theorem of ordered groupoids arising from my construction. This enables me to make a link with a construction of inverse semigroups described in [11]. In later paper [16], I shall show how Dehornoy's structural monoids [4] can be analysed using the constructions of this paper. In Section 5, I make some remarks of a historical nature.

Finally, I need to say a few words about the relationship between ordered groupoids and inverse semigroups. Let G be an ordered groupoid. If $x, y \in G$ are such that $e = \mathbf{d}(x) \wedge \mathbf{r}(y)$ exists, then Ehresmann defined

$$x \otimes y = (x \mid e)(e \mid y),$$

called the *pseudoproduct* of x and y. It can be proved [10] that if $x \otimes (y \otimes z)$ and $(x \otimes y) \otimes z$ are both defined, then they are equal. An ordered groupoid is said to be *inductive* if the order on the set of identities is a meet semilattice.[2] An inductive groupoid gives rise to an inverse semigroup $(G, \otimes)$ using the pseudoproduct, and every inverse semigroup arises in this way. Ordered functors between inductive groupoids that preserve the meet operation on the set of identities give rise to homomorphisms between the associated inverse semigroups. An ordered groupoid is said to be $*$-*inductive* if the following condition holds for each pair of identities: if they have a lower bound, they have a greatest lower bound. A $*$-inductive groupoid gives rise to an inverse semigroup with zero $(G^0, \otimes)$: adjoin a zero to the set G, and extend the pseudoproduct on G to G^0 in such a way that if $s, t \in G$ and $s \otimes t$ is not defined then put $s \otimes t = 0$, and define all products with 0 to be 0. Every inverse semigroup with zero arises in this way. The details of

[2] The term 'inductive' is used in inverse semigroup theory in a way quite different from that used in Ehresmann's work.

the ordered groupoid approach to inverse semigroup theory are described in [10].

Acknowledgements
I would like to thank Claas Röver of the University of Newcastle, England for referring me to Dehornoy's paper [4] which led directly to this new description of ordered groupoids.

2 Categories acting on groupoids

In this section, I shall define a class of actions of categories on combinatorial groupoids, and show that they can be used to construct ordered groupoids. The definition of what it means for a category to act (on the left) on a groupoid is essentially the one to be found in, say, [2] although I find it more convenient to express the definition in terms of actions rather than functors.

Let C be a category and G a groupoid. Let $\pi \colon G \to C_o$ be a function to the set of identities of C. Define

$$C * G = \{(a, x) \in C \times G \colon \mathbf{d}(a) = \pi(x)\}.$$

We say that C *acts on* G if there is a function from $C * G$ to G, denoted by $(a, x) \mapsto a \cdot x$, which satisfies the axioms (A1)–(A6) below. Note that I write $\exists a \cdot x$ to mean that $(a, x) \in C * G$. I shall also use $\exists$ to denote the existence of products in the categories C and G.

(A1) $\exists \pi(x) \cdot x$ and $\pi(x) \cdot x = x$.

(A2) $\exists a \cdot x$ implies that $\pi(a \cdot x) = \mathbf{r}(a)$.

(A3) $\exists a \cdot (b \cdot x)$ iff $\exists (ab) \cdot x$, and if they exist they are equal.

(A4) $\exists a \cdot x$ iff $\exists a \cdot \mathbf{d}(x)$, and if they exist then $\mathbf{d}(a \cdot x) = a \cdot \mathbf{d}(x)$;
$\exists a \cdot x$ iff $\exists a \cdot \mathbf{r}(x)$, and if they exist then $\mathbf{r}(a \cdot x) = a \cdot \mathbf{r}(x)$.

(A5) If $\pi(x) = \pi(y)$ and $\exists xy$ then $\pi(xy) = \pi(x)$.

(A6) If $\exists a \cdot (xy)$ then $\exists (a \cdot x)(a \cdot y)$ and $a \cdot (xy) = (a \cdot x)(a \cdot y)$.

We write (C, G) to indicate the fact that C acts on G.

Remarks

(1) The usual definition of a category acting on a set is a special case of the definition of a category acting on a groupoid: a set can be regarded as a groupoid in which each element is an identity. In this case, axioms (A4)–(A6) are automatic.

(2) Let C act on the groupoid G. Then C acts on the groupoid G_o. To prove this, it is enough to show that if x is an identity in G and $\exists a \cdot x$ then $a \cdot x$ is an identity in G. This follows by (A4), since $\mathbf{d}(a \cdot x) = a \cdot \mathbf{d}(x) = a \cdot x$. Thus if C acts on the groupoid G then an action of C on the set G_o is induced; this observation will be important later.

(3) Let C act on the groupoid G. If $x \in G$ and $a \in C$ then $\exists a \cdot x$ iff $\exists a \cdot x^{-1}$, in which case $(a \cdot x)^{-1} = a \cdot x^{-1}$. It is straightforward to check that $G_e = \pi^{-1}(e)$ is a subgroupoid of G, and that if $f \xleftarrow{a} e$ in C, then the function $x \mapsto a \cdot x$ from G_e to G_f is a functor. These observations provide the connection with the approach to category actions described in [2].

(4) Let C act on the groupoids G and G'. We say that (C, G) is *isomorphic* to (C, G') iff there is an isomorphism $\alpha \colon G \to G'$ such that $\exists a \cdot x$ iff $\exists a \cdot \alpha(x)$ in which case $\alpha(a \cdot x) = a \cdot \alpha(x)$.

There is one further piece of notation we shall use. If C acts on G and $x \in G$ then define

$$C \cdot x = \{a \cdot x \colon \exists a \cdot x\}.$$

Observe that if x is an identity then $C \cdot x$ consists entirely of identities.

Let C act on the groupoid G. Define $x \preceq y$ in G iff there exists $a \in C$ such that $x = a \cdot y$. It is easy to check that $\preceq$ is a preorder on G. Let $\equiv$ be the associated equivalence: $x \equiv y$ iff $x \preceq y$ and $y \preceq x$.

Remarks

(1) Observe that $x \preceq y$ iff $C \cdot x \subseteq C \cdot y$. Thus $x \equiv y$ iff $C \cdot x = C \cdot y$.

(2) If $x \equiv y$ then $\mathbf{d}(x) \equiv \mathbf{d}(y)$ and $\mathbf{r}(x) \equiv \mathbf{r}(y)$ by axiom (A4).

Denote the $\equiv$-equivalence class containing x by $[x]$, and denote the set of $\equiv$-equivalence classes by $J(C, G)$. The set $J(C, G)$ is ordered by $[x] \leq [y]$ iff $x \preceq y$.

We shall be interested in actions of categories C on groupoids G that satisfy two further conditions:

(A7) G is combinatorial.

(A8) $\mathbf{r}(a \cdot x) = \mathbf{r}(b \cdot x)$ iff $\mathbf{d}(a \cdot x) = \mathbf{d}(b \cdot x)$.

Condition (A7) is to be expected; condition (A8) will make everything work, as will soon become clear.

Theorem 2.1 *Let C be a category acting on the groupoid G, and suppose in addition that both (A7) and (A8) hold. Then*

(i) *$J(C, G)$ is an ordered groupoid.*

(ii) *$J(C, G)$ is $*$-inductive iff for all identities $e, f \in G$ we have that $C \cdot e \cap C \cdot f$ non-empty implies there exists an identity i such that $C \cdot e \cap C \cdot f = C \cdot i$.*

Proof (i) We begin with some preliminary definitions and results. Define

$$\mathbf{d}[x] = [\mathbf{d}(x)] \text{ and } \mathbf{r}[x] = [\mathbf{r}(x)].^3$$

These are well-defined by (A4).

We claim that $\mathbf{d}[x] = \mathbf{r}[y]$ iff there exists $x' \in [x]$ and $y' \in [y]$ such that $\exists x'y'$. To prove this, suppose first that $x' \in [x]$ and $y' \in [y]$ are such that $\exists x'y'$. Then $x' \equiv x$, $y \equiv y'$ and $\mathbf{d}(x') = \mathbf{r}(y')$. From (A4), we deduce that $\mathbf{d}(x') \equiv \mathbf{d}(x)$ and $\mathbf{r}(y') \equiv \mathbf{r}(y)$, which gives $\mathbf{d}(x) \equiv \mathbf{r}(y)$. Hence $\mathbf{d}[x] = \mathbf{r}[y]$, as claimed. Conversely, suppose that $\mathbf{d}[x] = \mathbf{r}[y]$. Then $\mathbf{d}(x) \equiv \mathbf{r}(y)$. There exist elements $a, b \in C$ such that $\mathbf{d}(x) = a \cdot \mathbf{r}(y)$ and $\mathbf{r}(y) = b \cdot \mathbf{d}(x)$. By (A4), we may deduce that both $a \cdot y$ and $b \cdot x$ are defined. By (A2) and (A3), we have that $b \cdot (a \cdot y)$ is defined. By (A4),

$$\mathbf{r}(b \cdot (a \cdot y)) = \mathbf{r}(y).$$

By (A8), this implies that

$$\mathbf{d}(b \cdot (a \cdot y)) = \mathbf{d}(y).$$

By (A7), this means that $y = b \cdot (a \cdot y)$. Hence $y \equiv a \cdot y$ and, in addition, $\exists x(a \cdot y)$, as required.

We define a partial product on $J(C, G)$ as follows: if $\mathbf{d}[x] = \mathbf{r}[y]$ then

$$[x][y] = [x'y'] \text{ where } x' \in [x],\ y' \in [y] \text{ and } \exists x'y',$$

otherwise the partial product is not defined. To show that this partial product is well-defined we shall use (A7) and (A8). Let $x'' \in [x]$ and $y'' \in [y]$ be such that $\exists x''y''$. We need to show that $x'y' \equiv x''y''$. By definition there exist $a, b, c, d \in C$ such that

$$x' = a \cdot x, \quad x = b \cdot x', \quad x'' = c \cdot x, \quad x = d \cdot x''$$

and there exist $s, t, u, v \in C$ such that

$$y' = s \cdot y, \quad y = t \cdot y', \quad y'' = u \cdot y, \quad y = v \cdot y''.$$

Now $x = b \cdot x'$ and $x'' = c \cdot x$. Thus $x'' = (cb) \cdot x'$ by (A3). Now $\exists x'y'$ and so $\pi(x'y') = \pi(x')$ by (A5). Thus $\exists (cb) \cdot (x'y')$. Hence $(cb) \cdot (x'y') = [(cb) \cdot x'][(cb) \cdot y']$ by (A6). The latter is $x''[(cb) \cdot y']$. We shall show that $(cb) \cdot y' = y''$. This will prove that $x''y'' \preceq x'y'$; the fact that $x'y' \preceq x''y''$ holds by a similar argument so that $x'y' \equiv x''y''$ as required. We now prove that $(cb) \cdot y' = y''$. We have that $y'' = (ut) \cdot y'$ and $\mathbf{d}(x'') = \mathbf{r}(y'')$. Thus $\mathbf{d}(x'') = \mathbf{r}(y'') = (ut) \cdot \mathbf{r}(y')$ by (A4). But $\mathbf{d}(x'') = (cb) \cdot \mathbf{r}(y')$. Thus $(ut) \cdot \mathbf{r}(y') = (cb) \cdot \mathbf{r}(y')$. Hence

$$\mathbf{r}((ut) \cdot y') = \mathbf{r}((cb) \cdot y')$$

by (A4). Therefore

$$\mathbf{d}((ut) \cdot y') = \mathbf{d}((cb) \cdot y')$$

[3] Strictly speaking, I should write $\mathbf{d}([x])$ but I shall omit the outer pair of brackets.

by (A8). It follows that the elements $(ut)\cdot y'$ and $(cb)\cdot y'$ have the same domains and codomains, and so are equal by (A7). It follows that $(cb)\cdot y' = (ut)\cdot y' = y''$. Thus the partial product is well-defined.

It is now easy to check that $J(C,G)$ is a groupoid: $[x]^{-1} = [x^{-1}]$, and the identities are the elements of the form $[x]$ where $x \in G_o$. The order on $J(C,G)$ is defined by $[x] \leq [y]$ iff $x = a\cdot y$ for some $a \in C$.

It remains to show that $J(C,G)$ is an ordered groupoid with respect to this order.

(OG1) holds: suppose that $[x] \leq [y]$. Then $x = a\cdot y$ and so $x^{-1} = (a\cdot y)^{-1} = a\cdot y^{-1}$. Thus $[x^{-1}] \leq [y^{-1}]$ and so $[x]^{-1} \leq [y]^{-1}$.

(OG2) holds: let $[x] \leq [y]$ and $[u] \leq [v]$ and suppose that the partial products $[x][u]$ and $[y][v]$ exist. Then there exist $x' \in [x]$, $u' \in [u]$, $y' \in [y]$ and $v' \in [v]$ such that $[x][u] = [x'u']$ and $[y][v] = [y'v']$. By assumption, $[x'] \leq [y']$ and $[u'] \leq [v']$ so that there exist $a, b \in C$ such that $x' = a\cdot y'$ and $u' = b\cdot v'$. We need to show that $x'u' \preceq y'v'$. Now $\mathbf{d}(x') = \mathbf{r}(u')$ and so $a\cdot\mathbf{d}(y') = b\cdot\mathbf{r}(v')$. But $\mathbf{d}(y') = \mathbf{r}(v')$. Thus $a\cdot\mathbf{d}(y') = b\cdot\mathbf{d}(y')$. Hence

$$\mathbf{d}(a\cdot y') = \mathbf{d}(b\cdot y').$$

By (A8), we therefore have that

$$\mathbf{r}(a\cdot y') = \mathbf{r}(b\cdot y'),$$

and so $a\cdot y' = b\cdot y'$ by (A7). Thus $x'u' = (a\cdot y')(b\cdot v') = (b\cdot y')(b\cdot v')$. Now $\exists y'v'$ and so by (A5) and (A6) we have that $(b\cdot y')(b\cdot v') = b\cdot(y'v')$. Thus $x'u' = b\cdot(y'v')$ and so $x'u' \preceq y'v'$, as required.

(OG3) holds: let $[e] \leq \mathbf{d}[x]$ where $e \in G_o$. Then $e \preceq \mathbf{d}(x)$ and so $e = a\cdot\mathbf{d}(x)$ for some $a \in C$. Now $\exists a\cdot x$ by (A4). Define

$$([x]\,|\,[e]) = [a\cdot x].$$

Clearly $[a\cdot x] \leq [a]$, and $\mathbf{d}[a\cdot x] = [a\cdot\mathbf{d}(x)] = [e]$. It is also unique with these properties as we now show. Let $[y] \leq [x]$ such that $\mathbf{d}[y] = [e]$. Then $y = b\cdot x$ for some $b \in C$ and $\mathbf{d}(y) \equiv e$. Because of the latter, there exists $c \in C$ such that $e = c\cdot\mathbf{d}(y)$. Thus $e = (cb)\cdot\mathbf{d}(x)$. But $e = a\cdot\mathbf{d}(x)$. Thus $(cb)\cdot\mathbf{d}(x) = a\cdot\mathbf{d}(x)$. Hence $(cb)\cdot\mathbf{r}(x) = a\cdot\mathbf{r}(x)$ by (A8). So by (A7), $c\cdot(b\cdot x) = a\cdot x$, giving $c\cdot y = a\cdot x$. It follows that we have shown that $a\cdot x \preceq y$. From $\mathbf{d}(y) \equiv e$, there exists $d \in C$ such that $\mathbf{d}(y) = d\cdot e$. Using (A7) and (A8), we can show that $y = d\cdot(a\cdot x)$, and so $y \preceq a\cdot x$. We have therefore proved that $y \equiv a\cdot x$. Hence $[y] = [a\cdot x]$, as required.

(OG3)* holds: although this axiom follows from the others, we shall need an explicit description of the corestriction. Let $[e] \leq \mathbf{r}[x]$ where $e \in G_o$. Then $e \preceq \mathbf{r}(x)$ and so $e = b\cdot\mathbf{r}(x)$ for some $b \in C$. Now $\exists b\cdot x$ by (A4). Define

$$([e]\,|\,[x]) = [b\cdot x].$$

The proof that this has the required properties is similar to the one above.

(ii) We now turn to the properties of the pseudoproduct in $J(C,G)$. Let $[e],[f]$ be a pair of identities in $J(C,G)$. It is immediate from the definition of the partial order that $[e]$ and $[f]$ have a lower bound iff $C\cdot e\cap C\cdot f\neq\emptyset$. Next, a simple calculation shows that $[i]\leq[e],[f]$ iff $C\cdot i\subseteq C\cdot e\cap C\cdot f$. Observe that $C\cdot j\subseteq C\cdot i$ iff $j\preceq i$. It is now easy to deduce that $[i]=[e]\wedge[f]$ iff $C\cdot i=C\cdot e\cap C\cdot f$.

It will be useful to have a description of the pseudoproduct itself. If $C\cdot i=C\cdot e\cap C\cdot f$ then denote by

$$e*f \text{ and } f*e$$

elements of C, not necessarily unique, such that

$$i=(e*f)\cdot f=(f*e)\cdot e.$$

Suppose that $[x],[y]$ are such that the pseudoproduct $[x]\otimes[y]$ exists. Then by definition $[\mathbf{d}(x)]\wedge[\mathbf{r}(y)]$ exists. Thus $C\cdot\mathbf{d}(x)\cap C\cdot\mathbf{r}(y)=C\cdot e$ for some $e\in G_o$. It follows that

$$[x]\otimes[y]=([x]\,|\,[e])([e]\,|\,[y]).$$

Now

$$([x]\,|\,[e])=[(\mathbf{r}(y)*\mathbf{d}(x))\cdot x]$$

and

$$([e]\,|\,[y])=[(\mathbf{d}(x)*\mathbf{r}(y))\cdot y].$$

Hence

$$[x]\otimes[y]=[((\mathbf{r}(y)*\mathbf{d}(x))\cdot x)((\mathbf{d}(x)*\mathbf{r}(y))\cdot y)].$$

■

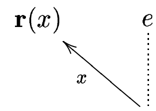
The condition that if $C\cdot e\cap C\cdot f$ is non-empty, where e and f are identities, then there exists an identity i such that $C\cdot e\cap C\cdot f=C\cdot i$ will be called the *orbit condition* for the pair (C,G). Part (ii) of Theorem 2.1 can therefore be stated thus: $J(C,G)$ is $*$-inductive iff (C,G) satisfies the orbit condition.

3 Universality of the construction

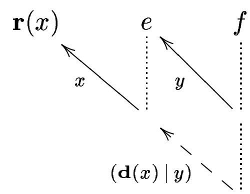
In this section, I shall show that every ordered groupoid is isomorphic to one of the form $J(C,H)$ for some action of a category C on a combinatorial groupoid H.

Let G be an ordered groupoid. There are three ingredients needed to construct $J(C,H)$: a category, which I shall denote by $C'(G)$, a combinatorial groupoid, which I shall denote by $R(G)$, and a suitable action of the former on the latter. We define these as follows:

- We define the category $C'(G)$ as follows: an element of $C'(G)$ is an ordered pair (x, e) where $(x, e) \in G \times G_o$ and $\mathbf{d}(x) \leq e$. This element can be represented thus



  We define a partial product on $C'(G)$ as follows: if $(x, e), (y, f) \in C'(G)$ and $e = \mathbf{r}(y)$ then $(x, e)(y, f) = (x \otimes y, f)$. This product can be represented thus



  since in this case $x \otimes y = x(\mathbf{d}(x) \,|\, y)$. It is easy to check that in this way $C'(G)$ becomes a right cancellative category with identities $(e, e) \in G_o \times G_o$. Further details of this construction may be found in [14].

- We define the groupoid $R(G)$ as follows: its elements are pairs (x, y) where $\mathbf{r}(x) = \mathbf{r}(y)$. Define $\mathbf{d}(x, y) = (y, y)$ and $\mathbf{r}(x, y) = (x, x)$. The partial product is defined by $(x, y)(y, z) = (x, z)$. Evidently, $R(G)$ is the groupoid associated with the equivalence relation that relates x and y iff $\mathbf{r}(x) = \mathbf{r}(y)$.

- We shall now define what will turn out to be an action of $C'(G)$ on $R(G)$. Define π: $R(G) \to C'(G)_o$ by $\pi(x, y) = (\mathbf{r}(x), \mathbf{r}(y))$, a well-defined function. Define $(g, e) \cdot (x, y) = (g \otimes x, g \otimes y)$ iff $e = \mathbf{r}(x) = \mathbf{r}(y)$. This is a well-defined function from $C'(G) * R(G)$ to $R(G)$.

Proposition 3.1 *Let G be an ordered groupoid. With the above definition, the pair $(C'(G), R(G))$ satisfies axioms (A1)–(A8).*

Proof The verification of axioms (A1)–(A7) is routine. We show explicitly that (A8) holds. Suppose that

$$\mathbf{r}[(s, e) \cdot (x, y)] = \mathbf{r}[(t, e) \cdot (x, y)].$$

Then $s \otimes x = t \otimes x$. The groupoid product $x^{-1}y$ is defined, and the two ways of calculating the pseudoproduct of the triple $(s, x, x^{-1}y)$ are defined, and the two ways of calculating the pseudoproduct of the triple $(t, x, x^{-1}y)$ are defined. It follows that $s \otimes y = t \otimes y$; that is,

$$\mathbf{d}[(s, e) \cdot (x, y)] = \mathbf{d}[(t, e) \cdot (x, y)].$$

The converse is proved similarly. ■

The next theorem establishes what we would hope to be true is true.

Theorem 3.2 *Let G be an ordered groupoid. Then $J(C'(G), R(G))$ is isomorphic to G.*

Proof Define α: $G \to J(C'(G), R(G))$ by $\alpha(g) = [(\mathbf{r}(g), g)]$. We show first that α is a bijection. Suppose that $\alpha(g) = \alpha(h)$. Then $(\mathbf{r}(g), g) \equiv (\mathbf{r}(h), h)$. Thus $(a, \mathbf{r}(g)) \cdot (\mathbf{r}(g), g) = (\mathbf{r}(h), h)$ and $(b, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h) = (\mathbf{r}(g), g)$ for some category elements $(a, \mathbf{r}(g))$ and $(b, \mathbf{r}(h))$. Hence

$$a \otimes \mathbf{r}(g) = \mathbf{r}(h), \quad b \otimes \mathbf{r}(h) = \mathbf{r}(g), \quad a \otimes g = h, \text{ and } b \otimes h = g.$$

It follows that a and b are identities and so $h \leq g$ and $g \leq h$, which gives $g = h$. Thus α is injective. To prove that α is surjective, observe that if $[(x, y)]$ is an arbitrary element of $J(C'(G), R(G))$, then $(x, y) \equiv (\mathbf{d}(x), x^{-1}y)$ because

$$(x^{-1}, \mathbf{r}(x)) \cdot (x, y) = (\mathbf{d}(x), x^{-1}y) \text{ and } (x, \mathbf{d}(x)) \cdot (\mathbf{d}(x), x^{-1}y) = (x, y).$$

Next we show that α is a functor. It is clear that identities map to identities. Suppose that gh is defined in G. Now $\alpha(g) = [(\mathbf{r}(g), g)]$ and $\alpha(h) = [(\mathbf{r}(h), h)]$. We have that $\mathbf{d}[(\mathbf{r}(g), g)] = [(g, g)]$ and $\mathbf{r}[(\mathbf{r}(h), h)] = [(\mathbf{r}(h), \mathbf{r}(h))]$. Now $(g, g) \equiv (\mathbf{d}(g), \mathbf{d}(g))$ because

$$(g^{-1}, \mathbf{d}(g)) \cdot (g, g) = (\mathbf{d}(g), \mathbf{d}(g))$$

and

$$(g, \mathbf{d}(g)) \cdot (\mathbf{d}(g), \mathbf{d}(g)) = (g, g).$$

Thus $\alpha(g)\alpha(h)$ is also defined. Now $(\mathbf{r}(h), h) \equiv (g, gh)$ because

$$(g, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h) = (g, gh)$$

and

$$(g^{-1}, \mathbf{r}(g)) \cdot (g, gh) = (\mathbf{r}(h), h).$$

Thus $\alpha(g)\alpha(h) = [(\mathbf{r}(g), gh)] = \alpha(gh)$. It follows that α is a functor.

Finally, we prove that α is an order isomorphism. Suppose first that $g \leq h$ in G. Then $g^{-1} \leq h^{-1}$ and $(\mathbf{d}(g)|h^{-1}) \leq h^{-1}$ and $\mathbf{r}(\mathbf{d}(g)|h^{-1}) = \mathbf{d}(g) = \mathbf{r}(g^{-1})$. Thus $(\mathbf{d}(g)|h^{-1}) = g^{-1}$. It is now easy to check that $(\mathbf{r}(g), g) = (g \otimes h^{-1}, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h)$. Thus $\alpha(g) \leq \alpha(h)$. Now suppose that $\alpha(g) \leq \alpha(h)$. Then $(\mathbf{r}(g), g) = (a, \mathbf{r}(h)) \cdot (\mathbf{r}(h), h)$. It follows that a is an identity and that $g = a \otimes h$ and so $g \leq h$. We have proved that α is an order isomorphism.

Hence α is an isomorphism of ordered groupoids. ■

The first application of the theory is to provide a way of describing inverse semigroups. I shall show how the theory can be used to describe inverse semigroups with zero; the case of inverse semigroups without zero is similar.

Let S be an inverse semigroup with zero. We denote by S^* the set $S \setminus \{0\}$ regarded as an ordered groupoid: the partial product of s and t is defined iff $s^{-1}s = tt^{-1}$ in which case it is equal to the usual product st; the partial order is the natural partial order.

The category $C' = C'(S^*)$ consists of those ordered pairs (s, e) where $s \in S^*$ and $e \in E(S^*)$, the set of non-zero idempotents of S, such that $s^{-1}s \leq e$. The product of (s, e) and (t, f) is defined iff $e = tt^{-1}$ in which case $(s, e)(t, f) = (st, f)$.

The combinatorial groupoid $R = R(S^*)$ consists of those pairs (s, t) such that s and t are both non-zero and $ss^{-1} = tt^{-1}$. Now the relation $\mathcal{R}$ is defined on S by $s \, \mathcal{R} \, t$ iff $ss^{-1} = tt^{-1}$ and is one of Green's relations.

By Proposition 3.1, the ordered groupoid S^* is isomorphic to $J(C', R)$. Thus the inverse semigroup S is isomorphic to $J(C', R)^0$ equipped with the pseudo-product. We may summarise these results as follows.

Theorem 3.3 *Every inverse semigroup with zero S is determined upto isomorphism by three ingredients: the category $C'(S^*)$, Green's $\mathcal{R}$-relation, and the action of the category on the groupoid determined by Green's $\mathcal{R}$-relation.* ■

The results of Section 8 of [11] can be translated easily into this new formalism. Here is one important example. An ordered groupoid G is said to be E^*-*unitary* if $e \leq g$, where e is an identity, implies that g is an identity. An inverse semigroup with zero S is E^*-unitary iff the ordered groupoid S^* is E^*-unitary. The class of E^*-unitary inverse semigroups is important; see [13].

Proposition 3.4 *Let (C, G) satisfy axioms (A1)–(A8). Then $J(C, G)$ is E^*-unitary iff $a \cdot g$ an identity implies that g is an identity.*

Proof Suppose that $J(C, G)$ is E^*-unitary. Let $a \cdot g = e$, an identity. Then $[e] \leq [g]$ in the ordered groupoid $J(C, G)$. But $[e]$ is an identity and so $[g]$ is an identity using the fact that $J(C, G)$ is an ordered groupoid. It follows from the proof of Theorem 2.1, that g is an identity, as required.

Now suppose that $a \cdot g$ an identity implies that g is an identity. Let $[e] \leq [g]$ in $J(C, G)$ where $[e]$ is an identity. Then $e = a \cdot g$ for some $a \in C$. It follows that g is an identity and so $[g]$ is an identity, as required. ■

4 A representation

In this section, I shall prove some results that link the construction described in this paper to an earlier construction by the author described in [11].

Ordered groupoids can be regarded as axiomatisations of ordered groupoids of partial bijections. It is noteworthy, therefore, that the ordered groupoid $J(C, G)$ can be naturally represented by means of partial bijections between certain subsets of G_o. To prove this, we shall need some preliminaries.

Let (C, G) satisfy the axioms (A1)–(A8). Let $e \in G_o$. We have already observed that the set $C \cdot e$ consists entirely of identities. A function $\alpha : C \cdot e \to C \cdot f$ is called a C-*isomorphism* if it satisfies the following three conditions:

(IM1) It is a bijection.

(IM2) $\pi(\alpha(x)) = \pi(x)$ for all $x \in C \cdot e$.

(IM3) $\alpha(a \cdot x) = a \cdot \alpha(x)$ for all $x \in C \cdot e$ and $a \in C$.

Denote the set of all C-isomorphisms of G by $I(C, G)$.

Proposition 4.1 *$I(C, G)$ is an ordered groupoid. The identities are the identity functions on the subsets of G of the form $C \cdot e$ where e is an identity.*

Proof Clearly if α is a C-isomorphism so is α^{-1}. If α: $C \cdot e \to C \cdot f$ and β: $C \cdot f \to C \cdot i$ are C-isomorphisms then so is $\beta\alpha$: $C \cdot e \to C \cdot i$. It follows that $I(C, G)$ is a groupoid. We order the elements of $I(C, G)$ using the usual order of partial functions. Let α: $C \cdot e \to C \cdot f$ be an element of $I(C, G)$, and let $C \cdot i \subseteq C \cdot e$. Let β be the restriction of α to $C \cdot i$. Using the fact that $\pi(\alpha(x)) = \pi(x)$, it is easy to check that $\alpha(C \cdot e) = C \cdot \alpha(e)$. It is now clear that $\beta \in I(C, G)$. It follows readily that with these definitions $I(C, G)$ is an ordered groupoid. ■

The next lemma is the key to obtaining a representation of $J(C, G)$ in $I(C, G)$.

Lemma 4.2 *Let (C, G) satisfy the axioms (A1)–(A8). Let $g \in G$ be such that $e \xrightarrow{g} f$. Define θ_g: $C \cdot e \to C \cdot f$ by $\theta_g(a \cdot e) = a \cdot f$. Then θ_g is a well-defined C-isomorphism such that $\theta_g(e) = f$. In addition, if $g \equiv h$ then $\theta_g = \theta_h$.*

Proof We begin by showing that θ_g is well-defined. If $\exists a \cdot e$ then $\mathbf{d}(a) = \pi(e)$. It follows easily from (A4), that $\pi(e) = \pi(f)$ so that $a \cdot f$ is defined. Suppose that $a \cdot e = b \cdot e$. Then by (A4), $\mathbf{d}(a \cdot g) = \mathbf{d}(b \cdot g)$. It follows by (A8) that $\mathbf{r}(a \cdot g) = \mathbf{r}(b \cdot g)$. Thus $a \cdot f = b \cdot f$. We have show that θ_g is well-defined. The function θ_g is injective by (A8). To show that it is surjective, let $a \cdot f \in C \cdot f$. Then $\mathbf{d}(a) = \pi(f) = \pi(\mathbf{r}(g))$. It follows that $\exists a \cdot e$ and so $\theta_g(a \cdot e) = a \cdot f$. It is easy to check from the definition that $\pi(\theta_g(x)) = \pi(x)$.

Suppose that $g \equiv h$. Then $\mathbf{d}(g) \equiv \mathbf{d}(h)$ and $\mathbf{r}(g) \equiv \mathbf{r}(h)$, so that $C \cdot \mathbf{d}(g) = C \cdot \mathbf{d}(h)$ and $C \cdot \mathbf{r}(g) = C \cdot \mathbf{r}(h)$. Thus θ_g and θ_h have the same domains and codomains. It remains to show that they have the same effects.

By assumption, $g \equiv h$ and so $g = u \cdot h$ and $h = v \cdot g$ for some $u, v \in C$. Let $x \in C \cdot \mathbf{d}(g)$. Then $x = a \cdot \mathbf{d}(g)$ and so

$$\theta_g(x) = \theta_g(a \cdot \mathbf{d}(g)) = a \cdot \mathbf{r}(g).$$

Now $\mathbf{d}(g) = u \cdot \mathbf{d}(h)$, and so $x = a \cdot (u \cdot \mathbf{d}(h)) = (au) \cdot \mathbf{d}(h)$. It follows that

$$\theta_h(x) = (au) \cdot \mathbf{r}(h).$$

But $u \cdot \mathbf{r}(h) = \mathbf{r}(g)$ and so $\theta_g(x) = \theta_h(x)$, as required. ■

In view of the above lemma, we may define a function

$$\phi\colon J(C, G) \to I(C, G) \text{ by } \phi_{[x]} = \theta_x.$$

Theorem 4.3 *Let C be a category acting on the groupoid G satisfying axioms (A1)–(A8). Then ϕ: $J(C,G) \to I(C,G)$ is an ordered embedding, which induces an isomorphism between the posets of identities.*

Proof Suppose that $\exists xy$. Then it is easy to check that $\phi_{[x][y]} = \phi_{[x]}\phi_{[y]}$. Thus ϕ is a functor between groupoids. Suppose that $[x] \leq [y]$. Then $x \preceq y$. It follows that $\mathbf{d}(x) \preceq \mathbf{d}(y)$ and $\mathbf{r}(x) \preceq \mathbf{r}(y)$. Thus $C \cdot \mathbf{d}(x) \subseteq C \cdot \mathbf{d}(y)$ and $C \cdot \mathbf{r}(x) \subseteq C \cdot \mathbf{r}(y)$. It is now easy to check that $\phi_{[x]}$ is the restriction of $\phi_{[y]}$. Conversely, suppose that $\phi_{[x]}$ is the restriction of $\phi_{[y]}$. We prove that $[x] \leq [y]$. By assumption, $\mathbf{d}(x) \in C \cdot \mathbf{d}(y)$. Thus $\mathbf{d}(x) = a \cdot \mathbf{d}(y)$ for some $a \in C$. Thus $\mathbf{d}(x) = \mathbf{d}(a \cdot y)$. Now $\phi_{[x]}(\mathbf{d}(x)) = \mathbf{r}(x)$. Also $\phi_{[y]}(a \cdot \mathbf{d}(y)) = a \cdot \mathbf{r}(y)$. Thus $\mathbf{r}(x) = \mathbf{r}(a \cdot y)$. Hence $x = a \cdot y$ by (A7).

Finally, observe that for each identity $e \in G_o$, the identity function on $C \cdot e$ is a C-isomorphism which is equal to ϕ_e. The last claim is now clear. ■

We say that the action of C on G is *complete* if the function ϕ defined in Theorem 4.3 is surjective, in which case $J(C,G)$ and $I(C,G)$ are isomorphic ordered groupoids. This does not appear to be true in general.

Let (C,G) satisfy axioms (A1)–(A8). Define the relation $\mathcal{R}^*$ on the set G_o as follows: $e\,\mathcal{R}^*\,f$ iff $\pi(e) = \pi(f)$ and for all $a,b \in C$ we have that, when defined, $a \cdot e = b \cdot e \Leftrightarrow a \cdot f = b \cdot f$. Observe that $\mathcal{R}^*$ is an equivalence relation on the set G_o. In addition, $e\,\mathcal{R}^*\,f$ implies that $c \cdot e\,\mathcal{R}^*\,c \cdot f$ for all $c \in C$ where $c \cdot e$ and $c \cdot f$ are defined.

Lemma 4.4 *Let (C,G) satisfy the axioms (A1)–(A8). Then the following are equivalent:*

(i) $e\,\mathcal{R}^*\,f$.

(ii) *There is a C-isomorphism α: $C \cdot e \to C \cdot f$ such that $\alpha(e) = f$.*

Proof (i) $\Rightarrow$ (ii). Suppose that $e\,\mathcal{R}^*\,f$. Define α: $C \cdot e \to C \cdot f$ by $\alpha(a \cdot e) = a \cdot f$. Observe that $\exists a \cdot e$ iff $\mathbf{d}(a) = \pi(e)$. By assumption $\pi(e) = \pi(f)$ and so $\exists a \cdot f$; thus the right-hand side really is defined. Suppose that $a \cdot e = a' \cdot e$. Then by assumption $a \cdot f = a' \cdot f$. It follows that α is a well-defined function. Suppose that $\alpha(a \cdot e) = \alpha(b \cdot e)$. Then $a \cdot f = b \cdot f$ and so $a \cdot e = b \cdot e$. Hence α is injective. Let $a \cdot f \in C \cdot f$. Then $\mathbf{d}(a) = \pi(f)$ and so $\mathbf{d}(a) = \pi(e)$, which implies that $a \cdot e$ is defined. It is immediate that $\alpha(a \cdot e) = a \cdot f$. We have shown that α is a bijection. Observe that $\pi(\alpha(a \cdot e)) = \pi(a \cdot f) = \mathbf{r}(a)$ and $\pi(a \cdot e) = \mathbf{r}(a)$ by (A2). It follows that $\pi(\alpha(a \cdot e)) = \pi(a \cdot e)$. It is immediate from the definition that $\alpha(a \cdot x) = a \cdot \alpha(x)$. Finally, we have that $\alpha(e) = \alpha(\pi(e) \cdot e) = \pi(e) \cdot f = f$.

(ii) $\Rightarrow$ (i). Suppose there is a C-isomorphism α: $C \cdot e \to C \cdot f$ such that $\alpha(e) = f$. It is immediate that $\pi(e) = \pi(f)$. Suppose that $a \cdot e = b \cdot e$. Then $a \cdot \alpha(e) = b \cdot \alpha(e)$, and so $a \cdot f = b \cdot f$. Suppose that $a \cdot f = b \cdot f$. Then $\alpha(a \cdot e) = \alpha(b \cdot e)$, and so $a \cdot e = b \cdot e$. Hence $e\,\mathcal{R}^*\,f$. ■

We now have the following criterion for the completeness of (C,G).

Proposition 4.5 *Let (C,G) satisfy axioms (A1)–(A8). Then (C,G) is complete if and only if ($e\,\mathcal{R}^*\,f$ iff $e \xrightarrow{g} f$, for some element g of G).*

Proof Suppose that (C,G) is complete. By Lemmas 4.2 and 4.4, from $e \xrightarrow{g} f$ we can deduce that $e\,\mathcal{R}^*\,f$, always. We shall prove that $e\,\mathcal{R}^*\,f$ implies $e \xrightarrow{g} f$ for some $g \in G$. By Lemma 4.4, there exists a C-isomorphism α: $C \cdot e \to C \cdot f$ such that $\alpha(e) = f$. By assumption, there exists $g \in G$ such that $\alpha = \phi_{[g]} = \theta_g$, and so, in particular, $e \xrightarrow{g} f$, as required.

Suppose now that $e\,\mathcal{R}^*\,f$ implies $e \xrightarrow{g} f$ for some $g \in G$. We shall prove that (C,G) is complete. Let α: $C \cdot e \to C \cdot f$ be a C-isomorphism. We can assume without loss of generality that $\alpha(e) = f$; to see why, put $\alpha(e) = f'$. Then $f' \in C \cdot f$ and so $f' \preceq f$. On the other hand, α is surjective and $f \in C \cdot f$ and so there exists $x \in C \cdot e$ such that $\alpha(x) = f$. Since $x = a \cdot e$ for some $a \in C$, we have that $\alpha(x) = a \cdot \alpha(e) = a \cdot f' = f$ and so $f \preceq f'$. It follows that $f \equiv f'$ and so $C \cdot f = C \cdot f'$. By Lemma 4.4, we deduce that $e\,\mathcal{R}^*\,f$. Thus by assumption, there exists $g \in G$ such that $e \xrightarrow{g} f$. But then $\theta_g = \alpha$ and so $\phi_{[g]} = \alpha$, and so (C,G) is complete. ■

I can now explain the connection between this paper and [11]. Let (C,X) be a pair consisting of a category C acting on a set X where we denote by π: $X \to C_o$ the function used in defining the action. We may define the equivalence relation $\mathcal{R}^*$ on X, and so we get a combinatorial groupoid

$$G(C,X) = \{(x,y) \colon x\,\mathcal{R}^*\,y\}.$$

Define π': $G(C,X) \to C_o$ by $\pi'(x,y) = \pi(x)$, and define an action of C on $G(C,X)$ by $a \cdot (x,y) = (a \cdot x, a \cdot y)$ when $\mathbf{d}(a) = \pi'(x,y)$. It is easy to check that axioms (A1)–(A8) hold. In addition, $(C, G(C,X))$ is complete: we use Proposition 4.5 to prove this. It is equivalent to the fact that $(x,x)\,\mathcal{R}^*\,(y,y)$ iff $x\,\mathcal{R}^*\,y$. We have proved the following except the last claim which is straightforward to prove.

Proposition 4.6 *Let C be a category acting on the set X. Then, with the definition above, C acts on combinatorial groupoid $G(C,X)$ in a complete way. In addition, the action of C on X is isomorphic to the action of C on $G(C,X)_o$.* ■

We now go in the other direction. Let C act on the combinatorial groupoid G. From the remarks made after the statement of (A6), it follows that there is an induced action of C on the set G_o.

Proposition 4.7 *Let (C,G) satisfy axioms (A1)–(A8). If (C,G) is complete, then the action of C on G is determined by the action of C on G_o.*

Proof Let (C,G) satisfy axioms (A1)–(A8). The action of the category C restricts to an action of the category C on the set G_o, where the set is regarded as a discrete category consisting only of identities. The relation $\mathcal{R}^*$ is defined entirely in terms of the action (C, G_o). Let G' denote the groupoid corresponding to the equivalence relation $\mathcal{R}^*$. Define $\xi\colon G \to G'$ by $\xi(g) = (\mathbf{r}(g), \mathbf{d}(g))$. Then ξ is a functor, which is injective by (A7). The functor ξ is surjective iff (C, G) is complete, by Proposition 4.5. Thus completeness implies that the groupoid G can be recovered upto isomorphism from the action (C, G_o).

Assuming completeness, define an action of C on G' by $\exists a \cdot (e, f)$ iff $\exists a \cdot e$ — which is equivalent to $\exists a \cdot f$ since $\pi(e) = \pi(f)$ — in which case, $a \cdot (e, f) = (a \cdot e, a \cdot f)$. It is easy to check that we have an action satisfying axioms (A1)–(A8). Suppose that $\exists a \cdot g$. Then by (A4), it is immediate that $a \cdot \xi(g)$ is defined. Furthermore, $\xi(a \cdot g) = a \cdot \xi(g)$, again by (A4). Now suppose that $\exists a \cdot \xi(g)$. Then by (A4), we have that $\exists a \cdot g$. It follows that the actions (C, G) and (C, G') are isomorphic. ■

We say that the pair (C, G) satisfies the *right cancellation condition* if $a \cdot e = b \cdot e$, where $a, b \in C$ and $e \in G_o$, implies that $a = b$. The following is easy to check.

Lemma 4.8 *Let (C, G) satisfy axioms (A1)–(A8) and the right cancellation condition. Then $e \, \mathcal{R}^* \, f$ iff $\pi(e) = \pi(f)$.* ■

Let G be an ordered groupoid. It is easy to check that the pair $(C'(G), R(G))$ of Theorem 3.2 satisfies the right cancellation condition. It follows that on $R(G)$ we have that $(x, x) \, \mathcal{R}^* \, (y, y)$ iff $\mathbf{r}(x) = \mathbf{r}(y)$ where $x, y \in G$. We therefore immediately deduce the following from Proposition 4.7.

Corollary 4.9 *Every ordered groupoid G is isomorphic to an ordered groupoid of the form $J(C, H)$ where (C, H) is complete.* ■

We conclude this section by classifying a class of substructures of $J(C, G)$. We say that a subgroupoid G' of a groupoid G is *wide* if $G'_o = G_o$.

Proposition 4.10 *Let C act on G and satisfy all the axioms. Let H be a wide subgroupoid of G such that $C \cdot H \subseteq H$. Then $J(C, H)$ is a wide subgroupoid of $J(C, G)$ which is also an order ideal. Every wide subgroupoid of $J(C, G)$ which is an order ideal is constructed in this way.*

Proof Let H be a wide subgroupoid of G such that $C \cdot H \subseteq H$. Then the action of C on H satisfies all the axioms and so we can form the groupoid $J(C, H)$. This is a subset of $J(C, G)$ because if $x \in H$ and $y \equiv x$ then $y \in H$. So the equivalence class of x with respect to the action of C on H is the same as the equivalence class of x with respect to the action of C on G. It is now easy to check that $J(C, H)$ is a wide subgroupoid of $J(C, G)$ and an order ideal.

Now let $J' \subseteq J(C, G)$ be a wide subgroupoid and order ideal. Let $H = \{x \in G \colon [x] \in J'\}$. It is now routine to check that H is a wide subgroupoid of G and that $C \cdot H \subseteq H$. It is easy to check that $J(C, H) = J'$. ■

5 Concluding remarks

I would like to say a few words about the origins of the constructions described in Sections 2 and 3. Let G be an ordered groupoid. The category $C'(G)$ is one of a pair of categories that can be associated with an ordered groupoid G. The other, denoted $C(G)$, is left rather than right cancellative. The origin of these categories goes back to one of the founding papers of inverse semigroup theory written by Clifford [3]. However the explicit connection between Clifford's work and category theory seems to have been discovered by Leech [17]. He showed that in the case of inverse monoids, the whole structure of the semigroup could be reconstituted from either of these two categories. The importance of these categories was further underlined in the discovery by Loganathan [18] that the cohomology of inverse semigroups introduced by Lausch [9] was the same as the usual cohomology of one of its categories. Further applications of these categories can be found in [13, 14, 15]. As I indicated above, these categories completely determine the structure of the semigroup in the case of inverse monoids. This raises the question of what can be said in general. The semigroup background to this question is discussed in [11]. As a result of reading a paper by Girard on linear logic, I was led to the construction described in [11], which shows how inverse semigroups can be constructed from categories acting on *sets*. I thought this was the final word on this construction until Claas Röver pointed out to me the paper by Dehornoy [4]. Dehornoy constructs an inverse semigroup from any variety, in the sense of universal algebra, that is described by equations which are balanced, meaning that the same variables occur on either side of the equation. This construction was clearly related to my construction in [11], but I felt the fit was not quite good enough. It was an analysis of the connections between the two that led me to the construction of this paper. Section 4 describes the connection between my old paper [11] and my new constructions of Sections 2 and 3; the details are spelt out in Propositions 4.6 and 4.7.

References

[1] S. Abramsky, A structural approach to reversible computation, Preprint.

[2] M. Barr, C. Wells, *Category theory for computing science*, Prentice Hall, 1990.

[3] A. Clifford, A class of d-simple semigroups, *Amer. J. Math.* **75** (1953), 547-556.

[4] P. Dehornoy, Structural monoids associated to equational varieties, *Proc. Amer. Math. Soc.* **117** (1993), 293–304.

[5] C. Ehresmann, *Oeuvres complètes et commentées*, (ed A. C. Ehresmann) Supplements to *Cahiers de Topologie et Géométrie Différentielle* Amiens, 1980-83.

[6] N. D. Gilbert, HNN extensions of inverse semigroups, accepted for publication in *J. Alg.*

[7] J.-Y. Girard, The geometry of interaction III: accommodating the additives, in *Advances in linear logic* (eds J.-Y. Girard, Y. Lafont, L. Regnier) Cambridge University Press, 1995.

[8] J. Kellendonk, M. V. Lawson, Partial actions of groups, to appear in *Inter. J. of Alg. and Computation.*

[9] H. Lausch, Cohomology of inverse semigroups, *J. Alg.* **35** (1975), 273–303.

[10] M. V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific, 1998.

[11] M. V. Lawson, Constructing inverse semigroups from category actions, *J. of Pure and Applied Alg.* **137** (1999), 57–101.

[12] M. V. Lawson, J. Matthews, T. Porter, The homotopy theory of inverse semigroups, *Inter. J. of Alg. and Computation* **12** (2002), 755–790.

[13] M. V. Lawson, E^*-unitary inverse semigroups, in *Semigroups, algorithms, automata and languages* (eds G. M. S. Gomes, J.-E. Pin, P. V. Silva) World Scientific, 2002, 195–214.

[14] M. V. Lawson, Ordered groupoids and left cancellative categories, to appear in in *Semigroup Forum.*

[15] M. V. Lawson, B. Steinberg, Etendues and ordered groupoids, accepted for publication by *Cahiers de Topologie et Géométrie Différentielle Catégoriques.*

[16] M. V. Lawson, Dehornoy's structural monoids, in preparation.

[17] J. Leech, Constructing inverse semigroups from small categories, *Semigroup Forum* **36** (1987), 89–116.

[18] M. Loganathan, Cohomology of inverse semigroups, *J. Alg.* **70** (1981), 375–393.

[19] K. S. S. Nambooripad, *Structure of regular semigroups I*, Memoirs of the American Mathematical Society **224** (1979).

[20] A. L. T. Paterson, *Groupoids, inverse semigroups, and their C^*-algebras*, Birkhäuser, 1998.

[21] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics 793, Springer-Verlag, Berlin, 1980.

[22] B. M. Schein, On the theory of inverse semigroups and generalized grouds, *Amer. Math. Soc. Translations (2)* **113** (1979), 89–122.

[23] B. Steinberg, Factorization theorems for morphisms of ordered groupoids and inverse semigroups, *Proc. Edinburgh Math. Soc.* **44** (2001), 549-569.