Zappa-Szép products of free monoids and groups

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Abstract

We prove that left cancellative right hereditary monoids satisfying the dedekind height property are precisely the Zappa-Szép products of free monoids and groups. The 'fundamental' monoids of this type are in bijective correspondence with faithful self-similar group actions.
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1 A class of left cancellative monoids

This paper develops some ideas that were touched on first in our paper [6], where we corrected an error in Nivat and Perrot's [8] generalisation of some pioneering work by David Rees [9]. In this section, we define the class of monoids we shall be interested in.

An important role in this paper will be played by free monoids. If X is a set then X^* denotes the free monoid generated by X. Elements of X^* are *strings* and the length of a string x is denoted by $|x|$. The *prefix order* on X^* is defined by $x \leq y$ iff $x = yz$ for some string z.

An S-*act* or *act* (X, S) is an action of a monoid S on a set X on the right. If S is a monoid then (S, S) is an act by right multiplication. If $Y \subseteq X$ is a subset such that $YS \subseteq Y$ then we say that Y is an S-*subact* or just a *subact*. Right ideals of S are subacts under right multiplication. If X and Y are acts then a function θ from X to Y is an S-*homomorphism* or just a *homomorphism* if $\theta(xs) = \theta(x)s$ for all $x \in X$ and $s \in S$. For a fixed S, we can form the category consisting of S-acts and the homomorphisms between them. The usual definitions from module theory can be adapted to the theory of acts. In particular, we can define when an act is *projective*. A monoid S is said to be *right PP* if all its principal right ideals are projective as right S-acts, and *right hereditary* if all its right ideals are projective as right S-acts. The following was proved by Dorofeeva [4].

Theorem 1.1 *A monoid S is right hereditary iff it is right PP, incomparable principal right ideals are disjoint, and S has the ascending chain condition for principal right ideals.* ■

We do not need the general characterisation of right PP monoids for this paper; it is enough to know that the right PP monoids with a single idempotent are precisely the left cancellative monoids.

Remarks

1. We shall often use the fact that (ACC) on principal right ideals is equivalent to the condition that every non-empty set of principal right ideals has a maximal element.
2. From now on, 'ideal' will always mean 'principal right ideal' unless otherwise stated, and 'maximal ideal' will always mean 'maximal proper principal right ideal'.
3. If two maximal ideals intersect in a left cancellative right hereditary monoid then they are equal; this is because they must be comparable, but both are maximal.
4. We denote the group of units of a monoid S by $G(S)$ or just G.
5. In a left cancellative monoid S we have that $aS = bS$ iff $a = bg$ for some unit g; we say that a and b are *associates*.
6. In a left cancellative monoid S we have that $aS = S$ iff a is invertible.
7. Generators of maximal ideals will be called *irreducible elements*.
8. Let S be a monoid and $a \in S$. A *left factor* of a is an element $b \in S$ such that $a \in bS$.

We shall study left cancellative right hereditary monoids satisfying a further finiteness condition. Let S be a left cancellative right hereditary monoid and $a \in S$. Then the set of all principal right ideals containing a need not be finite, but if it is we say that S satisfies the *dedekind height property* [1].

Let aS and bS be two principal right ideals. A *chain of length n from aS to bS* is a sequence

$$aS = a_0S \subset a_1S \subset a_2S \subset \ldots \subset a_nS = bS.$$

Lemma 1.2 *Let S be a left cancellative right hereditary monoid. Then the following are equivalent.*

(i) *S has the dedekind height property.*

(ii) *For each $a \in S$ there exists a unique chain of maximum finite length starting at aS and concluding at S.*

Proof (i) $\Rightarrow$ (ii). The set of all principal right ideals containing a is finite. Thus there is a bound on the length of chains starting at aS and ending at S. Given two such chains of maximum length they must be equal. To prove this, we show that the two chains must agree term by term. We use the fact that if two ideals have a non-empty intersection, then they must be comparable. Let

$$aS = a_0S \subset a_1S \subset \ldots \subset a_mS = S$$

and

$$aS = b_0S \subset b_1S \subset \ldots \subset b_nS = S$$

be two such chains. We claim that $a_1S = b_1S$. To see why observe that they are comparable because both contain aS. Thus either $a_1S \subset b_1S$ or vice-versa. If the former we could refine the second chain, if the latter we could refine the first chain. But neither refinement is possible since each chain is of maximum length. Thus $a_1S = b_1S$. This process continues. If $m > n$ then we could use the first chain to refine the second. If $n > m$ then we could use the second chain to refine the first. So the two chains must have the same length and the same terms.

(ii) $\Rightarrow$ (i). All the distinct principal right ideals containing aS must be comparable so they will form a totally ordered set from aS to S. This will be a chain of maximum length and so equal to the unique such chain assumed to exist. Thus the set of all principal right ideals containing aS must be finite. ■

The next lemma provides us with a class of examples of monoids satisfying the dedekind height property.

Lemma 1.3 *Let S be a left cancellative right hereditary monoid equipped with a monoid homomorphism λ: $S \to \mathbb{N}$ such that $\lambda^{-1}(0) = G(S)$. Then S satisfies the dedekind height property.*

Proof Let $aS \subseteq bS$. Then $a = bs$ and so $\lambda(a) = \lambda(b) + \lambda(s)$. Thus, in particular, $\lambda(a) \geq \lambda(b)$. Suppose, in addition, that $\lambda(a) = \lambda(b)$. Then $\lambda(s) = 0$ and so s is a unit. It follows that in this case, $aS = bS$. We deduce that if $aS \subset bS$ then $\lambda(a) > \lambda(b)$. Thus the length of any chain of principal right ideals starting at aS is bounded by $\lambda(a)$. ■

We define a *length function* on an arbitrary monoid S to be a surjective homomorphism λ: $S \to \mathbb{N}$ such that $\lambda^{-1}(0) = G(S)$.

Let S be a left cancellative and right hereditary satisfying the dedekind height property. *In addition, we shall assume throughout this paper that S is not a group.* Let $aS = S_0 \subset S_1 \subset S_2 \ldots \subset S_n = S$ be a chain of principal right ideals of maximum length. We define $\lambda(a) = n$.

Lemma 1.4 *Let S be a left cancellative right hereditary monoid satisfying the dedekind height property. Let*

$$bS = b_0S \subset b_1S \subset b_2S \subset \ldots \subset b_{\lambda(b)}S = S$$

be a chain of maximum length joining bS to S. Then

$$abS = ab_0S \subset ab_1S \subset ab_2S \subset \ldots \subset ab_{\lambda(b)}S = aS$$

is a chain of maximum length joining abS to aS.

Proof We show first that the inclusions really are distinct. Suppose that $ab_iS = ab_{i+1}S$ for some i. Then $ab_i = ab_{i+1}g$ for some unit g. By left cancellation, $b_i = b_{i+1}g$ giving $b_iS = b_{i+1}S$, which contradicts our assumption. Next we show that the chain is of maximum length. Suppose not. Then we can interpolate a principal right ideal somewhere

$$ab_iS \subset cS \subset ab_{i+1}S.$$

Let $ab_i = cf$ for some f and $c = ab_{i+1}d$ for some d. Thus by left cancellation, $b_i = b_{i+1}df$. We therefore have

$$b_iS \subseteq b_{i+1}dS \subseteq b_{i+1}S.$$

Suppose that $b_iS = b_{i+1}dS$. Then $b_i = b_{i+1}dg$ for some unit g. By left cancellation, it follows that $g = f$ and is a unit. Thus $ab_iS = cS$, which is contradiction. Suppose that $b_{i+1}dS = b_{i+1}S$. Then $b_{i+1}d = b_{i+1}h$ for some unit h. By left cancellation, $d = h$ and so $cS = ab_{i+1}S$, which is a contradiction. However, we now have

$$b_sS \subset b_{i+1}dS \subset b_{i+1}S$$

which contradicts the fact that our original chain was of maximum length. It follows that our new chain is of maximum length. ∎

Lemma 1.5 *Let S be a left cancellative right hereditary monoid satisfying the dedekind height property. Then the function λ defined before Lemma 1.4 is a length function.*

Proof By Lemma 1.4, if

$$bS = b_0S \subset b_1S \subset b_2S \subset \ldots \subset b_{\lambda(b)}S = S$$

is a chain of maximum length joining bS to S, then

$$abS = ab_0S \subset ab_1S \subset ab_2S \subset \ldots \subset ab_{\lambda(b)}S = aS$$

is a chain of maximum length joining abS to aS. Now glue this to a chain

$$aS = a_0 \subset a_1S \subset \ldots \subset a_{\lambda(a)}S = S$$

of maximum length. The resulting chain links abS to S and has maximum length, and this length is $\lambda(a) + \lambda(b)$. Thus λ is a homomorphism. Those elements a of length 0 are precisely those where $aS = S$, which are just the invertible elements. Finally, to show that the length function is surjective, it is enough to show that the number 1 is in the image of λ, but this follows from the existence of maximal ideals. ■

We combine Lemmas 1.2, 1.3 and 1.5 in the following theorem.

Theorem 1.6 *Let S be a left cancellative, right hereditary monoid. Then the following are equivalent.*

(i) *S satisfies the dedekind height property.*

(ii) *For each $a \in S$ there exists a unique chain of maximum finite length starting at aS and concluding at S.*

(iii) *The monoid S is equipped with a length function.* ■

An arbitrary monoid M is said to be *equidivisible* if for all $a, b, c, d \in M$ the fact that $ab = cd$ implies that either $a = cu, ub = d$ for some $u \in M$ or $c = av, b = vd$ for some $v \in M$.

Lemma 1.7 *Let S be a left cancellative monoid. Then the following are equivalent*

(i) *Incomparable principal right ideals are disjoint.*

(ii) *S is equidivisible.*

If either holds, then incomparable principal left ideals are disjoint.

Proof (i)⇒(ii). Suppose that $ab = cd$. Then $aS \cap cS \neq \emptyset$. Thus $aS \subseteq cS$ or $cS \subseteq aS$. Suppose the former. Then $a = cu$ for some $u \in S$. But $ab = cd$ and so $cub = cd$. By left cancellation, $ub = d$. Suppose the latter. Then $c = av$ for some $v \in S$. But $ab = cd$ and so $ab = avd$. By left cancellation, $b = vd$. Thus S is equidivisible.

(ii)⇒(i). This is immediate.

To prove the last assertion, suppose that $Sb \cap Sd \neq \emptyset$. Then $ab = cd$ for some $b, c \in S$. The result now follows by equidivisibility. ■

The following is immediate from Theorem 2.6 and Lemma 2.7 and Corollary 5.1.6 of [5] and the fact that free monoids are left cancellative, right hereditary and their length functions really are length functions in our sense. It tells us that left cancellative right hereditary monoid satisfying the dedekind height property are natural generalisations of free monoids.

Corollary 1.8 *Let S be a left cancellative right hereditary monoid satisfying the dedekind height property. Then S is a free monoid if and only if the group of units is trivial.* ■

To conclude this section, we shall look at two different ways in which examples of monoids satisfying our conditions might arise.

The class of left cancellative right hereditary monoids satisfying the dedekind height property is a proper subclass of the class of all left cancellative monoids. We shall now show how closely these two classes are related. We shall use the theory of Rhodes-expansions described in [1] adapted to our situation. Let S be a left cancellative monoid. We shall be interested in finite sequences of elements of S

$$\mathbf{x} = (x_1, \ldots, x_n)$$

where $x_{i+1} \in x_i S$ but $x_{i+1}S \neq x_n S$ and where x_1 is a unit. We denote by $\hat{S}$ the set of all such sequences. We shall now define a product on such sequences. Let

$$\mathbf{x} = (x_1, \ldots, x_m) \text{ and } \mathbf{y} = (y_1, \ldots, y_n).$$

Consider the sequence

$$x_1, \ldots, x_{m-1}, x_m, x_m y_1, \ldots, x_m y_n.$$

Because y_1 is a unit, we have that $x_m S = x_m y_1 S$. Clearly, $x_m y_1 S \subset x_{m-1} S$. Also from $y_{i+1}S \subset y_i S$ we get $x_m y_{i+1} S \subseteq x_m y_i S$. Observe that if $x_m y_{i+1} S = x_m y_i S$ then $x_m y_{i+1} = x_m y_i g$ for some unit g. Thus by left cancellation, $y_{i+1} = y_i g$ implying that $y_{i+1}S = y_i S$, contradicting our assumption. It follows that

$$\mathbf{xy} = (x_1, \ldots, x_{m-1}, x_m y_1, \ldots, x_m y_n)$$

is a well-defined element of $\hat{S}$. This defines a binary operation on $\hat{S}$. The fact that this is a semigroup follows from the general theory in [1]. It is easy to check that it is a monoid with identity (1), and that left cancellation in S is inherited by $\hat{S}$.

Proposition 1.9 *For each left cancellative monoid S, the monoid $\hat{S}$ is left cancellative, right hereditary and equipped with a dedekind height function. There is a surjective homomorphism from $\hat{S}$ onto S.*

Proof We first characterise the left factors of an element of $\hat{S}$. Suppose that $\mathbf{x} \in \mathbf{y}\hat{S}$. Then

$$(x_1, \ldots, x_m) = (y_1, \ldots, y_n)(z_1, \ldots, z_p).$$

Thus $m \geq n$, $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ and $y_n S = x_n S$. Conversely, suppose that $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ are such that $m \geq n$, $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ and $y_n S = x_n S$. For $0 \leq i \leq m-n$ define z_{i+1} by $x_{n+i} = y_n z_{i+1}$. Observe that z_1 is a unit. It is easy to check that $\mathbf{z} = (z_1, \ldots, z_p)$ is a well-defined element of $\hat{S}$ and that $\mathbf{x} = \mathbf{yz}$.

We can now show that $\hat{S}$ is right hereditary and satisfies the dedekind height property. Suppose that $\mathbf{x}\hat{S} \cap \mathbf{y}\hat{S} \neq \emptyset$. Then there is a $\mathbf{z}$ which has both $\mathbf{x}$ and $\mathbf{y}$ as left factors. Let $\mathbf{z} = (z_1, \ldots, z_p)$, $\mathbf{x} = (x_1, \ldots, x_m)$, and $\mathbf{y} = (y_1, \ldots, y_n)$. Then $p \geq m, n$ and $x_1 = z_1, \ldots, x_{m-1} = z_{m-1}, z_m S = x_m S$ and $y_1 = z_1, \ldots, y_{n-1} =$

$z_{n-1}, z_n S = y_n S$. Without loss of generality, suppose that $m \leq n$. Then $x_1 = y_1, \ldots, x_{m-1} = y_{m-1}$ and $x_m S = z_m S = y_m S$. Thus $\mathbf{y} \in \mathbf{x}\hat{S}$.

From the above we can easily derive the criterion for $\mathbf{y}\hat{S} = \mathbf{x}\hat{S}$: $\mathbf{x}$ and $\mathbf{y}$ have the same length, all the components are the same except the rightmost ones which are associate.

It follows from the above two characterisations that the dedekind height property is satisfied. Define $\eta_S \colon \hat{S} \to S$ by $(x_1, \ldots, x_n) \mapsto x_n$. Then this is a surjective homomorphism. Observe that restricted to the $\mathcal{R}$-classes of $\hat{S}$, this homomorphism is injective. ■

A group G is said to be *indicable* if there is a surjective homomorphism $\theta \colon G \to \mathbb{Z}$. For each monoid S there is a group $U(S)$ and a homomorphism $\iota \colon S \to U(S)$ such that for each homomorphism $\phi \colon S \to G$ to a group there is a unique homomorphism $\bar{\phi} \colon S \to G$ such that $\phi = \bar{\phi}\iota$. The group $U(S)$ is called the *universal group* of S. The monoid S can be embedded in a group iff ι is injective. It can be deduced from the results of Section 0.5 of [3] that every cancellative monoid in which any two principal right ideals are either disjoint or comparable can be embedded in a group. It follows that the cancellative right hereditary monoids satisfying the dedekind height property can be embedded in their universal groups. If S is a cancellative right hereditary monoids satisfying the dedekind height property, then there is a homomorphism from S onto $\mathbb{N}$ and so a homomorphism from S to $\mathbb{Z}$. It follows that the universal group of S admits a homomorphism to $\mathbb{Z}$. Since the image of this homomorphism contains $\mathbb{N}$ it is in fact the whole of $\mathbb{Z}$ and so surjective. We have therefore proved the following.

Proposition 1.10 *The universal group of a cancellative right hereditary monoid satisfying the dedekind height property is indicable.* ■

2 Zappa-Szép products

The goal of this section is to obtain a structural description of the class of monoids introduced in Section 1. The tool we shall use is Zappa-Szép products. The paper by Matt Brin [2] is a useful introduction and contains further references. However, I shall prove almost everything from scratch here and so prior knowledge of this construction is not needed. The proof of the following was first given in [6], but I reproduce it here for the sake of completeness.

Proposition 2.1 *Let S be a left cancellative right hereditary monoid satisfying the dedekind height property with group of units $G(S)$. Let X be a transversal of the generators of the maximal proper principal right ideals, and denote by X^* the submonoid generated by the set X. Then the monoid X^* is free, $S = X^*G(S)$, and each element of S can be written uniquely as a product of an element of X^* and an element of $G(S)$.*

Proof We show first that $S = X^*G(S)$. Let $s \in S \setminus G(S)$. Consider the set of all proper ideals that contain s. This set contains a maximal element x_1S, which is necessarily a maximal ideal, and $x_1 \in X$. Thus $s = x_1s_1$. If s_1 is a unit or irreducible the process stops. Otherwise, repeat this process with s_1 to get $s_1 = x_2s_2$ and so on. Thus we can write $s = x_1 \dots x_is_i$. To show that this process terminates observe that

$$sS \subset x_1 \dots x_iS \subset \dots \subset x_1S.$$

Thus termination follows from the dedekind height property. It follows that we can write $s = x_1 \dots x_ng$ where g is a unit.

Next we prove that X^* is free. Suppose that

$$1 = x_1 \dots x_m$$

where $m \geq 1$ and $x_i \in X$. Then $S = x_1 \dots x_mS \subseteq x_1S$ and so x_1 is invertible, which is a contradiction. Now suppose that

$$x_1 \dots x_m = y_1 \dots y_n$$

where $x_i, y_j \in X$. By our result above, we can assume that $m, n \geq 1$. Now $x_1 \dots x_mS = y_1 \dots y_nS \subseteq x_1S, y_1S$. It follows that $x_1 = y_1$ and so, by left cancellation, $x_2 \dots x_m = y_2 \dots y_n$. This process can be repeated and because either $m < n$ or $n < m$ would lead to a contradiction, namely that an element of X is invertible, we must have that $m = n$ and $x_i = y_i$. Thus X^* is the free monoid on X.

Finally, we prove the uniqueness of the decomposition. Suppose that $xg = yh$ where $x, y \in X^*$, $g, h \in G(S)$, and $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$ where $x_i, y_j \in X$. Arguing as before, $xS = yS \subseteq x_1S, y_1S$ and so $x_1 = y_1$. By left cancellation $x_2 \dots x_mg = y_2 \dots y_nh$. If $m = n$ then we can repeat this argument to get $x = y$ and so $g = h$, by left cancellation. If $m < n$, then we can easily deduce that y_{m+1} is invertible, which is a contradiction. A similar argument shows that we cannot have $n < m$. ∎

We now make a key definition. Let G be a group and X^* a free monoid on the set X. We suppose that there two operations that link G and X^*. The first is defined by a function $G \times X^* \to X^*$, called the *action*, denoted by $(g, x) \mapsto g \cdot x$. The second is defined by a function $G \times X^* \to G$, called the *restriction*, denoted by $(g, x) \mapsto g|_x$. We require that the following eight axioms hold:[1]

(SS1) $1 \cdot x = x$.

(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$.

(SS3) $g \cdot 1 = 1$.

[1]Observe that we use 1 to denote both the identity of the group G and the empty string of X^*.

(SS4) $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

(SS5) $g|_1 = g$.

(SS6) $g|_{xy} = (g|_x)|_y$.

(SS7) $1|_x = 1$.

(SS8) $(gh)|_x = g|_{h \cdot x} h|_x$.

I shall call this a *ZS-action of G on the free monoid X^**.

An action of a group G on a free monoid X^* is *length preserving* if $|g \cdot x| = |x|$ for all $x \in A^*$, and *prefix preserving* if $x = yz$ in A^* implies that $g \cdot x = (g \cdot y)z'$ for some string z'. This means precisely that if $x \leq y$ then $g \cdot x \leq g \cdot y$. The following was proved as Lemma 2.5 of [6].

Lemma 2.2 *Let G act on X^* in such a way that the axioms (SS1)–(SS8) hold. Then the action is length-preserving and prefix-preserving.* ■

The following is a consequence of the theory of Zappa-Szép products and follows from the properties of the identity element and associativity. A proof can also be found as Proposition 2.4 of [6].

Proposition 2.3 *With each left cancellative right hereditary monoid satisfying the dedekind height property we can associate a ZS-action.* ■

We shall now look at the converse of the above result. Let G be an arbitrary group, and M an arbitrary left cancellative monoid (not necessarily free) equipped with a function $G \times M \to M$, denoted by $(g, m) \mapsto g \cdot m$, and a function $G \times M \to G$, denoted by $(g, m) \mapsto g|_m$, satisfying the obvious generalisations of (SS1)–(SS8). On the set $M \times G$ define the binary operation by

$$(x, g)(y, h) = (x(g \cdot y), g|_y h).$$

The following is part of the general theory of Zappa-Szép products, but we prove it anyway.

Proposition 2.4 *With the above product, $M \times G$ is a left cancellative monoid containing copies of M and G such that $M \times G$ can be written as a unique product of these copies.*

Proof We begin by proving associativity. We calculate first

$$[(x, g)(y, h)](z, k).$$

By (SS2), (SS8), and (SS6) we get

$$(x(g \cdot y)g|_y \cdot (h \cdot z), g|_{y(h \cdot z)} h|_z k).$$

We now calculate

$$(x, g)[(y, h)(z, k)].$$

Using (SS4), we get the same result.

We now show that $(1, 1)$ is the identity. We calculate

$$(1, 1)(x, g) = (1(1 \cdot 1), 1|_x g) = (x, g)$$

using (SS1) and (SS7). We calculate

$$(x, g)(1, 1) = (x(g \cdot 1), g|_1 1) = (x, g)$$

using (SS3) and (SS5). We have now used all the axioms (SS1)–(SS8).

Next we show that $M \bowtie G$ is left cancellative. Suppose that

$$(x, g)(y, h) = (x, g)(z, k).$$

Then

$$(x(g \cdot y), g|_y h) = (x(g \cdot z), g|_z k).$$

Left cancellation in M gives us $g \cdot y = g \cdot z$ and so because this is an action $y = z$. Hence $h = k$.

We now have to show that M and G are each embedded in $M \bowtie G$. Define $\iota_M \colon M \to M \bowtie G$ by $x \mapsto (x, 1)$. This is an injective homomorphism by (SS1) and (SS7). Denote its image by M'. Define $\iota_G \colon G \to M \bowtie G$ by $g \mapsto (1, g)$. This is an injective homomorphism by (SS3) and (SS5). Denote its image by G'. Observe that $(x, g) = (x, 1)(1, g)$. Thus $M \bowtie G = M'G'$. This decomposition is evidently unique. ■

The monoid constructed in Proposition 2.4 is called the *Zappa-Szép product of M and G* and is denoted $M \bowtie G$.

Proposition 2.5 *Let S be a monoid. Suppose that $S = MG$ uniquely where M is a left cancellative monoid and G is a group. Then S is a left cancellative monoid whose ideal structure is order isomorphic with the ideal structure of M. In particular, when M is a free monoid, the monoid S is right hereditary and equipped with a length function.*

Proof Observe that $\{1\} = G \cap M$. To see why if $g \in G \cap M$ then $g = 1g = g1$ and so we would lose uniqueness. We use the notation $gx = (g \cdot x)g|_x$. We prove first that S is left cancellative. Let $ab = ac$ where $a = mg$, $b = nh$, and $c = pk$. Then $mgnh = mgpk$. Thus $m(g \cdot n)g|_n h = m(g \cdot p)g|_p k$. In the monoid M we have that $m(g \cdot n) = m(g \cdot p)$, and in the group G we have that $g|_n h = g|_p k$. By left cancellation in M and properties of the group action we get $n = p$ and so $h = k$. Hence $b = c$, as required.

We now show that the ideal structures of M and S are order-isomorphic. If $a \in S$ then $a = xg$ and so $aS = xS$. We prove that $xS \subseteq yS$ iff $xM \subseteq yM$. Suppose that $x = yb$ for some $b \in S$. Let $b = zu$ where $z \in M$ and $u \in G$. Then $x = (yz)u$. By uniqueness, $u = 1$ and so $x \in yM$. The converse is clear.

Finally, when M is a free monoid, the monoid S will be right hereditary and satisfy the dedekind height property because S and M have isomorphic ideal structures. The fact that S has a length function follows from Theorem 1.6. ■

Remark It follows from the theory that the length functions of left cancellative, right hereditary monoids satisfying the dedekind height property always restrict to the usual length function on the free submonoid X^*.

Combining Propositions 2.1, 2.3, 2.4 and 2.5, we obtain the following.

Theorem 2.6 *A monoid is left cancellative, right hereditary and satisfies the dedekind height property if and only if it isomorphic to a Zappa-Szép product of a free monoid by a group. Furthermore, Zappa-Szép products of free monoids by groups determine, and are determined by, ZS-actions.* ■

A natural question at this point is whether the monoids of the kind we are considering in this paper 'occur in nature'. We shall prove that they do, and complete an argument begun in [6].

Let S be a left cancellative, right hereditary monoid satisfying the dedekind height property. A subgroup N of $G(S)$ is said to be a *right normal divisor* of S if $Ns \subseteq sN$ for all $s \in S$. Define

$$M(S) = \{g \in G(S) \colon gs \in sG(S) \quad \forall s \in S\}.$$

Then $M(S)$ is the greatest right normal divisor of S [9]. We shall say that S is *fundamental*[2] if $M(G) = \{1\}$. Associated with S we have the action of $G(S)$ on X^* defined by $(g, x) \mapsto g \cdot x$. The following was proved as Proposition 2.6 of [6].

Proposition 2.7 *The action of $G(S)$ on X^* is faithful if and only if S is fundamental.* ■

The next result tells us that faithful ZS-actions admit a simpler characterisation.

Lemma 2.8 *Let G act faithfully on the free monoid X^* in a length-preserving way and suppose that there is a function $G \times X^* \to G$ denoted by $(g, x) \mapsto g|_x$ such that $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$. Then this determines a ZS-action.*

Proof We are assuming that axioms (SS1), (SS2) and (SS4) hold. We prove that the remaining axioms hold. From the fact that $g \cdot x = g \cdot (1x)$ we deduce that (SS3) and (SS5) hold. The proof of (SS7) follows from that fact that $xy = 1 \cdot (xy)$. The fact that (SS6) holds follows from the fact that $g \cdot (x(yz)) = g \cdot ((xy)z)$. Finally, (SS8) follows from the fact that $(gh) \cdot (xy) = g \cdot (h \cdot (xy))$. ■

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[2] This is not a term used by Rees, but is adapted from its related usage in inverse semigroup theory.

Actions of the type described in the lemma are said to be *faithful self-similar actions* [7]. We therefore conclude that *fundamental left cancellative right hereditary monoids satisfying the dedekind height property correspond to faithful self-similar group actions.*

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