Representations of the Thompson group F via representations of the polycyclic monoid on two generators

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Abstract

We show how the Thompson group F can be constructed from the polycyclic monoid on two generators.

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1 Introduction

The Thompson group F almost needs no introduction. I shall assume the reader has met some construction of the group; a good reference is [2]. The polycyclic monoid on two generators was introduced by Nivat and Perrot [7], and is discussed in Chapter 9 of my book [6]. Birget [1] has described one connection between the Thompson group and the polycyclic monoid on two generators: he proved that the group is a subgroup of a quotient algebra of the monoid. Our approach is different, and more elementary. The idea is to show that the elements of the Thompson group, regarded as bijections, can be represented as disjoint unions of partial bijections that ultimately arise from a representation of the polycyclic monoid. The idea for this representation came in two stages. In Section 9.3 of my book [6], I wrote up work with Peter Hines, some of which appeared in his thesis [5], on an interpretation of some calculations by Girard [4] in linear logic in terms of representations of the polycyclic monoid. It was only when I read Dehornoy's paper [3] that I realised the calculations Peter and I carried out could be used to construct the Thompson group F.

2 The polycyclic monoid and its representations

Let S be an inverse semigroup with zero. If $s \in S$ we write $\mathbf{d}(s) = s^{-1}s$ and $\mathbf{r}(s) = ss^{-1}$. A pair of elements $s, t \in S$ are said to be *disjoint* if

$$s^{-1}t = 0 = st^{-1}.$$

If s and t are disjoint I shall write $s \perp t$. Observe that $s \perp t$ iff $\mathbf{d}(s)\mathbf{d}(t) = 0$ and $\mathbf{r}(s)\mathbf{r}(t) = 0$.

Let X be a set. The symmetric monoid on X, denoted $I(X)$, is the inverse semigroup of all partial bijections on X. Note that functional composition will be evaluated from right to left. If $f \perp g$ in $I(X)$ then f and g have disjoint domains and disjoint ranges. It follows that their union $f \cup g$ also belongs to $I(X)$. I shall denote the operation of the union of disjoint functions in $I(X)$ by $+$, and call it *disjoint union*. If $f_1, \ldots, f_n$ is a set of n elements of $I(X)$ such that $f_i \perp f_j$ for $i \neq j$ then we can form $f_1 + \ldots + f_n$ which I shall also write as $\sum_{i=1}^{n} f_i$. The following lemma summarises some important properties of the disjoint union of partial bijections. The proofs are all easy.

Lemma 2.1 *In the symmetric inverse monoid, the following hold.*

(i) *If* $\sum_{i=1}^{n} f_i$ *exists, then* $\sum_{i=1}^{n} f_i^{-1}$ *exists and*

$$(\sum_{i=1}^{n} f_i)^{-1} = \sum_{i=1}^{n} f_i^{-1}.$$

(ii) *If* $\sum_{i=1}^{n} f_i$ *exists and* $f \in I(X)$ *then* $\sum_{i=1}^{n} ff_i$ *exists and*

$$f(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} ff_i,$$

and dually.

(iii) *If* $\sum_{i=1}^{m} f_i$ *and* $\sum_{j=1}^{n} g_j$ *both exist, then* $\sum_{i,j} f_i g_j$ *exists and*

$$(\sum_{i=1}^{m} f_i)(\sum_{j=1}^{n} g_j) = \sum_{i,j} f_i g_j.$$

(iv) *If* $(\sum_{i=1}^{n} f_i)$ *exists, then both* $\sum_{i=1}^{n} \mathbf{d}(f_i)$ *and* $\sum_{i=1}^{n} \mathbf{r}(f_i)$ *exist and*

$$\mathbf{d}(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} \mathbf{d}(f_i) \text{ and } \mathbf{r}(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} \mathbf{r}(f_i).$$

■

The *polycyclic monoid on two generators*, P_2, is defined by means of the following monoid-with-zero presentation

$$P_2 = \langle p, q, p^{-1}, q^{-1} \colon pp^{-1} = 1 = qq^{-1} \text{ and } pq^{-1} = 0 = qp^{-1} \rangle.$$

Each non-zero element in P_2 can be written uniquely in the form $x^{-1}y$ where x and y are strings over the alphabet $\{p, q\}$. With respect to these normal forms, the product in P_2 is defined as follows:

$$(x^{-1}y)(u^{-1}v) = \begin{cases} x^{-1}(wv) & \text{if } y = wu \\ (wx)^{-1}v & \text{if } u = wy \\ 0 & \text{else} \end{cases}$$

It can be proved that P_2 is congruence-free.

We shall be interested in homomorphisms from P_2 to $I(X)$, where X is a non-empty set, which are monoid homomorphisms and map the zero of P_2 to the zero of $I(X)$. Because P_2 is congruence-free, θ will be an injection. We call θ a *representation* of P_2. We say that a representation is *strong* if

$$\theta(p^{-1}p) + \theta(q^{-1}q) = 1_X.$$

By $X \sqcup X$ I mean the set $(X \times \{0\}) \cup (X \times \{1\})$.

Theorem 2.2

(i) *Every injection from $X \sqcup X$ to X determines and is determined by a representation of P_2.*

(ii) *Every bijection from $X \sqcup X$ to X determines and is determined by a strong representation of P_2.*

Proof (i) Let $\phi \colon X \sqcup X \to X$ be an injective function. Define functions $p^{-1}, q^{-1} \colon X \to X$ as follows:

$$p^{-1}(x) = \phi(x, 0) \text{ and } q^{-1}(x) = \phi(x, 1).$$

It is easy to check that p^{-1} and q^{-1} are injective functions. Their partial inverses are denoted p and q respectively. Hence $pp^{-1} = 1_X = qq^{-1}$. The fact that ϕ is injective implies that the images of p^{-1} and q^{-1} are disjoint. Thus $p^{-1}p \perp q^{-1}q$. It follows that $pq^{-1} = 0 = qp^{-1}$.

Conversely, suppose we have injective functions $p^{-1}, q^{-1} \colon X \to X$ such that $pp^{-1} = 1_X = qq^{-1}$ and $p^{-1}p \perp q^{-1}q$. Define $\phi \colon X \sqcup X \to X$ by

$$\phi(x, 0) = p^{-1}(x) \text{ and } \phi(x, 1) = q^{-1}(x).$$

It is easy to check that ϕ is an injection.

(ii) The function ϕ is surjective iff each element of X lies either in the image of p^{-1} or in the image of q^{-1} iff $1_X = p^{-1}p + q^{-1}q$. ■

In what follows, I shall assume that a strong representation of P_2 has been chosen and fixed. Without loss of generality, let $p, q \in I(X)$ be such that $pq^{-1} = 0 = qp^{-1}$ and $pp^{-1} = 1 = qq^{-1}$ where 0 denotes the empty function and 1 the identity function in $I(X)$. Let $f, g \in I(X)$. Define

$$f \oplus g = p^{-1}fp + q^{-1}gq.$$

Theorem 2.3 *With the above definitions we have that*

(i) $1 \oplus 1 = 1$.

(ii) $(f \oplus g)(h \oplus k) = fh \oplus gk$.

(iii) $(f \oplus g)^{-1} = f^{-1} \oplus g^{-1}$

Proof (i) By definition

$$1 \oplus 1 = p^{-1}1p + q^{-1}1q = 1.$$

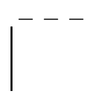
(ii) By definition

$$f \oplus g = p^{-1}fp + q^{-1}gq \text{ and } h \oplus k = p^{-1}hp + q^{-1}kq.$$

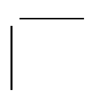
We now multiply these two expressions together using Lemma 2.1 and, after simplifying, obtain $fh \oplus gk$.

(iii) The proof is straightforward and uses Lemma 2.1. ■

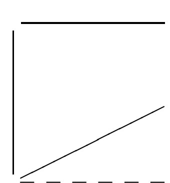
We shall now describe a way of representing the elements of P_2 by means of pictures, although not strictly necessary in what follows they provide useful motivation for otherwise arbitrary looking definitions. Each picture should be read *from right to left* and scaled to be the same size. The zero element has diagram



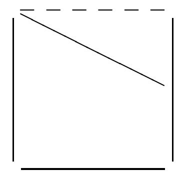
The identity $1 = \varepsilon^{-1}\varepsilon$ has the diagram



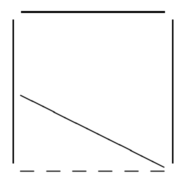
The element p has diagram



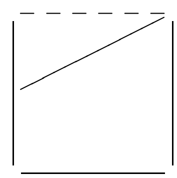
The element q has diagram



The element p^{-1} has diagram



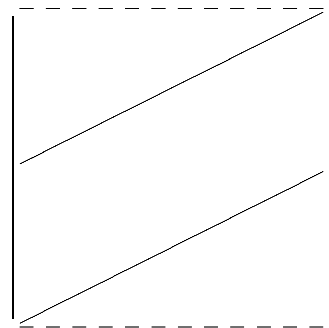
The element q^{-1} has diagram



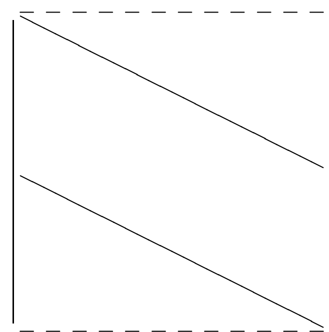
Here is the general recipe for constructing the diagram of $x^{-1}y$. The element x tells us about the range or left-hand side of the diagram, and y tells us about the domain or the right-hand side of the diagram. Let $|x| = m$ and $|y| = n$. Divide the range into 2^m equal intervals, and the domain into 2^n equal intervals. The string x tells us which of the 2^m to choose: read the string x from right-to-left; the letter p means 'top-half' and the letter q means 'bottom-half'. The same procedure is adopted for the string y. Let L be the interval that x describes, called the *range* of the diagram, and let R be the interval that y describes, called the *domain* of the diagram. The diagram is completed by drawing a line from the top of R to the top of L, and from the bottom of R to the bottom of L. Here are some examples.

Some examples of diagrams

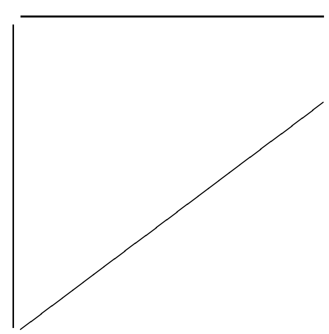
$q^{-1}p$



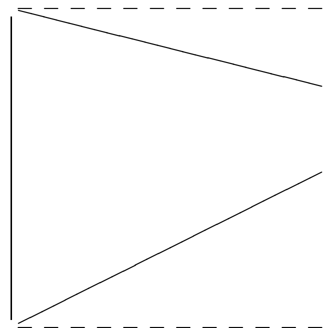
$p^{-1}q$



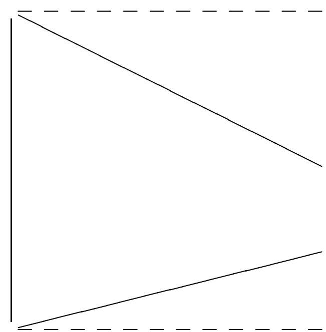
p^2



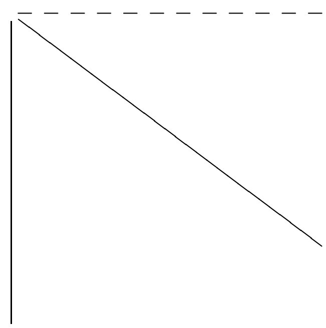
q^2



pq



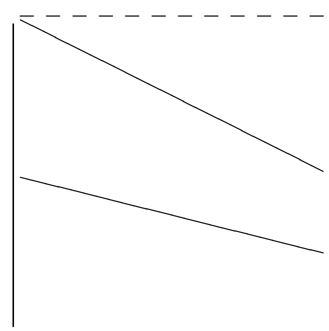
qp



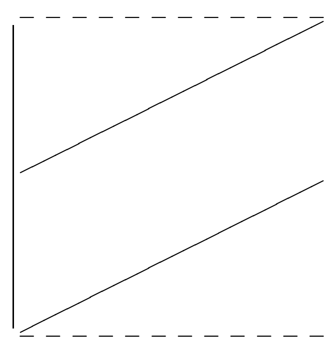
It is clear that there is a bijection between elements of P_2 and diagrams. It is possible to define a multiplication on diagrams so that this bijection becomes an isomorphism. Consider the diagrams D_1 and D_2 corresponding to $x^{-1}y$ and $u^{-1}v$ respectively:

- If the domain of D_1 does not contain or is not contained by the range of D_2 then the product of the two diagrams is the zero diagram.
- Suppose that $y = wu$. Then the domain of D_1 is wholly contained in the range of D_2 (or is equal to it if $w = \varepsilon$). The range of D_1D_2 is the same as the range of D_1; the domain of D_1D_2 is described by wv — it is therefore contained in the domain of D_2. This can be obtained by 'projecting back' the domain of D_1 to an interval within the domain of D_2.
- Suppose that $u = wy$. Then the range of D_2 is wholly contained within the domain of D_1. The domain of D_1D_2 is the same as the domain of D_2; the range is described by wx which is obtained by 'projecting forwards' the range of D_1.

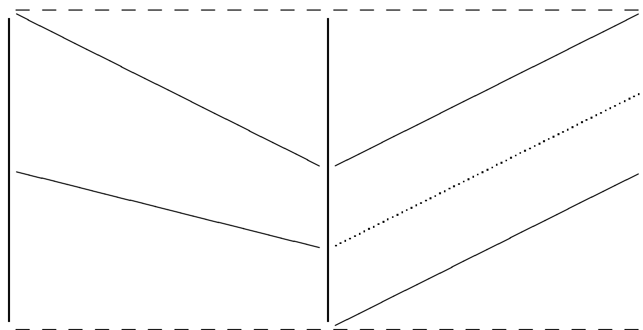
Here is an example of this procedure. Let the elements of P_2 we wish to multiply together be $p^{-1}pq$ and $q^{-1}p$ respectively. Their respective diagrams are



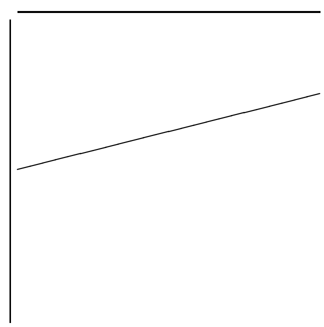
and



We now glue the diagrams together and, in this case, project back.



The diagram of the product is therefore



which corresponds to $p^{-1}p^2$.

The diagram notion for elements of P_2 has been introduced to make the following definitions more transparent. If P_2 is strongly embedded in $I(X)$ then I can *extend* the diagram notation to describe elements of $I(X)$ that are disjoint unions of elements of P_2.

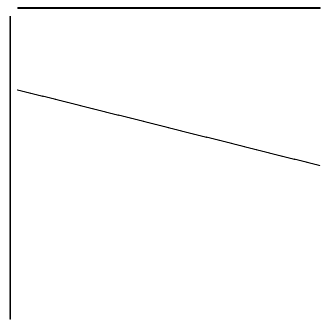
Proposition 2.4 *The element*

$$\alpha = p^{-2}p + (qp)^{-1}pq + q^{-1}q^2$$

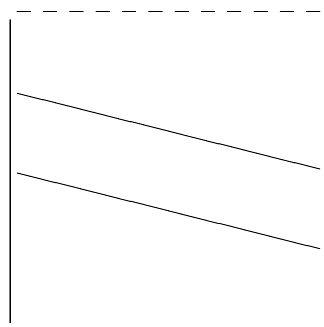
is a well-defined bijection in $I(X)$.

Proof The result can easily be checked algebraically. I shall show how an extended diagram can be constructed for α.

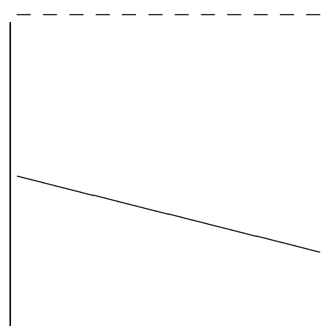
The diagram for $p^{-2}p$ is



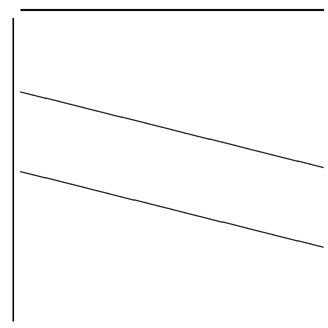
The diagram for $(qp)^{-1}pq$ is



The diagram for $q^{-1}q^2$ is



The extended diagram for α is therefore



■

The significance of the bijection α is explained by the following result.

Theorem 2.5 *For all $f, g, h \in I(X)$ we have that*

$$\alpha(f \oplus (g \oplus h))\alpha^{-1} = (f \oplus g) \oplus h.$$

Proof This is a straightforward calculation from the definitions. The details can be found in Theorem 14 of Section 9.3 of [6]. ■

Define

$$\beta = 1 \oplus \alpha.$$

It is clear that β is also a bijection. I leave it to the reader to construct an extended diagram of β.

Theorem 2.6 *For all $f, g, h, k \in I(X)$ we have that*

$$\beta(f \oplus (g \oplus (h \oplus k)))\beta^{-1} = f \oplus ((g \oplus h) \oplus k).$$

Proof We use Theorem 2.5 and Theorem 2.3. ■

3 The Thompson group F.

Theorem 3.1 *Given any strong representation of P_2 in $I(X)$, the elements α and β defined in Section 2 generate a subgroup of the symmetric group on X isomorphic to the Thompson group F.*

Proof Define $\chi_0 = \alpha$, $\chi_1 = \beta = 1 \oplus \alpha$, and, in general, $\chi_n = 1 \oplus \chi_{n-1}$ for $n \geq 2$. We first prove that

$$\chi_n = \alpha^{-1}\chi_{n-1}\alpha$$

for $n \geq 2$. For $n = 2$, we have that $\chi_2 = 1 \oplus (1 \oplus \alpha)$. Thus

$$\alpha\chi_2\alpha^{-1} = \alpha(1 \oplus (1 \oplus \alpha))\alpha^{-1} = (1 \oplus 1) \oplus \alpha = 1 \oplus \alpha = \chi_1.$$

For $n \geq 3$, the results can easily be proved by induction. It follows that the group generated by α and β is the same as the group generated by the χ_i where $i \in \mathbb{N}$. Next we prove that for all $k < n$ where $n \geq 1$ we have that

$$\chi_k^{-1}\chi_n\chi_k = \chi_{n+1}.$$

The claim is true for $k = 0$. To prove the general case, where $k \geq 1$ and $n \geq 1$, we verify that

$$\chi_k\chi_{n+1}\chi_k^{-1}$$

is equal to χ_n. This follows from Theorem 2.5, and the fact that

$$\chi_l = \overbrace{1 \oplus \ldots \oplus 1}^{l \text{ times}} \oplus \alpha$$

where I assume associativity to the right to avoid brackets.

From [2], we know that the Thompson group F has a presentation

$$F = \langle X_0, X_1, \ldots \colon X_k^{-1}X_nX_k = X_{n+1} \text{ for } k < n\rangle.$$

It follows that the group generated by α and β is a homomorphic image of F. It is known that every proper quotient of F is abelian. So if we can prove that the group generated by α and β is not abelian, we shall have proved that it is isomorphic to F, as claimed.

I shall prove that $\alpha\beta\alpha^{-1}\beta^{-1}$ is not the identity on X. Since the element $\alpha\beta\alpha^{-1}\beta^{-1}$ is, by Lemma 2.1, a disjoint union of partial bijections of X, it is enough to prove that one of the summands in this disjoint union is not an idempotent. The elements α, β, α^{-1} and β^{-1} are, respectively:

- $\alpha = p^{-2}p + \overline{(qp)^{-1}pq} + q^{-1}q^2$.
- $\beta = p^{-1}p + (p^2q)^{-1}pq + \overline{(qpq)^{-1}pq^2} + q^{-2}q^3$.
- $\alpha^{-1} = p^{-1}p^2 + (pq)^{-1}qp + \overline{q^{-2}q}$.
- $\beta^{-1} = p^{-1}p + \overline{(pq)^{-1}p^2q} + (pq^2)^{-1}qpq + q^{-3}q^2$.

where I have highlighted one summand from each element. Multiplying the highlighted elements together in order we get

$$\mu = (q^2p)^{-1}p^2q.$$

This is clearly not an idempotent, but I shall prove it nevertheless. Observe that $\mu^2 = 0$. If $\mu = 0$ then $p = 0$, but $X \neq \emptyset$ and so this is false. Hence μ is a non-zero element whose square is zero; it follows that μ cannot be an idempotent. ■

The Thompson group arises in many different mathematical contexts. The above theorem tells us that to obtain a copy of the Thompson group in some context, we can try first to find a strong representation of P_2 in this context. An isomorphic representation of the Thompson group will then come along from free. The point is that finding a strong representation of P_2 in a given context should not be difficult. The following examples illustrate this idea.

Examples 3.2

1. Let $\mathcal{C}$ be the Cantor set. Define

$$p\colon [0, \frac{1}{3}] \cap \mathcal{C} \to \mathcal{C} \text{ by } x \mapsto 3x$$

and

$$q\colon [\frac{2}{3}, 1] \cap \mathcal{C} \to \mathcal{C} \text{ by } x \mapsto 3x - 2.$$

Then this gives us a strong representation of P_2 on the Cantor set. The bijections α and β are homeomorphisms of the Cantor set, and so our theory gives a representation of the Thompson group by homeomorphisms of the Cantor set.

2. Let $\mathbb{E}$ and $\mathbb{O}$ be the sets of even and odd natural numbers respectively. Define

$$p\colon \mathbb{E} \to \mathbb{N} \text{ by } n \mapsto \frac{n}{2}$$

and

$$q\colon \mathbb{O} \to \mathbb{N} \text{ by } n \mapsto \frac{n-1}{2}.$$

Then this gives us a strong representation of P_2 on the set of natural numbers. Thus we get a representation of the Thompson group by permutations of the natural numbers.

3. This example is a little more complex. Let $I = [0, 1]$. Define

$$p\colon [0, \frac{1}{2}] \to [0, 1] \text{ by } x \mapsto 2x$$

and

$$q\colon [\frac{1}{2}, 1] \to [0, 1] \text{ by } x \mapsto 2x - 1.$$

A *dyadic rational* in I is a rational number that can be written in the form $\frac{2^a}{2^b}$ for some natural numbers a and b. Let I' be the unit interval with the dyadic rationals removed. The maps p and q and their inverses map dyadic rationals to dyadic rationals. We may therefore define

$$p'\colon [0, \frac{1}{2}]' \to [0, 1]' \text{ by } x \mapsto 2x$$

and

$$q'\colon [\frac{1}{2}, 1]' \to [0,1]' \text{ by } x \mapsto 2x - 1$$

where the primes on the intervals mean that dyadic rationals have been removed. It follows that we get a strong representation of P_2 on I'. The maps α and β are bijections defined on I'. Define functions A and B on I as follows:

$$A(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4} & \text{for } \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1 & \text{for } \frac{3}{4} \le x \le 1 \end{cases}$$

and

$$B(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \text{for } \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8} & \text{for } \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1 & \text{for } \frac{7}{8} \le x \le 1 \end{cases}$$

It can be checked that A restricted to I' is α, and B restricted to I' is β. Thus the Thompson group is also isomorphic to a group of bijections of the unit interval, the bijections being piecewise affine.

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