# A correspondence between a class of monoids and self-similar group actions I

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**Abstract**

We describe a correspondence between a class of left cancellative monoids and self-similar group actions in the sense of Nekrashevych *et al.* This correspondence originated in Perrot's 1972 thesis, and developed the ideas to be found in Rees' 1948 paper.

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## 1 Introduction

The theory of self-similar group actions arose in the 1980's by considering groups generated by automata. Many of these groups have interesting properties, such as the famous Grigorchuk group. Within this theory, it is known that self-similar group actions give rise to monoids, the so-called 'tensor semigroups' of [23]. However the structure of such monoids and their exact connection with the actions with which they are associated has not been pursued by the group-theorists. In fact, these semigroups were studied much earlier and were introduced by Perrot [25, 26] in the course of generalising some work of Rees. So it is with Rees' work that we must begin.

David Rees' 1948 paper [29][1] deals with the structure of left cancellative monoids. In particular, Rees isolates an important property of their partially

[1]This paper has been influential in the development of the theory of inverse semigroups:

ordered sets (posets) of principal right ideals $\mathbf{P}(S)$. He observes that the posets that occur in this way are special, in that for each principal right ideal $R$ the whole poset $\mathbf{P}(S)$ is isomorphic to the subposet of all principal right ideals contained in $R$. Motivated by this, he defines an arbitrary poset $P$ to be *uniform* if every principal order ideal is order isomorphic to $P$. Thus the posets $\mathbf{P}(S)$ are uniform. In fact, all uniform posets $P$ arise in this way: define $\mathbf{S}(P)$ to be the semigroup of all order isomorphisms from $P$ to its principal order ideals. Then $P$ is order isomorphic to $\mathbf{P}(\mathbf{S}(P))$. Uniformity is, of course, a notion of self-similarity. The simplest kinds of uniform posets are those order-isomorphic to the natural numbers with their dual ordering. Accordingly, Rees studies the left cancellative monoids whose posets have this property. It is interesting to note that such monoids are analogous to discrete valuation rings. More precisely, let $S$ be a left cancellative monoid in which the poset of principal right ideals has the following structure:

$$S = R_0 \supset R_1 \supset R_2 \supset \ldots$$

Let $a$ be a generator of $R_1$. Then $a^n$ is a generator of $R_n$. It follows that each element of $S$ can be written *uniquely* in the form $a^n g$ for some $n \geq 0$ and some $g \in G(S)$, the group of units of $S$. This uniqueness property has important consequences for the structure of the monoid. In particular, if $g$ is invertible then $ga = ah$ for a unique invertible element $h$. In the light of this result, define $\alpha \colon G(S) \to G(S)$ by $ga = a\alpha(g)$. Then $\alpha$ is an endomorphism of the group $G(S)$, the proof being a consequence of the fact that $(gh)a = g(ha)$ combined with the uniquenes of the decomposition. We can identify the submonoid of $S$ generated by $a$ with the monoid $\mathbb{N}$ under addition. It follows that $S$ is isomorphic to the set $\mathbb{N} \times G(S)$ equipped with the multiplication defined by

$$(m, g)(n, h) = (m + n, \alpha^n(g)h).$$

Rees therefore obtains a semidirect product decomposition of the class of left cancellative monoids under consideration.

Perrot [25, 26] generalised Rees' results. The first step was to generalise the condition on the poset of principal right ideals. To explain how he did this we shall need free monoids. The *free monoid* $X^*$ on a set $X$ consists of all finite sequences of elements of $X$ called *strings*, including the empty string $\varepsilon$, usually denoted by 1, with multiplication given by *concatenation* of strings. The monoid $\mathbb{N}$ is the free monoid on one generator. The key observaton is that its poset of principal right ideals is the decreasing chain

$$\mathbb{N} \supset 1 + \mathbb{N} \supset 2 + \mathbb{N} \supset \ldots.$$

Accordingly, Perrot introduced the more general class of left cancellative monoids in which we assume that $\mathbf{P}(S)$ is order isomorphic to the infinite tree of principal right ideals of the free monoid on $n$ generators where $n \geq 2$: the infinite

together with Clifford's 1953 paper [5], it provided one of the key ideas that led to the theory of 0-bisimple inverse semigroups [30, 19, 20, 21, 18]. One of the corollaries of this paper is that McAlister's theory of 0-bisimple inverse semigroups [18] can be viewed as a generalisation of the theory of self-similar group actions.

regular $n$-ary tree. The second step was to analyse this class of monoids in the spirit of Rees' paper.

For the rest of this section, $S$ will be a left cancellative monoid and $\mathbf{P}(S)$ will be isomorphic to $\mathbf{P}(A^*)$ where $A$ is a set with $n$ elements and $A^*$ is the free monoid on $A$.

Our monoid $S$ has $n$ maximal proper principal right ideals which I shall denote by $a_1S, \ldots, a_nS$. We denote the set of $a_i$'s by $A$. The set $A$ replaces the single element $a$ in the case Rees considered. Perrot shows that the submonoid $A^*$ of $S$ generated by the set $A$ is free, that $S = A^*G(S)$, and that each element of $S$ can be written *uniquely* as a product of an element of $A^*$ followed by an element of $G(S)$. This result is the direct generalisation of Rees'. However, unlike the case considered by Rees, the further analysis is more complex. This hinges on the fact that in general $gx = xh$ does *not* hold.[2] What we do have is that for each $x \in A^*$ and $g \in G(S)$ there is a unique $x' \in A^*$ and a unique $g' \in G(S)$ such that $gx = x'g'$. We follow Perrot and introduce some notation for the elements $x'$ and $g'$, although ours differs from his. We denote $x'$ by $g \cdot x$, and $g'$ by $g|_x$, and we call $g \cdot x$ 'action' and $g|_x$ 'restriction'. Thus

$$gx = (g \cdot x)g|_x.$$

Let $xg$ and $yh$ be elements of $S$ where $x, y \in A^*$ and $g, h \in G(S)$. Using the definitions above we see that

$$(xg)(yh) = x(g \cdot y)g|_y h.$$

Thus the multiplication in $S$ can be completely described in terms of action and restriction. Perrot axiomatised the properties of these two operations, and as a result was able to describe the multiplication in $S$ purely in terms of the action and restriction operations. In modern terminology, Perrot showed that his monoids are precisely Zappa-Szép products of free monoids and groups, where such products can be regarded as being two-sided semidirect products and so once again generalise Rees' construction.

The key point is this: *the actions Perrot obtains are* precisely *the self-similar group actions of [22], and as a consequence, Chapter 6 of Perrot's thesis is an independent discovery of self-similar group actions* avant la lettre.

I shall not continue with this historical approach here. Instead, my goal is to describe the correspondence between the class of left cancellative monoids introduced by Perrot and the self-similar group actions of Nekrashevych using any modern ideas that clarify the theory. However, the theory I describe is due, in all essentials, to Perrot.

Sections 2,3 and 4 of this paper describe Perrot's theory in modern dress. Section 5 makes the link between Perrot's work and the contemporary theory of self-similar group actions. Sections 6 and 7 provide alternative characterisations of self-similar group actions.

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[2] This is the result stated in [24] and is wrong.

## 2 Left Rees monoids

We begin with some definitions.

In a free monoid, the length $|x|$ of a string $x$ is the total number of elements of $X$ that occur in it. If $x = yz$ then $y$ is called a *prefix* of $x$. The *prefix order* on $X^*$ is defined by $x \leq y$ iff $x = yz$ for some string $z$.

Let $S$ be a monoid with group of units $G(S)$. A *length function* for $S$ is a surjective homomorphism $\lambda \colon S \to \mathbb{N}$ such that $\lambda^{-1}(0) = G(S)$. Free monoids have length functions: their usual length functions.

In a left cancellative monoid $S$, we have that $aS = bS$ iff $a = bg$ for some unit $g$, and we say that $a$ and $b$ are *associates*; the relation of being associates is an equivalence relation. We have that $aS = S$ iff $a$ is invertible.

A monoid $S$ is said to be *equidivisible* if for all $a, b, c, d \in S$ the fact that $ab = cd$ implies that either $a = cu, ub = d$ for some $u \in S$ or $c = av, b = vd$ for some $v \in S$.

**Lemma 2.1** *Let $S$ be a left cancellative monoid. Then the following are equivalent.*

(i) *Incomparable principal right ideals are disjoint.*

(ii) *$S$ is equidivisible.*

*If either holds, then incomparable principal left ideals are disjoint.*

**Proof** (i)⇒(ii). Suppose that $ab = cd$. Then $aS \cap cS \neq \emptyset$. Thus $aS \subseteq cS$ or $cS \subseteq aS$. Suppose the former. Then $a = cu$ for some $u \in S$. But $ab = cd$ and so $cub = cd$. By left cancellation, $ub = d$. Suppose the latter. Then $c = av$ for some $v \in S$. But $ab = cd$ and so $ab = avd$. By left cancellation, $b = vd$. Thus $S$ is equidivisible.

(ii)⇒(i). This is immediate.

To prove the last assertion, suppose that $Sb \cap Sd \neq \emptyset$. Then $ab = cd$ for some $b, c \in S$. The result now follows by equidivisibility. ■

A monoid $S$ will be called a *left Rees monoid*[3] if it satisfies the following conditions:

(LR1) $S$ is a left cancellative monoid.

(LR2) Incomparable principal right ideals are disjoint.

(LR3) Each principal right ideal is properly contained in only a finite number of principal right ideals.

We define *right Rees monoids* dually.

**Remark** If $a \in bS$ we say that $b$ is a *left factor* of $a$. Thus (LR3) says that each element has only a finite number of left factors upto associates. Condition

[3]They could, with equal justice, be called 'left Perrot monoids'.

(LR2) says that if $a$ and $b$ are both left factors of $c$ then one of $a$ and $b$ is a left factor of the other. Thus the definition of left Rees monoids is essentially an arithmetical one.

Let $aS$ and $bS$ be two principal right ideals. A *chain of length $n$ from $aS$ to $bS$* is a sequence

$$aS = a_0S \subset a_1S \subset a_2S \subset \ldots \subset a_nS = bS.$$

**Lemma 2.2** *Let $S$ be a left cancellative monoid in which incomparable principal right ideals are disjoint. Then the following are equivalent.*

(i) *Each principal right ideal is properly contained in only a finite number of principal right ideals.*

(ii) *For each $a \in S$ there exists a unique chain of maximum finite length starting at $aS$ and concluding at $S$.*

**Proof** (i) $\Rightarrow$ (ii). The set of all principal right ideals containing $a$ is finite. Thus there is a bound on the length of chains starting at $aS$ and ending at $S$. Given two such chains of maximum length they must be equal. To prove this, we show that the two chains must agree term by term. We use the fact that if two ideals have a non-empty intersection, then they must be comparable. Let

$$aS = a_0S \subset a_1S \subset \ldots \subset a_mS = S$$

and

$$aS = b_0S \subset b_1S \subset \ldots \subset b_nS = S$$

be two such chains of maximum length. We claim that $a_1S = b_1S$. To see why observe that they are comparable because both contain $aS$. Thus either $a_1S \subset b_1S$ or vice-versa. If the former we could refine the second chain, if the latter we could refine the first chain. But neither refinement is possible since each chain is of maximum length. Thus $a_1S = b_1S$. This process continues. If $m > n$ then we could use the first chain to refine the second. If $n > m$ then we could use the second chain to refine the first. So the two chains must have the same length and the same terms.

(ii) $\Rightarrow$ (i). All the distinct principal right ideals containing $aS$ must be comparable so they will form a totally ordered set from $aS$ to $S$. This will be a chain of maximum length and so equal to the unique such chain assumed to exist. Thus the set of all principal right ideals containing $aS$ must be finite. ■

The next lemma provides us with a class of examples of left Rees monoids.

**Lemma 2.3** *Let $S$ be a left cancellative monoid in which incomparable principal right ideals are disjoint equipped with a length function. Then each principal right ideal is properly contained in only a finite number of distinct principal right ideals.*

**Proof** Let $aS \subseteq bS$. Then $a = bs$ and so $\lambda(a) = \lambda(b)+\lambda(s)$. Thus, in particular, $\lambda(a) \geq \lambda(b)$. Suppose, in addition, that $\lambda(a) = \lambda(b)$. Then $\lambda(s) = 0$ and so $s$ is a unit. It follows that in this case, $aS = bS$. We deduce that $aS = bS$ iff $\lambda(a) = \lambda(b)$.

Let $aS$ be a fixed principal right ideal. By the above results, the distinct principal right ideals containing $aS$ correspond bijectively to the distinct natural numbers strictly less than $\lambda(a)$. Thus there are only finitely many principal right ideals containing $aS$. ■

Let $S$ be a left Rees monoid. Let

$$aS = S_0 \subset S_1 \subset S_2 \ldots \subset S_n = S$$

be a chain of principal right ideals of maximum length. We define $\lambda(a) = n$.

**Lemma 2.4** *Let $S$ be a left Rees monoid. Let*

$$bS = b_0S \subset b_1S \subset b_2S \subset \ldots \subset b_{\lambda(b)}S = S$$

*be a chain of maximum length joining $bS$ to $S$. Then for any $a \in S$ we have that*

$$abS = ab_0S \subset ab_1S \subset ab_2S \subset \ldots \subset ab_{\lambda(b)}S = aS$$

*is a chain of maximum length joining $abS$ to $aS$.*

**Proof** We show first that the inclusions really are distinct. Suppose that $ab_iS = ab_{i+1}S$ for some $i$. Then $ab_i = ab_{i+1}g$ for some unit $g$. By left cancellation, $b_i = b_{i+1}g$ giving $b_iS = b_{i+1}S$, which contradicts our assumption. Next we show that the chain is of maximum length. Suppose not. Then we can interpolate a principal right ideal somewhere

$$ab_iS \subset cS \subset ab_{i+1}S.$$

Let $ab_i = cf$ for some $f$ and $c = ab_{i+1}d$ for some $d$. Thus by left cancellation, $b_i = b_{i+1}df$. We therefore have

$$b_iS \subseteq b_{i+1}dS \subseteq b_{i+1}S.$$

Suppose that $b_iS = b_{i+1}dS$. Then $b_i = b_{i+1}dg$ for some unit $g$. By left cancellation, it follows that $g = f$ and is a unit. Thus $ab_iS = cS$, which is contradiction. Suppose that $b_{i+1}dS = b_{i+1}S$. Then $b_{i+1}d = b_{i+1}h$ for some unit $h$. By left cancellation, $d = h$ and so $cS = ab_{i+1}S$, which is a contradiction. However, we now have

$$b_sS \subset b_{i+1}dS \subset b_{i+1}S$$

which contradicts the fact that our original chain was of maximum length. It follows that our new chain is of maximum length. ■

**Lemma 2.5** *Let $S$ be a left Rees monoid which is not a group. Then the function $\lambda$ defined before Lemma 2.4 is a length function which sends an element to 1 if and only if it is irreducible.*

**Proof** We show first that $S$ has an infinitely descending chain of principal right ideals. Let $a \in S$ be non-invertible. Let $aS \subseteq bS$ where $bS$ is a maximal proper principal right ideal. Then $b$ is non-invertible. We therefore have an infinitely descending sequence of ideals

$$bS \supset b^2S \supset b^3S \supset \dots$$

We show that this is a maximal such chain. Suppose that $b^{n+1}S \subseteq cS \subseteq b^nS$. Then $b^{n+1} = cx$ and $c = b^ny$. Thus $b^{n+1} = b^nyx$ giving $b = yx$. Thus $bS \subseteq yS$. It follows that either $bS = yS$ or $y$ is invertible. Suppose the former. Then $b = yg$ for some invertible element $g$. It follows that $x$ is invertible and so $cS = b^{n+1}S$. Suppose the latter. Then $y$ is invertible and so $cS = b^nS$. Thus the chain is maximal. It follows that the function $\lambda$ is surjective.

By Lemma 2.4, if

$$bS = b_0S \subset b_1S \subset b_2S \subset \dots \subset b_{\lambda(b)}S = S$$

is a chain of maximum length joining $bS$ to $S$, then

$$abS = ab_0S \subset ab_1S \subset ab_2S \subset \dots \subset ab_{\lambda(b)}S = aS$$

is a chain of maximum length joining $abS$ to $aS$. Now glue this to a chain

$$aS = a_0 \subset a_1S \subset \dots \subset a_{\lambda(a)}S = S$$

of maximum length. The resulting chain links $abS$ to $S$ and has maximum length, and this length is $\lambda(a) + \lambda(b)$. Thus $\lambda$ is a homomorphism.

Those elements $a$ of length 0 are precisely those where $aS = S$, which are just the invertible elements.

The elements of length 1 are the irreducibles by construction. ■

We combine Lemmas 2.2, 2.3 and 2.5 in the following theorem.

**Theorem 2.6** *Let $S$ be a left cancellative monoid, which is not a group, in which incomparable principal right ideals are disjoint. Then the following are equivalent.*

(i) *Each principal right ideal is properly contained in only a finite number of principal right ideals.*

(ii) *For each $a \in S$ there exists a unique chain of maximum finite length starting at $aS$ and concluding at $S$.*

(iii) *The monoid $S$ is equipped with a length function in which generators of maximal proper principal right ideals are mapped to 1.*

(iv) *The monoid $S$ is equipped with a length function.* ■

One final ambiguity concerns the status of cancellative left Rees monoids. Theoretically, they might not be right Rees monoids. We now show this cannot happen.

**Proposition 2.7** *Let $S$ be a cancellative monoid admitting a length function. Then $S$ is a left Rees monoid if and only if it is a right Rees monoid.*

**Proof** Let $S$ be a left Rees monoid. By Lemma 2.1, we know that if two principal left ideals have a non-empty intersection then they are comparable. Thus $S$ is right cancellative and incomparable principal left ideals are disjoint. By the left-right dual of Lema 2.3, each principal left ideal is properly contained in only a finite number of distinct principal right ideals. Thus $S$ is a right Rees monoid. The converse follows by symmetry. ■

We define a monoid to be a *Rees monoid* if it is both a left Rees monoid and a right Rees monoid. It follows by the above proposition, that cancellative left Rees monoids are Rees monoids.

We conclude by describing some examples of left Rees monoids.

**Examples 2.8**

(i) From Theorem 2.6, Proposition 2.7 and Corollary 5.1.6 of [12] and the fact that free monoids are left Rees monoids we deduce that free monoids are precisely the left Rees monoids in which the group of units is trivial. This is the first indication that we might be able to construct arbitrary left Rees monoids from free monoids and groups.

(ii) Left Rees monoids provide natural examples of monoids defined in 'homological' terms. We recall some definitions first. An *$S$-act* or *act* $(X, S)$ is an action of a monoid $S$ on a set $X$ on the right. If $S$ is a monoid then $(S, S)$ is an act by right multiplication. If $Y \subseteq X$ is a subset such that $YS \subseteq Y$ then we say that $Y$ is an *$S$-subact* or just a *subact*. Right ideals of $S$ are subacts under right multiplication. If $X$ and $Y$ are acts then a function $\theta$ from $X$ to $Y$ is an *$S$-homomorphism* or just a *homomorphism* if $\theta(xs) = \theta(x)s$ for all $x \in X$ and $s \in S$. For a fixed $S$, we can form the category consisting of $S$-acts and the homomorphisms between them. The usual definitions from module theory can be adapted to the theory of acts. In particular, we can define when an act is *projective*. A monoid $S$ is said to be *right PP* if all its principal right ideals are projective as right $S$-acts, and *right hereditary* if all its right ideals are projective as right $S$-acts. Dorofeeva [6] proved that a monoid $S$ is right hereditary iff it is right PP, incomparable principal right ideals are disjoint, and $S$ has the ascending chain condition for principal right ideals. Right PP monoids with a single idempotent are precisely the left cancellative monoids. Thus left Rees monoids are examples of right hereditary monoids with a single idempotent.

(iii) The idea for the following example comes from [28] where the similarity semigroup of the Sierpinski gasket is defined. The proof uses some ideas to be found in the next section. Centre the Sierpinski gasket at the origin, and consider the monoid $S$ of all similarities of the plane that map the gasket into itself. The group of units of this monoid is just the six element group of symmetries of the equilateral triangle. I shall now pick out certain important elements of $S$: a clockwise rotation by $\frac{2\pi}{3}$ denoted by $\rho$; a reflection in the vertical denoted by $\sigma$; and three similarities denoted $T$, $L$ and $R$ which halve the size of the gasket and then translate it to the top, left and right parts of the original gasket. It is not hard to see that the monoid generated by these similarities is $S$ and that the submonoid of $S$ generated by $T$, $L$ and $R$ is the free monoid on three generators. Simple calculations show that

$$\rho T = R\rho, \quad \rho L = T\rho, \quad \rho R = L\rho$$

and

$$\sigma T = T\sigma, \quad \sigma L = R\sigma, \quad \sigma R = L\sigma.$$

Thus every element of $S$ can be written as a product of an element of a free monoid and a group element. This representation is unique: if $xg = yh$ where $g, h \in G(S)$ and $x, y \in \{T, L, R\}^*$ then $x = yhg^{-1}$. However elements of $\{T, L, R\}^*$ do not change the orientation of a triangle whereas non-identity elements of $G(S)$ do. Thus $g = h$ and so $x = y$. It is now easy to check that $S$ is a Rees monoid. In particular, the relations above show that we have defined a non-trivial action of the group $G(S)$ on the free monoid on three generators generated by $T$, $L$ and $R$ that satisfies axioms (SS1)–(SS8) of Section 2.

(iv) The class of left Rees monoids is a subclass of the class of all left cancellative monoids. We shall now show how closely these two classes are related. We shall use the theory of Rhodes-expansions described in [3] adapted to our situation. Let $S$ be a left cancellative monoid. We shall be interested in finite sequences of elements of $S$

$$\mathbf{x} = (x_1, \ldots, x_n)$$

where $x_{i+1} \in x_i S$ but $x_{i+1}S \neq x_n S$ and where $x_1$ is a unit. We denote by $\hat{S}$ the set of all such sequences. We shall now define a product on such sequences. Let

$$\mathbf{x} = (x_1, \ldots, x_m) \text{ and } \mathbf{y} = (y_1, \ldots, y_n).$$

Consider the sequence

$$x_1, \ldots, x_{m-1}, x_m, x_m y_1, \ldots, x_m y_n.$$

Because $y_1$ is a unit, we have that $x_m S = x_m y_1 S$. Clearly, $x_m y_1 S \subset x_{m-1}S$. Also from $y_{i+1}S \subset y_i S$ we get $x_m y_{i+1} S \subseteq x_m y_i S$. Observe that if $x_m y_{i+1} S = x_m y_i S$ then $x_m y_{i+1} = x_m y_i g$ for some unit $g$. Thus by left cancellation, $y_{i+1} = y_i g$ implying that $y_{i+1}S = y_i S$, contradicting our assumption. It follows that

$$\mathbf{xy} = (x_1, \ldots, x_{m-1}, x_m y_1, \ldots, x_m y_n)$$

is a well-defined element of $\hat{S}$. This defines a binary operation on $\hat{S}$. The fact that this is a semigroup follows from the general theory in [3]. It is easy to check that it is a monoid with identity (1), and that left cancellation in $S$ is inherited by $\hat{S}$. We claim that for each left cancellative monoid $S$, the monoid $\hat{S}$ is a left Rees monoid. There is a surjective homomorphism from $\hat{S}$ onto $S$. We now prove our claim. Suppose that $\mathbf{x} \in \mathbf{y}\hat{S}$. Then

$$(x_1, \ldots, x_m) = (y_1, \ldots, y_n)(z_1, \ldots, z_p).$$

Thus $m \geq n$, $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ and $y_n S = x_n S$. Conversely, suppose that $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ are such that $m \geq n$, $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}$ and $y_n S = x_n S$. For $0 \leq i \leq m - n$ define $z_{i+1}$ by $x_{n+i} = y_n z_{i+1}$. Observe that $z_1$ is a unit. It is easy to check that $\mathbf{z} = (z_1, \ldots, z_p)$ is a well-defined element of $\hat{S}$ and that $\mathbf{x} = \mathbf{yz}$. Suppose that $\mathbf{x}\hat{S} \cap \mathbf{y}\hat{S} \neq \emptyset$. Then there is a $\mathbf{z}$ which has both $\mathbf{x}$ and $\mathbf{y}$ as left factors. Let $\mathbf{z} = (z_1, \ldots, z_p)$, $\mathbf{x} = (x_1, \ldots, x_m)$, and $\mathbf{y} = (y_1, \ldots, y_n)$. Then $p \geq m, n$ and $x_1 = z_1, \ldots, x_{m-1} = z_{m-1}, z_m S = x_m S$ and $y_1 = z_1, \ldots, y_{n-1} = z_{n-1}, z_n S = y_n S$. Without loss of generality, suppose that $m \leq n$. Then $x_1 = y_1, \ldots, x_{m-1} = y_{m-1}$ and $x_m S = z_m S = y_m S$. Thus $\mathbf{y} \in \mathbf{x}\hat{S}$. From the above we can easily derive the criterion for $\mathbf{y}\hat{S} = \mathbf{x}\hat{S}$: $\mathbf{x}$ and $\mathbf{y}$ have the same length, all the components are the same except the rightmost ones which differ by a unit. Define $\eta_S \colon \hat{S} \to S$ by $(x_1, \ldots, x_n) \mapsto x_n$. Then this is a surjective homomorphism. Observe that restricted to the $\mathcal{R}$-classes of $\hat{S}$, this homomorphism is injective. Thus $\hat{S}$ is a left Rees monoid.

■

# 3 Zappa-Szép products

**Assumption** We shall assume that our left Rees monoids are not groups. By Lemma 2.5 this is equivalent to assuming that they are equipped with length functions.

The goal of this section is to obtain a structural description of the class of left Rees monoids. The tool we shall use goes under many names. Within the theory of quantum groups, it is known as the 'bicrossed product' [10], within semigroup theory as the 'Zappa-Szép product', the term I shall use. The paper by Matt Brin [4] is a useful introduction and contains further references. However, I shall prove almost everything from scratch here and so prior knowledge of this construction is not needed. The key result on the structure of left Rees monoids is the following.

**Proposition 3.1** *Let $S$ be a left Rees monoid. Let $X$ be a transversal of the generators of the maximal proper principal right ideals, and denote by $X^*$ the submonoid generated by the set $X$. Then the monoid $X^*$ is free, $S = X^*G(S)$, and each element of $S$ can be written uniquely as a product of an element of $X^*$ and an element of $G(S)$.*

**Proof** We show first that $S = X^*G(S)$. Let $s \in S \setminus G(S)$. Consider the set of all proper ideals that contain $s$. This set contains a maximal element $x_1S$, which is necessarily a maximal proper principal right ideal, and $x_1 \in X$. Thus $s = x_1s_1$. If $s_1$ is a unit or generates a maximal proper principal right ideal the process stops. Otherwise, repeat this process with $s_1$ to get $s_1 = x_2s_2$ and so on. Thus we can write $s = x_1 \dots x_is_i$. To show that this process terminates observe that

$$sS \subset x_1 \dots x_iS \subset \dots \subset x_1S.$$

Thus termination follows from Lemma 2.2. It follows that we can write $s = x_1 \dots x_ng$ where $g$ is a unit.

Next we prove that $X^*$ is free. Suppose that

$$1 = x_1 \dots x_m$$

where $m \geq 1$ and $x_i \in X$. Then $S = x_1 \dots x_mS \subseteq x_1S$ and so $x_1$ is invertible, which is a contradiction and so $m = 1$ and $x_1 = 1$.

Now suppose that

$$x_1 \dots x_m = y_1 \dots y_n$$

where $x_i, y_j \in X$. By our result above, we can assume that $m, n \geq 1$. Now $x_1 \dots x_mS = y_1 \dots y_nS \subseteq x_1S, y_1S$. It follows that $x_1 = y_1$ and so, by left cancellation, $x_2 \dots x_m = y_2 \dots y_n$. This process can be repeated and because either $m < n$ or $n < m$ would lead to a contradiction, namely that an element of $X$ is invertible, we must have that $m = n$ and $x_i = y_i$. Thus $X^*$ is the free monoid on $X$.

Finally, we prove the uniqueness of the decomposition. Suppose that $xg = yh$ where $x, y \in X^*$, $g, h \in G(S)$, and $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$ where $x_i, y_j \in X$. Arguing as before, $xS = yS \subseteq x_1S, y_1S$ and so $x_1 = y_1$. By left cancellation $x_2 \dots x_mg = y_2 \dots y_nh$. If $m = n$ then we can repeat this argument to get $x = y$ and so $g = h$, by left cancellation. If $m < n$, then we can easily deduce that $y_{m+1}$ is invertible, which is a contradiction. A similar argument shows that we cannot have $n < m$. ■

The fact that each element of $S$ can be written uniquely as a product of an element from a free monoid and an element from the group of units we call the *uniqueness property*.

Having proved our key result, we now come to our key definition. Let $G$ be a group and $X^*$ a free monoid on the set $X$. We suppose that there are two operations that link $G$ and $X^*$. The first is defined by a function $G \times X^* \to X^*$, called the *action*, denoted by $(g, x) \mapsto g \cdot x$. The second is defined by a function $G \times X^* \to G$, called the *restriction*, denoted by $(g, x) \mapsto g|_x$. We require that the following eight axioms hold:[4]

(SS1) $1 \cdot x = x$.

(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$.

(SS3) $g \cdot 1 = 1$.

(SS4) $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

(SS5) $g|_1 = g$.

(SS6) $g|_{xy} = (g|_x)|_y$.

(SS7) $1|_x = 1$.

(SS8) $(gh)|_x = g|_{h \cdot x} h|_x$.

The axioms (SS1)–(SS3) tell us that we have a left action $G \times X^* \to X^*$ in which the empty string is a fixed point; the axioms (SS5)–(SS7) tell us that we have a right actions $G \times X^* \to G$ in which the group identity is a fixed point; and axioms (SS4) and (SS8) tell us how the two actions interact.

**Proposition 3.2** *With each left Rees monoid, we can associate an action satisfying the axioms (SS1)–(SS8).*

**Proof** We use the uniqueness property established in Proposition 3.1. The proof follows by considering properties of the identity element and different cases of the associativity law. From $1x = x$, we deduce both (SS1) and (SS7). From $g1 = g$, we deduce both (SS3) and (SS5). From $(gh)x = g(hx)$, we deduce both (SS2) and (SS8). Finally, from $(gx)y = g(xy)$, we deduce both (SS4) and (SS6).■

An action of a group $G$ on a free monoid $X^*$ is *length-preserving* if $|g \cdot x| = |x|$ for all $x \in X^*$, and *prefix-preserving* if $x = yz$ in $X^*$ implies that $g \cdot x = (g \cdot y)z'$ for some string $z'$. This means precisely that if $x \leq y$ then $g \cdot x \leq g \cdot y$.

**Lemma 3.3** *Let $G$ act on $X^*$ in such a way that the axioms (SS1)–(SS8) hold. Then the action is length-preserving and prefix-preserving.*

**Proof** Prefix-preserving follows from (SS4). We now prove that the action is length-preserving. Observe first that by (SS3), if $x$ is the empty string so too is $g \cdot x$. Conversely, if $g \cdot x = 1$ then $x = g^{-1} \cdot 1 = 1$ by (SS3). Thus $g \cdot x$ is the empty string iff $x$ is. Let $x \in X$. Suppose that $g \cdot x = yz$ where $y$ is a letter and $z$ is a string, possibly empty. Then by (SS4), we have that

$$x = (g^{-1} \cdot y)(g^{-1}|_y \cdot z).$$

[4] Observe that we use 1 to denote both the identity of the group $G$ and the empty string of $X^*$.

We know that $g^{-1} \cdot y$ cannot be empty and so has length at least one. Since the leftthand side has length one and lengths add, we deduce that $(g^{-1}|_y \cdot z)$ has length zero. Thus $z$ is the empty string. It follows that letters are mapped to letters. The result now follows by (SS4) and induction. ■

We define $G_x$ to be the stabiliser of $x \in X^*$ under the (left) action of $G$.

**Lemma 3.4** *Let $G$ act on $X^*$ in such a way that the axioms (SS1)–(SS8) hold.*

(i) $(g|_x)^{-1} = g^{-1}|_{g \cdot x}$.

(ii) $(g^{-1}|_x)^{-1} = g|_{g^{-1} \cdot x}$.

(iii) *The function $\phi_x$: $G_x \to G$ given by $g \mapsto g|_x$ is a homomorphism.*

(iv) *Let $y = g \cdot x$. Then $G_y = gG_xg^{-1}$ and*

$$\phi_y(h) = g|_x \phi_x(g^{-1}hg)(g|_x)^{-1}.$$

**Proof** (i) By (SS7), $1 = 1|_x$. Thus $1 = 1|_x = (g^{-1}g)|_x = (g^{-1}|_{g \cdot x})g|_x$ using (SS8). Hence $(g|_x)^{-1} = g^{-1}|_{g \cdot x}$.

The proof of (ii) follows of course from (i)

(iii) Let $g, h \in G_x$. Then

$$\phi_x(gh) = (gh)|_x = g|_{h \cdot x} h|_x = g|_x h|_x = \phi_x(g)\phi_x(h),$$

using (SS8), as required.

(iv) We have that $h \cdot y = y$ iff $h \cdot (g \cdot x) = g \cdot x$ iff $g^{-1}hg \cdot x = x$ iff $g^{-1}hg \in G_x$. Hence iff $h \in gG_xg^{-1}$. The proof of the second part is obtained by direct calculation. ■

We shall now show how to construct left Rees monoids from actions satisfying axioms (SS1)–(SS8). Let $G$ be an arbitrary group, and $M$ an arbitrary left cancellative monoid (not necessarily free) equipped with a function $G \times M \to M$, denoted by $(g, m) \mapsto g \cdot m$, and a function $G \times M \to G$, denoted by $(g, m) \mapsto g|_m$, satisfying the obvious generalisations of (SS1)–(SS8). On the set $M \times G$ define the binary operation by

$$(x, g)(y, h) = (x(g \cdot y), g|_y h).$$

The following is part of the general theory of Zappa-Szép products, but we prove it anyway.

**Proposition 3.5** *With the above product, $M \times G$ is a left cancellative monoid containing copies of $M$ and $G$ such that $M \times G$ can be written as a unique product of these copies.*

**Proof** We begin by proving associativity. We calculate first

$$[(x,g)(y,h)](z,k).$$

By (SS2), (SS8), and (SS6) we get

$$(x(g\cdot y)g|_y\cdot(h\cdot z), g|_{y(h\cdot z)}h|_z k).$$

We now calculate

$$(x,g)[(y,h)(z,k)].$$

Using (SS4), we get the same result.

We now show that $(1,1)$ is the identity. We calculate

$$(1,1)(x,g) = (1(1\cdot 1), 1|_x g) = (x,g)$$

using (SS1) and (SS7). We calculate

$$(x,g)(1,1) = (x(g\cdot 1), g|_1 1) = (x,g)$$

using (SS3) and (SS5). We have now used all the axioms (SS1)–(SS8).

Next we show that $M\times G$ is left cancellative. Suppose that

$$(x,g)(y,h) = (x,g)(z,k).$$

Then

$$(x(g\cdot y), g|_y h) = (x(g\cdot z), g|_z k).$$

Left cancellation in $M$ gives us $g\cdot y = g\cdot z$ and so because this is an action $y = z$. Hence $h = k$.

We now have to show that $M$ and $G$ are each embedded in $M\times G$. Define $\iota_M\colon M\to M\times G$ by $x\mapsto(x,1)$. This is an injective homomorphism by (SS1) and (SS7). Denote its image by $M'$. Define $\iota_G\colon G\to M\times G$ by $g\mapsto(1,g)$. This is an injective homomorphism by (SS3) and (SS5). Denote its image by $G'$. Observe that $(x,g) = (x,1)(1,g)$. Thus $M\times G = M'G'$. This decomposition is evidently unique. ■

The monoid constructed in Proposition 3.5 is called the *Zappa-Szép product of $M$ and $G$* and is denoted $M\bowtie G$.

**Proposition 3.6** *Let $S$ be a monoid. Suppose that $S = MG$ uniquely where $M$ is a left cancellative monoid and $G$ is a group. Then $S$ is a left cancellative monoid whose poset of principal right ideals is order isomorphic with that of $M$. In particular, when $M$ is a free monoid, the monoid $S$ is a left Rees monoid.*

**Proof** Observe that $\{1\} = G\cap M$. To see why if $g\in G\cap M$ then $g = 1g = g1$ and so we would lose uniqueness. We use the notation $gx = (g\cdot x)g|_x$. We prove first that $S$ is left cancellative. Let $ab = ac$ where $a = mg$, $b = nh$, and $c = pk$. Then $mgnh = mgpk$. Thus $m(g\cdot n)g|_n h = m(g\cdot p)g|_p k$. In the monoid $M$ we have that $m(g \cdot n) = m(g \cdot p)$, and in the group $G$ we have that $g|_n h = g|_p k$. By left cancellation in $M$ and properties of the group action we get $n = p$ and so $h = k$. Hence $b = c$, as required.

We now show that the posets of principal right ideals of $M$ and $S$ are order-isomorphic. If $a \in S$ then $a = xg$ and so $aS = xS$. We prove that $xS \subseteq yS$ iff $xM \subseteq yM$. Suppose that $x = yb$ for some $b \in S$. Let $b = zu$ where $z \in M$ and $u \in G$. Then $x = (yz)u$. By uniqueness, $u = 1$ and so $x \in yM$. The converse is clear.

Finally, when $M$ is a free monoid, the monoid $S$ will be a left Rees monoid since it inherits the order on its principal right ideals from that on the principal right ideals of $M$. ∎

**Remark** It follows from the theory that the length functions of left Rees monoids can always be chosen to restrict to the usual length function on the free submonoid $X^*$.

Combining Propositions 3.1, 3.2, 3.5 and 3.6, we obtain the following important theorem which is just a modern way of expressing Perrot's original work.

**Theorem 3.7 (Perrot)** *A monoid is a left Rees monoid (which is not a group) if and only if it isomorphic to a Zappa-Szép product of a free monoid by a group. Furthermore, Zappa-Szép products of free monoids by groups determine, and are determined by, actions satisfying axioms (SS1)–(SS8).* ∎

In Section 5, I shall examine actions satisfying axioms (SS1)–(SS8) in more detail.

# 4 Right normal divisors

In this section, we shall adapt some results from [29] to the case of left Rees monoids. Throughout this section, $S$ is a left Rees monoid and we fix a decomposition $S = X^*G$ which gives us the definitions of action and restriction.

Let $S$ be a left Rees monoid. Define

$$K(S) = \{g \in G(S) \colon gs \in sG(S) \text{ for all } s \in S\}.$$

This is a normal subgroup of $G(S)$. There is no agreed terminology for this subgroup, but we suggest calling it the *kernel* of the left Rees monoid. Left Rees monoids $S$ for which $K(S) = \{1\}$ are said to be *fundamental*.[5]

**Lemma 4.1** $K(S) = \bigcap_{x \in X^*} G_x$.

[5] This is not a term used by Rees, but is adapted from its related usage in inverse semigroup theory.

**Proof** Let $g \in K(S)$. Let $x \in X^*$. Then $gx \in xG$. Thus $gx = xh$ for some $h$. But $gx = (g \cdot x)g|_x$ and so $gx = (g \cdot x)g_x = xh$. By uniqueness, $g \cdot x = x$.

Conversely, let $g \in \bigcap_{x \in X^*} G_x$. Let $s \in S$. Then $s = xh$. Thus

$$gs = g(xh) = (g \cdot x)g|_x h = (xh)h^{-1}g|_x h \in sG.$$

It follows that $g \in K(S)$. ■

The following is now immediate and is the first step in linking the structure of the left Rees monoid to the structure of the associated action, a link we explore in more detail in [15].

**Corollary 4.2** *The action of $G$ on $X^*$ is faithful iff $K(S) = \{1\}$. Thus fundamental left Rees monoids correspond to faithful actions satisfying axioms (SS1)–(SS8).* ■

Put $K_1 = \bigcap_{x \in X} G_x$. This is a normal subgroup of $G$ and contains $K$.

**Lemma 4.3** *We have that $g \in K(S)$ iff $g|_x \in K_1$ for all $x \in X^*$.*

**Proof** Let $g \in K(S)$, $x \in X^*$ and $y \in X$. By assumption, $g(xy) \in xyG$ and so $(g \cdot x)(g|_x \cdot y) = xy$. By length considerations, $g|_x \cdot y = y$ and so $g|_x \in \bigcap_{x \in X} G_x$. Conversely, let $g|_x \in \bigcap_{x \in X} G_x$ for all $x \in X^*$. We prove that $g \in K(S)$. Clearly $g$ fixes all elements of length zero and one. Assume that it fixes all elements of length $n$. Let $y$ be a string of length $n+1$. Then $y = xv$ where $x$ has length $n$ and $v$ has length 1. Then $g \cdot y = (g \cdot x)(g|_x \cdot v)$. By assumption $g \cdot x = x$ and $g|_x \cdot v = v$. Thus $g \cdot y = y$, as required. ■

Let $S$ be a left Rees monoid with group of units $G$. A subgroup $N$ of $G$ is said to be a *right normal divisor* if for all $s \in S$ we have that $Ns \subseteq sN$. Clearly, $N$ is a normal subgroup of $G$.

**Lemma 4.4** *Let $S$ be a left Rees monoid, and let $N$ be a normal subgroup of $G(S)$. Then the following are equivalent.*

(i) *$N$ is a right normal divisor.*

(ii) *$N$ is a subgroup of the kernel*

(iii) *For all $g \in N$ and $x \in X$ we have that $g \cdot x = x$ and $g|_x \in N$.*

(iv) *For all $g \in N$ and $x \in X^*$ we have that $g \cdot x = x$ and $g|_x \in N$.*

**Proof** (i) $\Leftrightarrow$ (ii). This is immediate from the definitions.

(i) $\Rightarrow$ (iii). By (i), if $g \in N$ and $x \in X$ then $gx = xk$ for some $k \in G$. But $gx = (g \cdot x)g|_x$. Thus by uniqueness, $x = g \cdot x$ and $g|_x = k \in G$. Hence (iii) holds.

(iii) $\Rightarrow$ (iv). We prove this by induction. Suppose (iv) is true for all strings of length $n$. Let $x$ be a string of length $n+1$, and let $g \in N$. Let $x = wy$ where $w$ has length $n$, and $y$ has length one. Then $gx = g(wy) = (g \cdot w)(g|_w \cdot y)g|_w|_y$. By the induction hypothesis $g \cdot w = w$ and $g|_w \in N$. By (iii), we have that $g_w \cdot y = y$ and $g|_x = g|_w|_y \in N$. It follows that $g \cdot x = x$ and $g|_x \in N$, as required.

(iv) $\Rightarrow$ (i). Let $s \in S$. We calculate $gs$. Let $s = xh$. Then $gs = g(xh) = (g \cdot x)g_x h$. By assumption $g \cdot x = x$ and $g|_x \in N$. Thus $gs = xh(h^{-1}g_x h)$ and so $gs = sk$ where $k = h^{-1}g_x h \in N$, since $N$ is normal. Thus (i) holds. ∎

The significance of right normal divisors is that we can use them to form quotient monoids. Let $S$ be a a left Rees monoid and let $N$ be a right normal divisor. Put $S/N = \{sN : s \in S\}$. Then we may define a multiplication on this set by $(sN)(tN) = stN$. The following can be quickly deduced from Rees [29].

**Proposition 4.5** *Let $S$ be a left Rees monoid isomorphic to $X^* \bowtie G$, and $N$ a right normal divisor of $S$. Then $S/N$ is a left Rees monoid isomorphic to $X^* \bowtie G/N$.* ∎

# 5 The work of Nekrashevych et al

The work of Sections 1 to 4 is a reformulation of Perrot's thesis. In this section, we introduce something new. Specifically, the goal of this section is to make the connection between left Rees monoids and self-similar group actions, as described in [22].

## 5.1 Self-similar group actions

Let $G$ act on $X^*$ in such a way that axioms (SS1)–(SS8) hold. Because the action of $G$ on $X^*$ is length-preserving it restricts to an action on $X$. In verifying that axioms (SS1)–(SS8) hold, it is essentially enough to know the action of $G$ restricted to $X$, and the function from $G \times X$ to $G$ defined by $(g, x) \mapsto g|_x$. The following lemma spells out the precise result.

**Lemma 5.1** *Let $G$ act on $X$ on the left, with the action denoted by $(g, x) \mapsto g \cdot x$. We assume also that $g \cdot 1$ is defined and equals $1$. Let there be a function $G \times X \to X$ given by $(g, x) \mapsto g|_x$ such that $1|_x = 1$ for all $x \in X$ and $(gh)|_x = g|_{h \cdot x} h|_x$. Then this data can be used to define a unique action of $G$ on $X^*$ extending the given data and satisfying axioms (SS1)–(SS8).*

**Proof** Uniqueness follows from the fact that by using (SS4) and induction we have that

$$g \cdot (x_1 \dots x_n) = (g \cdot x_1)(g|_{x_1} \cdot x_2) \dots (g|_{x_1 \dots x_{n-1}} \cdot x_n)$$

and by (SS6) we have that

$$g|_{y_1 \dots y_m} = (\dots (g|_{y_1})|_{y_2} \dots |_{y_m}).$$

Thus given an action satisfying axioms (SS1)–(SS8) and knowing $g \cdot x$ and $g|_x$ for all letters we have determined the action.

We now sketch out the existence proof. The function $G \times X \to X$ given by $(g, x) \mapsto g|_x$ can be regarded as a function from $X$ to the full transformation monoid $T(G)$ on the set $G$. Therefore this extends to a unique monoid homomorphism from the free monoid $X^*$ to $T(G)$ where we write arguments on the the left. Thus $g|_x$ now describes the effect of the string $x$ on the element $g$. We have therefore defined an action of $X^*$ on $G$ on the right and axioms (SS5), (SS6) and (SS7) are satisfied.

Now we extend the left action of $G$ on $X$ to the whole of $X^*$ by induction. Assume that the action is defined on all strings of length $n$. Let $z$ be a string of length $n+1$. We write $z = xy$ where $x$ has length $n$ and $y$ has length one. Define $g \cdot z = (g \cdot x)(g|_x \cdot y)$ which is well-defined since $g|_x$ is defined by the previous paragraph and $g|_x \cdot y$ is defined since $y$ has length one. The element $g \cdot x$ is defined by the induction hypothesis. We now have a function $G \times X^* \to X^*$. Observe that by construction the 'action' is length-preserving.

To finish off we then check that (SS8), (SS2) and (SS4) each hold. ■

We now shall show that our definition of a self-similar group action is the same as the one given in Definition 1.5.6 of [22]. We use the same notation as there. Let $X = \{x_1, \dots, x_d\}$.

**Proposition 5.2** *$G$ acts on $X^*$ in such a way that axioms (SS1)–(SS8) hold if and only if $(G, X)$ is a self-similar group action in the sense of Definition 1.5.6 of [22].*

**Proof** Let $\psi \colon G \to S_X \wr G$ be a homomorphism into the wreath product, where $S_X$ is the permutation group on the set $X$. Thus for each $g \in G$ we have

$$\psi(g) = \alpha(g|_{x_1}, \dots, g|_{x_d})$$

where $\alpha \in S_X$ and $(g|_{x_1}, \dots, g|_{x_d})$ denotes the function from $X$ to $G$. We write $\alpha(x) = g \cdot x$. Let $h \in G$ where

$$\psi(h) = \beta(h|_{x_1}, \dots, h|_{x_d}).$$

Then because $\psi$ is a homomorphism we have that $\psi(gh) = \psi(g)\psi(h)$. Put

$$\psi(gh) = \gamma(gh|_{x_1}, \dots, gh|_{x_d}).$$

Then we have that $(gh) \cdot x = g \cdot (h \cdot x)$ for all $x \in X$ and $(gh)|_x = g|_{h \cdot x} h|_x$ for all $x \in X$. If we calculate $\psi(1)$ then we find that $1 \cdot x = x$ and $1|_x = 1$ for all $x \in X$. The conditions of Lemma 5.1 are satisfied and so the axioms (SS1)–(SS8) are satisfied.

Conversely, if we start with an action satisfying axioms (SS1)–(SS8), then we can clearly define a homomorphism into a suitable wreath product and so deduce that the action is self-similar in the sense of [22]. ■

**Remark** In the light of the above result, it is now legitimate to call actions satisfying axioms (SS1)–(SS8) *self-similar group actions*. Thus Theorem 3.7 can now be phrased in the following way: *left Rees monoids (which are not groups) determine and are determined by self-similar group actions*.

It remains to try to motivate the definition of self-similar group actions. We begin with a lemma that characterises faithful actions satisfying axioms (SS1)–(SS8).

**Lemma 5.3** *Let $G$ act faithfully on the free monoid $X^*$ in a length-preserving way and suppose that there is a function $G \times X^* \to G$ denoted by $(g,x) \mapsto g|_x$ such that $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$. Then this determines an action satisfying axioms (SS1)–(SS8).*

**Proof** We are assuming that axioms (SS1), (SS2) and (SS4) hold. We prove that the remaining axioms hold. From the fact that $g \cdot x = g \cdot (1x)$ we deduce that (SS3) and (SS5) hold. The proof of (SS7) follows from that fact that $xy = 1 \cdot (xy)$. The fact that (SS6) holds follows from the fact that $g \cdot (x(yz)) = g \cdot ((xy)z)$. Finally, (SS8) follows from the fact that $(gh) \cdot (xy) = g \cdot (h \cdot (xy))$. ■

The key to faithful actions satisfying axioms (SS1)–(SS8) is thus the axiom (SS4). We shall now show how this condition arises; we are here adapting ideas to be found in [22].

We begin with an arbitrary automorphism $\theta$ of $(A^*, \leq)$, the free monoid regarded as a poset with respect to its prefix ordering. The automorphism $\theta$ is prefix-preserving, by assumption, and evidently length-preserving. For each $a \in A$, we see that $ax \leq a$ implies $\theta(ax) \leq \theta(a)$ and so $\theta$ induces an order isomorphism $\theta'$ from the principal order ideal $aA^*$ to the principal order ideal $\theta(a)A^*$. There are order isomorphisms $\lambda_a \colon A^* \to aA^*$ and $\lambda_{\theta(a)} \colon A^* \to \theta(a)A^*$ which arise from the fact that $(A^*, \leq)$ is a *uniform* poset. Define $\phi = \lambda_{\theta(a)}^{-1}\theta'\lambda_a$, an order automorphism of $A^*$. Then

$$\theta(ax) = \theta(a)\phi(x),$$

where $\phi$ is uniquely determined by $a$ and $\theta$. If we define $\theta \cdot x = \theta(x)$ and $\theta|_a = \phi$ then we have exactly axiom (SS4). It follows by Lemma 5.3 that the automorphism group of the poset $(A^*, \leq)$ satisfies the axioms (SS1)–(SS8). Faithful actions satisfying axioms (SS1)–(SS8) therefore correspond to subgroups of the group of automorphisms of $(A^*, \leq)$ which are closed with respect to the restriction structure.

**Remark** Perrot [25], having defined self-similar group actions, takes the first steps in their theory although, as he says, the calculations are 'souvent délicate'.

He considers the case where $|X| = 2$ and obtains some information about those groups which have faithful self-similar actions on $X^*$.

## 5.2 Tensor monoids

In this section, we shall prove that left Rees monoids are precisely 'tensor monoids'.[6] In Chapter 2 of [22], the algebraic properties of self-similar group actions are handled using covering bimodules. We show how to construct the monoid associated with the self-similar group action from the covering bimodule.

Let $X$ be a set and $S$ and $T$ monoids. We say that $X$ is a $(S,T)$-*biact* if $X$ is a left $S$-act, a right $T$-act and if $(sx)t = s(xt)$ for all $s \in S$, $t \in T$ and $x \in X$.[7] If $X$ and $Y$ are $(S,T)$-biacts then a function $\theta$: $X \to Y$ is called a *bihomomorphism* if $\theta(sxt) = s\theta(x)t$ for all $s \in S$, $t \in T$ and $x \in X$. We shall be interested in biacts where both acting monoids are the same and are groups.

Let $S$ be a monoid with group of units $G$. Then under left and right multiplication $S$ is also a $(G,G)$-biact.

**Lemma 5.4** *Let $S$ be a left Rees monoid. Let $M$ be the set of all generators of the maximal proper principal right ideals of $S$. Then $M$ is a $(G,G)$-biact under left and right multiplication by $G$, and the right $G$-action is free.*

**Proof** Let $x$ be a generator of a maximal proper principal right ideal. Then $xS = xgS$ and so $xg$ is a generator of a maximal proper principal right ideal. Consider now $gx$. We prove that $gxS$ is a maximal proper principal ideal. If it is not maximal then there is a maximal proper principal right ideal $yS$ such that $gxS \subseteq yS$. Thus $xS \subseteq g^{-1}yS$. Now $xS$ is maximal and so either $g^{-1}yS = S$ or $gxS = yS$. The former cannot occur because $y$ is not invertible. Thus $gxS = yS$. Thus $gx$ is also a generator of a maximal proper principal right ideal. Observe that by left cancellation, the right $G$-action is free. ■

**Remark** Let $S = X^* \bowtie G$. In this case, the set $M$ is $M = X \times G$. Observe that

$$(1,h)(x,g) = (h \cdot x, h|_x g) \text{ and } (x,g)(1,h) = (x,gh).$$

Thus if we define left and right actions by $G$ on $M$ as follows: $G \times M \to M$ is given by $h(x,g) = (h{\cdot}x, h|_x g)$, and $M \times G \to M$ is given by $(x,g)h = (x,gh)$ then we get a $(G,G)$-biact. In [22], biacts such as this are called 'covering bimodules'.

We define a *covering biact* to be a $(G,G)$-biact $M$ where the righthand action is free. In Lemma 5.4, we showed how to construct a covering biact $M$ from a left Rees monoid.

We shall now investigate the relationship between the original monoid $S$ and the covering biact $M$ constructed from it. It is convenient to assume that

[6] The term 'tensor semigroup' is used in [23]).

[7] In [22], the term 'commuting' is used for the last condition.

$S = X^*G$, uniquely. In this case, $M = XG$. Define $\iota\colon M \to S$ by $\iota(xg) = xg$. Recall that $S$ is a $(G, G)$-biact for left and right multiplication by $G$. The function $\iota$ is a $(G, G)$-bihomomorphism: this is simply because $M$ is a $(G, G)$-subact of $S$. The relationship between $M$ and $S$ is characterised by the following theorem.

**Theorem 5.5** *Let $S$ be a left Rees monoid with group of units $G$. Let $M$ be the covering biact associated with $S$. Let $T$ be a monoid with group of units $G$. Let $\alpha\colon M \to T$ be a $(G, G)$-bihomomorphism. Then there is a unique monoid homomorphism $\bar{\alpha}\colon S \to T$ such that $\alpha = \bar{\alpha}\iota$ and which is the identity on the group of units of $S$.*

**Proof** Define $\bar{\alpha}$ by

$$\bar{\alpha}(x_1 \dots x_n g) = \alpha(x_1) \dots \alpha(x_n) g.$$

It is clear that $\bar{\alpha}\iota = \alpha$. We need to prove that $\bar{\alpha}$ is a homomorphism. Let $\mathbf{x} = x_1 \dots x_m g$ and $\mathbf{y} = y_1 \dots y_n h$ be elements of $S$. Their product is

$$x_1 \dots x_m (g \cdot y_1)(g|_{y_1} \cdot y_2) \dots (g|_{y_1 \dots y_i} \cdot y_{i+1}) \dots (g|_{y_1 \dots y_{n-1}} \cdot y_n) g|_{y_1 \dots y_n} h.$$

We now calculate $\bar{\alpha}(\mathbf{xy})$. This is equal to

$$\alpha(x_1) \dots \alpha(x_m) \alpha(g \cdot y_1) \alpha(g|_{y_1} \cdot y_2) \dots \alpha(g|_{y_1 \dots y_i} \cdot y_{i+1}) \dots \alpha(g|_{y_1 \dots y_{n-1}} \cdot y_n) \alpha(1, g|_{y_1 \dots y_n} h).$$

We shall now use the fact that $\alpha$ is a bihomomorphism. Consider

$$\alpha(g \cdot y_1) \alpha(g|_{y_1} \cdot y_2).$$

We write this as

$$\alpha(g \cdot y_1) g|_{y_1} (g|_{y_1})^{-1} \alpha(g|_{y_1} \cdot y_2)$$

and now use the fact that $\alpha$ is a bihomomorphism and that

$$(g|_{y_1})^{-1}|_{g|_{y_1} \cdot y_2} = (g|_{y_1 y_2})^{-1}$$

by Lemma 3.4 to get

$$\alpha((g \cdot y_1) g|_{y_1}) \alpha(y_2 (g|_{y_1 y_2})^{-1})$$

which is equal to

$$g \alpha(y_1) \alpha(y_2) (g|_{y_1 y_2})^{-1}.$$

We now consider the remaining product

$$(g|_{y_1 y_2})^{-1} \alpha(g|_{y_1 y_2} \cdot y_3) \dots \alpha(g|_{y_1 \dots y_i} \cdot y_{i+1}) \dots \alpha(g|_{y_1 \dots y_{n-1}} \cdot y_n) \alpha(g|_{y_1 \dots y_n} h).$$

We now push the leftmost group element through the product using the fact that

$$(g|_{y_1 \dots y_i})^{-1}|_{g|_{y_1 \dots y_i} \cdot y_{i+1}} = (g|_{y_1 \dots y_{i+1}})^{-1}.$$

The last term is

$$(g|_{y_1\ldots y_n})^{-1}\alpha(g|_{y_1\ldots y_n}h) = h.$$

It follows that

$$\bar{\alpha}(\mathbf{xy}) = \bar{\alpha}(\mathbf{x})\bar{\alpha}(\mathbf{y}).$$

It remains to prove uniqueness. Let $\alpha'\colon S \to T$ be another monoid homomorphism such that $\alpha'\iota = \alpha$. Then $\bar{\alpha}(xg) = \alpha'(xg)$ for all $x \in X$ and $g \in G$. By definition $\bar{\alpha}(x_1 \ldots x_n g) = \alpha(x_1)\ldots\alpha(x_n)g$. By assumption this is equal to $\alpha'(x_1)\ldots\alpha'(x_n)g$. But $\alpha'$ is a homomorphism and so this is equal to $\alpha'(x_1 \ldots x_n g)$, as required. ∎

Let $M$ be an arbitrary covering $(G, G)$-biact. We may form the tensor product $M \otimes M$ whose elements we denote by $x \otimes y$. The bihomomorphism $\otimes\colon M \times M \to M \otimes M$ has the property that $xg \otimes y = x \otimes gy$; such maps are called *bimaps*. The tensor product is the universal such bimap. Observe that $a \otimes b = c \otimes d$ iff $a = cg$ and $b = g^{-1}d$. The theory of tensor products of monoid acts is described in [9].

Define $M^{\otimes 0} = G$ and $M^{\otimes n} = M^{\otimes n-1} \otimes M$. For $p, q > 0$ there are isomorphisms $\phi_{p,q}\colon M^{\otimes p} \otimes M^{\otimes q} \to M^{\otimes p+q}$ which map $(u, v)$ to $u \otimes v$. Observe that all tensor products are free right $G$-acts. Put $S = \bigcup_{n=0}^{\infty} M^{\otimes n}$. There is the obvious embedding $\iota\colon M \to S$. The $(G, G)$-biact $S$ becomes a monoid under tensor products and left and right actions by $G$: we use the isomorphisms above to define the multiplication. We call $S$ the *tensor monoid* of the $(G, G)$-biact $M$ by analogy with the tensor algebra of a module [17].

More informally, the elements of $S$ can be regarded as the elements of $G$ together with all formal products $x_1 \otimes \ldots \otimes x_n$ where $x_i \in M$. The product of two formal products $u$ and $v$ is just the formal product $u \otimes v$ and the product of $g \in G$ and a formal product $x_1 \otimes \ldots \otimes x_n$ is given by $g(x_1 \otimes \ldots \otimes x_n) = gx_1 \otimes \ldots \otimes x_n$ and $(x_1 \otimes \ldots \otimes x_n)g = x_1 \otimes \ldots \otimes x_n g$.

**Proposition 5.6** *The tensor monoid of a covering $(G, G)$-biact is a left Rees monoid.*

**Proof** We essentially use Lemmas 2.1 and 2.3 to show that the tensor monoid is a left Rees monoid. The group of units of $S$ is $G$, and there is a surjective homomorphism from $S$ to $\mathbb{N}$, in which the inverse image of 0 is $G$. Thus $S$ is equipped with a length function. Because the action is free on the right, it is easy to check that if $\mathbf{x} \otimes \mathbf{y} = \mathbf{x} \otimes \mathbf{y}'$ in $S$ then $\mathbf{y} = \mathbf{y}'$ and so $S$ is left cancellative (this works because lengths match). Thus $S$ is left cancellative. We finish off by showing that $S$ is equidivisible Suppose that $\mathbf{x} \otimes \mathbf{u} = \mathbf{y} \otimes \mathbf{v}$. There are three cases to consider depending on the relative lengths of $\mathbf{x}$ and $\mathbf{y}$. We shall just consider the case where the length of $\mathbf{x}$ is $m$, that of $\mathbf{y}$ is $n$ and where $m < n$. We therefore suppose that $\mathbf{u} = \mathbf{w} \otimes \mathbf{z}$ and that $\mathbf{x} \otimes \mathbf{w}$ has the same length as $\mathbf{y}$. Thus $\mathbf{x} \otimes \mathbf{w} = \mathbf{y}g$ and $\mathbf{z} = g^{-1}\mathbf{v}$. Thus $\mathbf{y} = \mathbf{x} \otimes (\mathbf{w}g^{-1})$. Once the argument is completed by the other two cases, it will follow that $S$ is equidivisible. ∎

The following theorem can be proved using the universal properties of tensor products.

**Theorem 5.7** *Let $S$ be the tensor monoid of the covering $(G,G)$-biact $M$. Let $T$ be any monoid with group of units $G$, and let $\alpha$: $M \to T$ be a bihomomorphism. Then there is a unique monoid homomorphism $\bar{\alpha}$: $S \to T$ such that $\alpha = \bar{\alpha}\iota$ and which is the identity on the group of units.* ■

The results of this section can be placed in a categorical framework. We fix a group $G$ and consider the category whose objects are left Rees monoids with $G$ as their groups of units and whose morphisms are the monoid homomorphisms which are the identity on the groups of units and which map generators of maximal proper principal right ideals to generators of maximal proper principal right ideals. Then there is a forgetful functor from this category to the category whose objects are the covering $(G,G)$-biacts and whose morphisms are the $(G,G)$-bihomomorphisms. This functor has a left adjoint which associates with a covering biact its tensor monoid.

# 6 Inverse monoids

Left Rees monoids can also be used to construct a class of inverse monoids — which is precisely what Perrot did [25, 26]. This is a well-known procedure forming part of the theory of 0-bisimple inverse semigroups so I shall simply sketch out the theory as it applies to our case. For more details see [14]. For all undefined terms from inverse semigroup theory see [13].

Let $S$ be a left Rees monoid. The inverse monoid $B(S)$ of all $S$-isomorphisms between the principal right ideals of $S$ together with the empty function is a 0-bisimple inverse monoid. There is a useful isomorphic representation of $B(S)$. Define an equivalence relation on the set of nonzero ordered pairs of elements of $S$ by $(a,b)$ is equivalent to $(au,bu)$ for all units $u \in S$. Denote by $[a,b]$ the equivalence class containing $(a,b)$. Consider now the set of all such equivalence classes together with a zero element. Define $[d,c][b,a]$ to be zero if $cS \cap bS$ is empty. If $cS \cap bS$ is not empty there are two possibilities. If $c = bs$ for some $s$ then we define the product to be $[d,as]$. If $b = cs$ for some $s$ then we define the product to be $[ds,a]$. It can be proved that the resulting structure is isomorphic to $B(S)$ and, from now on, we shall treat $B(S)$ in this way. The non-zero idempotents of $B(S)$ are the elements $[a,a]$. The natural partial order is given by $[a,b] \leq [c,d]$ iff $(a,b) = (c,d)p$ for some $p \in S$. The idempotent structure of $B(S)$ is isomorphic to the semilattice of principal right ideals of $S$ together with the empty set. It follows that if $e$ and $f$ are idempotents of $B(S)$ and $ef \neq 0$ then $e$ and $f$ are comparable with respect to the natural partial order. The identity of $B(S)$ is $[1,1]$ and the $\mathcal{L}$-class of the identity consists of elements of the form $[a,1]$ and forms a left cancellative monoid isomorphic to $S$.

It follows from the general theory of 0-bisimple inverse semigroups that there is a correspondence between the following two classes of monoids:

- Left Rees monoids.
- 0-bisimple inverse monoids with two properties: first, if $e$ and $f$ are idempotents and $ef \neq 0$ then $e$ and $f$ are comparable and second, there are only a finite number of idempotents above any non-zero idempotent.

Under this correspondence, fundamental monoids of the first class correspond to fundamental inverse monoids of the second. I shall call the inverse monoids that arise in this way the *associated inverse monoids*.

If $S$ is an inverse semigroup with zero, then $S^* = S \setminus \{0\}$. A *prehomomorphism* $\theta$ from an inverse semigroup $S$ to an inverse semigroup $T$ is a function $\theta\colon S^* \to T^*$ such that $ab \neq 0$ implies that $\theta(ab) = \theta(a)\theta(b)$. An inverse monoid is said to be *strongly $E^*$-unitary* if it admits a prehomomorphism to a group such that the inverse image of the identity consists only of idempotents. The associated inverse monoid is strongly $E^*$-unitary if and only if the associated monoid is cancellative [14].

The set of idempotents of an inverse semigroup is said to be 0-*disjunctive* if whenever $0 < e < f$ then there exists a nonzero idempotent $g$ such that $g \leq f$ and $ge = 0$.

**Lemma 6.1** *If $|X| > 1$, then the set of idempotents of an associated inverse monoid is* 0*-disjunctive.*

**Proof** Let $0 < [a,a] < [b,b]$ in $B(S)$. Then $a = bp$ in $S$. Let $a = xg$, $b = yh$ and $p = zk$. Then $xg = y(h \cdot z)h|_z k$. Thus by uniqueness $x = y(h \cdot z)$ and $g = h|_z k$. If $x = y$ then $h \cdot z$ is the identity and so $z$ would be the empty string. This would imply that $[a,a] = [b,b]$. It follows that $y$ is a proper prefix of $x$. Let $x = yw$ where $w$ has length at least one. Let $q \in X$ different from the first letter of $w$. Put $c = yq$. Then $0 < [c,c] \leq [b,b]$, and $[a,a][c,c] = 0$ by construction.■

Fundamental 0-bisimple inverse monoids with a 0-disjunctive set of indempotents are *congruence-free* (see page 181 of [27]). The following result follows from Corollary 4.2: a self-similar group action is faithful if and only if the associated left Rees monoid is fundamental.

**Proposition 6.2** *The inverse monoids associated with faithful self-similar group actions on free monoids with at least two letters are congruence-free.* ■

It is possible to write the elements of $B(S)$ in a more straightforward form. If $x \in X^*$ then $x^{-1}$ denotes the *reverse string of* $x$. Observe that $(xy)^{-1} = y^{-1}x^{-1}$. If $z = xy$ then we define $x^{-1}z = y$. We can identify the nonzero elements of $B(S)$ with the formal products $xgy^{-1}$. The product of $xgy^{-1}$ and $whz^{-1}$ is then: zero if neither $y$ nor $w$ is a prefix of the other; $x(g \cdot p)g|_p hz^{-1}$ if $w = yp$; and $xg(h|_{h^{-1} \cdot p})(h^{-1} \cdot p)^{-1}z^{-1}$ if $y = wp$. Monoids of the form $B(X^*)$ are called the *polycyclic monoids* [24]. The form of the elements and product in $B(S)$ just described generalises the usual way in which polycylic monoids are described.

The axioms (SS1)–(SS8) can be modified to take account of zeros as described by Kunze [11]: if $S$ and $T$ are arbitrary monoids with zero ($T$ replacing $G$ and $S$ replacing $X^*$) then we require that $t \cdot s = 0$ iff $t|_s = 0$ and we require that $0 \cdot s = 0 = t \cdot 0$ and $0|_s = 0 = t|_0$. It is possible to write polycyclic monoids and more generally the monoids $B(S)$ as Zappa-Szép products with zero. Every non-zero element $xy^{-1}$ in the polycyclic monoid is evidently uniquely of the form $x \in X^*$ times $y^{-1} \in (X^*)^{-1}$ where the latter is just the dual of the free monoid. Denote by $S_0$ the semigroup $S$ with a zero adjoined, and denote by $\overline{S}$ the dual semigroup of $S$. Thus the polycyclic monoid on $|X|$ generators is isomorphic to $X_0^* \bowtie \overline{X_0^*}$ More generally, if $S = X^*G$ is a left Rees monoid then $B(S) = S_0 \bowtie \overline{X_0^*}$.

Finally, the inverse monoids defined here lead to Cuntz-Pimsner algebras in the same way that the polycyclic monoids lead to Cuntz algebras. See Section 13.2 of [1].

# 7 Other characterisations

In this section, we show how left Rees monoids are also related to Mealy machines and double categories.

## 7.1 Mealy machines

The connection between self-similar group actions and automata is well-known [22]. However, in this section we shall describe a slightly different perspective on this connection.

A *(non-initial) (Mealy) machine* $\mathbf{A} = (S, X, |, \cdot)$ consists of the following information: a set of states $S$, an input/output alphabet $X$, a state transition function $S \times X \to S$, denoted by $(s, x) \mapsto s|_x$, where $x \in X$ and an output function $S \times X \to X$, denoted by $(s, x) \mapsto s \cdot x$, where $x \in X$. Machines are defined to process input and output *letters*, but can easily be extended to process input and output strings in the following way. First, state transitions for strings are defined by means of the following recursion:

- $s|_1 = s$ for all states $s$.
- $s|_{ax} = (s|_a)|_x$ where $a$ is a letter and $x$ a string.

Second, outputs are defined for strings by the following recursion:

- $s \cdot 1 = 1$.
- $s \cdot (ax) = (s \cdot a)(s|_a \cdot x)$ where $a$ is a letter and $x$ a string.

Observe that these conditions are actually axioms (SS5), (SS6), (SS3), and (SS4). A function $\theta\colon \mathbf{A} \to \mathbf{B}$ is a *homomorphism* of Mealy machines if $\theta\colon S \to T$, $\theta(s|_x) = \theta(s)|_x$, and $s \cdot x = \theta(s) \cdot x$. The composition of homomorphisms is a homomorphism and the identity function on a set of states is the identity function on the Mealy machine. An *isomorphism* is just a bijective homomorphism. We therefore have a category $\mathcal{A}$ whose objects are Mealy machines (over the same input/output alphabet $X$) and whose morphisms are the homomorphisms of Mealy machines.

This category is endowed with extra structure which we now describe. We denote by $\mathbf{I}$ the machine with one state, and which simply outputs the input. Next, given two machines $\mathbf{A}$ and $\mathbf{B}$ with sets of states $S$ and $T$ respectively, we define a new machine $\mathbf{A}*\mathbf{B}$ as follows: the set of states is $S \times T$; the input/output alphabet is $X$; the transition function is given by $(s,t)|_x = (s|_{t \cdot x}, t|_x)$; and the output function is given by $(s,t) \cdot x = s \cdot (t \cdot x)$. Intuitively, this machine is constructed by taking the output of $\mathbf{B}$ and using it as the input to $\mathbf{A}$ and is called the *cascade product of* $\mathbf{A}$ *and* $\mathbf{B}$. Observe that underlying the construction of $\mathbf{A}*\mathbf{B}$ is the product of sets, and that $\mathbf{I}$ has as underlying set the one-element set. Now the category of sets is a monoidal category with respect to products of sets and the one-element set as unit. Our homomorphisms are simply set functions satisfying certain algebraic conditions. We therefore have the following.

**Theorem 7.1** *The category $\mathcal{A}$ of Mealy machines over a fixed input/output alphabet $X$ is a monoidal category with respect to cascade product.* ■

For each state $s \in S$, there is an *initial Mealy machine* $\mathbf{A}_s$ where $s$ is the distinguished *initial state*. An initial Mealy machine $\mathbf{A}_s$ computes a function from $X^*$ to itself which maps $x$ to $s \cdot x$. A homomorphism between initial Mealy machines is required to map initial states to initial states. It can be shown that if there is a homomorphism between two initial Mealy machines then they compute the same function. If $\mathbf{A}_s$ is an initial Mealy machine computing $f\colon X^* \to X^*$ and $\mathbf{B}_t$ is an initial Mealy machine computing $g\colon X^* \to X^*$ then $\mathbf{A}_s * \mathbf{B}_t$ is an initial Mealy machine computing $fg$, composing from right-to-left.

**Theorem 7.2** *Let $G$ be a group and $X$ a set. Then $(G, X^*)$ is a self-similar group action if and only if $G$ is a Mealy machine with structure $(G, X, |, \cdot)$, the multiplication function in $G$ given by $G * G \to G$ is a homomorphism of Mealy machines, and the function $\eta\colon \mathbf{I} \to G$ which maps the single state of $\mathbf{I}$ to the identity of the group $G$ is a homomorphism of Mealy machines.*

**Proof** Suppose first that $(G, X^*)$ is a self-similar group action Let $(g,h)$ be a state in the Mealy machine $G * G$. Let $x$ be an input letter. Then by definition $(g,h)|_x = (g|_{h \cdot x}, h|_x)$. Thus $\mu((g,h)|_x) = (g|_{h \cdot x})(h|_x)$. On the other hand, $\mu(g,h)|_x = (gh)|_x$. These two are equal by axiom (SS8).

Let $(g,h)$ be a state and $x$ and input letter. Then $(g,h) \cdot x = g \cdot (h \cdot x)$ and $\mu(g,h) \cdot x = (gh) \cdot x$. These two are equal by axiom (SS2).

The fact that $\eta$ is a homomorphism follows by axioms (SS1) and (SS7).

The proof of the converse is straightforward. The information we are given is that $G$ is a Mealy machine with structure $(G, X, |, \cdot)$, the multiplication function in $G$ given by $G * G \to G$ is a homomorphism of Mealy machines, and the function $\eta\colon \mathbf{I} \to G$ which maps the single state of $\mathbf{I}$ to the identity of the group $G$ is a homomorphism of Mealy machines. All of this information is the information needed to apply Lemma 5.1 and so determines a (unique) self-similar group action of $G$ on $X^*$ extending the given data. ■

The theorem above says that every self-similar group action gives rise to a monoid in the monoidal category $(\mathcal{A}, *, I)$ in the sense of Section 3 of Chapter VII of [16].

## 7.2 Double categories

We now consider another interpretation of self-similar group actions involving categories which is closely related to the Mealy machine interpretation. The diagrams we draw were used, for example, in [2]. We adapt to our setting some of the ideas to be found in [7] where full definitions can be found if required. Let $G$ be a group with a self-similar action on $X$. We define a double category as follows. Its elements are squares of the form

$$\begin{tikzcd} \cdot \arrow[r, "g\cdot x"] \arrow[d, "g"'] & \cdot \arrow[d, "g|_x"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

We define *horizontal multiplication* as follows. Let

$$\begin{tikzcd} \cdot \arrow[r, "h\cdot y"] \arrow[d, "h"'] & \cdot \arrow[d, "h|_y"] \\ \cdot \arrow[r, "y"'] & \cdot \end{tikzcd}$$

be another square such that $g|_x = h$. Then their product is

$$\begin{tikzcd} \cdot \arrow[r, "(g\cdot x)(h\cdot y)"] \arrow[d, "g"'] & \cdot \arrow[d, "h|_y"] \\ \cdot \arrow[r, "xy"'] & \cdot \end{tikzcd}$$

This is well-defined because $g \cdot (xy) = (g \cdot x)(g|_x \cdot y) = (g \cdot x)(h \cdot y)$, and $h|_y = (g|_x)|_y = g|_{xy}$. We define *vertical multiplication* as follows. We suppose now that $x = h \cdot y$. Then their product is

$$\begin{tikzcd} \cdot \arrow[r, "g\cdot x"] \arrow[d, "gh"'] & \cdot \arrow[d, "g|_x h|_y"] \\ \cdot \arrow[r, "y"'] & \cdot \end{tikzcd}$$

This is well-defined because $(gh) \cdot y = g \cdot (h \cdot y) = g \cdot x$, and $(gh)|_y = g|_{h \cdot y} h|_y = g|_x h|_y$. It is easy to check that the interchange law holds, so we have defined a double category from a self-similar group action. This double category has the vertical structure of a group and the horizontal structure of a free monoid. The following emulates Proposition 2.4 of [7].

**Proposition 7.3** *Let* $\mathbf{B}$ *be a double category in which the vertical structure is a group* $G$*, the horizontal structure is a free monoid* $X^*$*, and such that the* star condition *holds: every pair*

$$\begin{tikzcd} \cdot \arrow[d, "g"'] & \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

*can be uniquely completed to a square*

$$\begin{tikzcd} \cdot \arrow[r, "g \cdot x"] \arrow[d, "g"'] & \cdot \arrow[d, "g|_x"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

*where* $g \cdot x$ *and* $g|_x$ *denote uniquely defined elements. Then there is a self-similar group action of* $G$ *on* $X^*$*.*

**Proof** From horizontal multiplication we get that (SS4) and (SS6) hold, from vertical multiplication we get that (SS2) and (SS8) hold. The remaining four axioms hold by considering the horizontal and vertical morphisms in the double category: squares of the form

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d, "1"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

are the horizontal morphisms and imply that axioms (SS1) and (SS7) hold, squares of the form

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "g"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "1"'] & \cdot \end{tikzcd}$$

are the vertical morphisms and imply that axioms (SS3) and (SS5) hold. ■

With each double category can be associated a bisimplicial complex. The diagonal of this bisimplicial set is a simplicial set which is actually the nerve of a category. In our case, this category is a monoid: its elements are diagrams of the form

$$\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd}$$

and the product with

$$\begin{tikzcd} {} \arrow[r, "y"] & {} \arrow[d, "h"] \\ & {} \end{tikzcd}$$

is given by

$$\begin{tikzcd} {} \arrow[r, "x"] & {} \arrow[r, dashed, "g\cdot y"] \arrow[d, "g"] & {} \arrow[d, dashed, "g|_y"] \\ & {} \arrow[r, "y"] & {} \arrow[d, "h"] \\ & & {} \end{tikzcd}$$

using the star condition of Proposition 3.5 and so is just

$$\begin{tikzcd} {} \arrow[r, "x(g\cdot y)"] & {} \arrow[d, "g|_y h"] \\ & {} \end{tikzcd}$$

This monoid is just the monoid associated with the self-similar group action. The argument of Proposition 2.6 of [7] therefore yields the following result.

**Proposition 7.4** *Let a self-similar group action be given. Then the classifying space of the double category associated with the action is canonically homotopically equivalent to the classifying space of the monoid associated with the action.* ■

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# References

[1] L. Bartholdi, R. Grigorchuk, V. Nekrashevych, From fractal groups to fractal sets, *Fractals in Graz* (eds P. Grabner, W. Woess) Trends in Mathematics, Birkhäuser Verlag, 2003, 25–118.

[2] L. Bartholdi, A. G. Henriques, V. V. Nekrashevych, Automata, groups, limit spaces, and tilings, preprint dated 19th December 2004.

[3] J.-C. Birget, Iteration of expansions — unambiguous semigroups, *J. Pure and Applied Algebra* **34** (1984), 1–55.

[4] M. G. Brin, On the Zappa-Szép product, *Comm. Algebra* **33** (2005), 393–424.

[5] A. H. Clifford, A class of $d$-simple semigroups, *Amer. J. Math.* **75** (1953), 547–556.

[6] M. P. Dorofeeva, Hereditary and semi-hereditary monoids, *Semigroup Forum* **9** (1975), 294–309.

[7] Z. Fiedorowicz, J-L. Loday, Crossed simplicial groups and their associated homology, *TAMS* **326** (1991), 57–87.

[8] R. I. Grigorchuk, V. V. Nekrashevich, V. I. Sushchanskii, Automata, dynamical systems and groups, *Proc. of the Steklov Inst. Maths* **231** (2000), 134–214.

[9] J. M. Howie, *Fundamentals of semigroup theory*, Clarendon Press, Oxford, 1995.

[10] C. Kassel, *Quantum groups*, Springer-Verlag, 1995.

[11] M. Kunze, Zappa products, *Acta Math. Hung.* **41** (1983), 225–239.

[12] G. Lallement, *Semigroups and combinatorial applications*, John Wiley, 1979.

[13] M. V. Lawson, *Inverse semigroups*, World Scientific, 1998.

[14] M. V. Lawson, The structure of 0-$E$-unitary inverse semigroups I: the monoid case, *Proc. Edin. Math. Soc.* **42** (1999), 497–520.

[15] M. V. Lawson, A correspondence between a class of monoids and self-similar group actions II, Preprint, Heriot-Watt University, 2008.

[16] S. Mac Lane, *Categories*, second edition, Springer, 1998.

[17] S. Mac Lane, G. Birkoff, *Algebra*, second edition, MacMillan, 1979.

[18] D. B. McAlister, 0-bisimple inverse semigroups, *Proc. London Math. Soc. (3)* **28** (1974), 193–221.

[19] W. D. Munn, Uniform semilattices and bisimple inverse semigroups, *Quart. J. Math. Oxford Ser.* **17** (1966), 151–170.

[20] W. D. Munn, Fundamental inverse semigroups, *Quart. J. Math. Oxford (2)* **21** (1970), 157–170.

[21] W. D. Munn, 0-bisimple inverse semigroups, *J. Algebra* **15** (1970), 570–588.

[22] V. Nekrashevych, *Self-similar groups*, AMS, 2005.

[23] V. Nekrashevych, Symbolic dynamics and self-similar groups, Preprint.

[24] M. Nivat, J.-F. Perrot, Une généralisation du monoïde bicyclique, *Comptes Rendus de l'Académie des Sciences de Paris* **271** (1970), 824-827.

[25] J.-F. Perrot, *Contribution a l'étude des monoïdes syntactiques et de certains groupes associés aux automates finis*, Thése Sc. math. Paris, 1972.

[26] J.-F. Perrot, Une famille de monoïdes inversifs 0-bisimples généralisant le monoide bicyclique, *Séminaire Dubreil. Algèbre* **25** (1971-1972), 1–15.

[27] M. Petrich, *Inverse semigroups*, John Wiley and Sons, 1984.

[28] P. Prusinkiewicz, A. Lindenmayer, *The algorithmic beauty of plants*, Springer, 1990.

[29] D. Rees, On the ideal structure of a semi-group satisfying a cancellation law, *Quart. J. Math. Oxford Ser.* **19** (1948), 101–108.

[30] N. R. Reilly, Bisimple inverse semigroups, *Trans. Amer. Math. Soc.* **132** (1968), 101–114.