Self-similar group actions and a paper of David Rees

Mark V. Lawson
Department of Mathematics
and the
Maxwell Institute for Mathematical Sciences
Heriot-Watt University
Riccarton
Edinburgh EH14 4AS
Scotland
M.V.Lawson@ma.hw.ac.uk

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Abstract

We show that David Rees' 1948 paper on the structure of left cancellative monoids stands at the threshold of the theory of self-similar group actions, and in doing so correct an error in a paper by Nivat and Perrot.
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1 Introduction

In this note, I shall discuss Rees' 1948 paper [10] dealing with a class of cancellation (that is, cancellative) semigroups. This paper has been influential in the development of the theory of inverse semigroups: together with Clifford's 1953 paper [2], it provided one of the key ideas that led to the theory of 0-bisimple inverse semigroups [11, 5, 6, 7, 4]. Rees' paper was generalised by Nivat and Perrot in 1970 [8]. However, their paper contains an important error. It was whilst I was correcting this error that I realised that Rees' paper could be viewed as standing at the threshold of the theory of self-similar group actions [9]. The goal of this note is to explain how. One of the corollaries of this paper is that McAlister's theory of 0-bisimple inverse semigroups can be viewed as a *generalisation* of the theory of self-similar group actions [4]. Rees' paper is divided into three sections. For the rest of this introduction, I shall describe in more detail the contents of the first two sections, and deal with the contents of the third in Section 2.

His paper deals with left cancellative monoids S and their partially ordered sets (posets) of principal right ideals $\mathbf{P}(S)$. Rees observes (Theorem 1.2 [10]) that the posets that occur in this way are special in that for each principal right ideal R the whole poset $\mathbf{P}(S)$ is isomorphic to the subposet of all principal right ideals contained in R. Motivated by this, he defines an arbitrary poset P to be *uniform* if every principal order ideal is order isomorphic to P. Thus the posets $\mathbf{P}(S)$ are uniform. In fact, all uniform posets P arise in this way: define $\mathbf{S}(P)$ to be the semigroup of all order isomorphisms from P to its principal order ideals. Then P is order isomorphic to $\mathbf{P}(\mathbf{S}(P))$ (Theorem 1.3 [10]).

Given a uniform poset P there are many semigroups S for which $\mathbf{P}(S)$ is order isomorphic to P. To get some handle on these semigroups, Rees introduces the following notion. Denote by $G(S)$ the group of units of S. A subgroup N of $G(S)$ is said to be a *right normal divisor* of S if $Ns \subseteq sN$ for all $s \in S$. It is easy to check that N is a normal subgroup of S. The importance of right normal divisors is that they can be used to construct quotient monoids: given a left cancellative monoid S and a right normal divisor N, one can form a left cancellative monoid S/N whose elements are the 'right cosets' of N, meaning elements of the form sN where $s \in S$. The monoid S/N is a homomorphic image of S and $\mathbf{P}(S/N)$ and $\mathbf{P}(S)$ are order isomorphic (Lemma 2.13 [10]). Define

$$M(S) = \{g \in G(S) \colon gs \in sG(S) \quad \forall s \in S\}.$$

Then $M(S)$ is the greatest right normal divisor of S (Lemma 2.11 [10]). We shall say that S is *fundamental*[1] if $M(G) = \{1\}$. The semigroup $S/M(S)$ is fundamental as is $\mathbf{S}(P)$. In addition, the semigroup $\mathbf{S}(P)$ has the property that if T is any semigroup such that $\mathbf{P}(T)$ is order isomorphic to P then $T/M(T)$ is isomorphic to a submonoid of $\mathbf{S}(P)$. Thus $\mathbf{S}(P)$ is the 'largest' fundamental semigroup whose poset of principal right ideals is order isomorphic to P.

2 Self-similar group actions

I shall now turn to the third and final section of Rees' paper, and it is here that we shall ultimately make contact with the theory of self-similar group actions. Let S be a left cancellative monoid. Following Rees, we assume that the poset of principal right ideals has the following structure:

$$S = R_0 \supset R_1 \supset R_2 \supset \ldots$$

Let a be a generator of R_1. Then a^n is a generator of R_n (Lemma 3.3 [10]). It follows that each element of S can be written *uniquely* in the form $a^n g$ for some $n \geq 0$ and some $g \in G(S)$. The following lemma is tacitly assumed in Rees' paper. We prove it for reasons that will become clear later. We use the result that $uS = vS$ iff $u = vg$ for some invertible element g.

Lemma 2.1 *If g is invertible then $ga = ah$ for a unique invertible element h.*

[1] This is not a term used by Rees, but is adapted from its related usage in inverse semigroup theory.

Proof The principal right ideal gaS must be one of the R_i. It cannot be $R_0 = S$, because that would imply that a is invertible. Thus certainly $gaS \subseteq R_1 = aS$. That is $aS \subseteq g^{-1}aS$. Given our assumption on the principal right ideals, there are two possibilities: either $g^{-1}aS = S$ or $g^{-1}aS = aS$. The former would imply that a is invertible, thus $aS = g^{-1}aS$, and so $ga = ah$ for some invertible element h, as required. ∎

Define the function α: $G(S) \to G(S)$ by $ga = a\alpha(g)$. The function α is an endomorphism of the group $G(S)$. The proof of this uses the fact that $(gh)a = g(ha)$ combined with the uniquenes of the decomposition. We can identify the submonoid of S generated by a with the monoid $\mathbb{N}$ under addition. It follows that S is isomorphic to the set $\mathbb{N} \times G(S)$ equipped with the multiplication defined by

$$(m, g)(n, h) = (m + n, \alpha^n(g)h).$$

Remark Observe that there is a surjective monoid homomorphism from $\mathbb{N} \times G(S)$ to $\mathbb{N}$, and that the full inverse image of zero under this homomorphism is the group of units of $\mathbb{N} \times G(S)$, which is isomorphic to $G(S)$.

This then is Rees' structure theorem for left cancellative monoids whose principal right ideals form a decreasing chain order isomorphic to the dual of the usual ordering on the natural numbers. Although tangential to our main interest, it is worth noting that such monoids are analogues of DVR's. The following result, due to Stuart Margolis (private communication), deepens this analogy. First, we need a definition. A group G is said to be *indicable* if there is a surjective homomorphism θ: $G \to \mathbb{Z}$. We shall call the full inverse image under θ of the natural numbers (and zero!) the *positive cone* of G.

Proposition 2.2 *The positive cones of indicable groups are precisely the* cancellative *monoids whose principal right ideals form a decreasing chain order isomorphic to the dual of the usual ordering on the natural numbers.*

Proof Let G be an indicable group and θ: $G \to \mathbb{Z}$ its surjective homomorphism. Put $S = \theta^{-1}(\mathbb{N})$, the positive cone of G. It is easy to check that the group of units of S is precisely the group $\theta^{-1}(0)$, and that for all $s, t \in S$, we have that $sS = tS$ if and only if $\theta(s) = \theta(t)$. Let a be a fixed element of S such that $\theta(a) = 1$, which exists by surjectivity. Now given $s, t \in S$, suppose that $\theta(s) \leq \theta(t)$. Then $\theta(sa^{\theta(t)-\theta(s)}) = \theta(t)$, and so $tS = sa^{\theta(t)-\theta(s)}S \subseteq sS$. It follows that S is a cancellative monoid whose principal right ideals form a decreasing chain order isomorphic to the dual of the usual ordering on the natural numbers.

Conversely assume that S is a cancellative monoid whose principal right ideals form a decreasing chain order isomorphic to the dual of the usual ordering on the natural numbers. From the theory developed by Rees described above, there is a surjective homomorphism θ: $S \to \mathbb{N}$ such that $\theta^{-1}(0)$ is the group of units of S. Now S satisfies the Ore conditions and thus has a group of fractions G such that each element of G is of the form st^{-1} where $s, t \in S$. It is easy to see from the structure of groups of fractions that θ extends uniquely to a surjective function G to $\mathbb{Z}$, proving that G is indicable and that S is its positive cone.■

To understand how self-similar group actions can follow from the line of argument pursued by Rees, we need to weaken Rees' assumption on the structure of the principal right ideals. Observe that the monoid $\mathbb{N}$ is the free monoid on one generator and its poset of principal right ideals is the decreasing chain

$$\mathbb{N} \supset 1 + \mathbb{N} \supset 2 + \mathbb{N} \supset \ldots.$$

More generally, we shall assume that $\mathbf{P}(S)$ is order isomorphic to the infinite tree of principal right ideals of the free monoid on n generators where $n \geq 2$: the infinite regular n-ary tree. This assumption was first explictly made by Nivat and Perrot in [8]. However, their analysis of this case is flawed: they essentially assume that the analogue of Lemma 2.1 holds. It does not. By analysing this case correctly in the spirit of Rees' paper we shall arrive at self-similar group actions. Thus for the rest of this paper S will be a left cancellative monoid and $\mathbf{P}(S)$ will be isomorphic to $\mathbf{P}(A_n^*)$ where A_n is a set with n elements and A_n^* is the free monoid on A_n.

Our monoid S has n maximal proper principal right ideals which I shall denote by $a_1S, \ldots, a_nS$. We denote the set of a_i's by A. The set A replaces the single a in the case Rees considered. Let s be a non-invertible element of S. Then sS is a proper principal right ideal. By our assumption on the principal ideal structure of S, sS is contained in exactly one of the a_iS. We suppose that $sS \subseteq x_1S$ where $x_1 \in A$. Thus $s = x_1s_1$. The element s_1 is invertible iff $sS = x_1S$. If it is not invertible then $sS \subset x_1S$. In this case, $s_1S \subseteq x_2S$ for some unique $x_2 \in A$. Thus $s_1 = x_2s_3$. This process can be continued and so we obtain the following sequence of principal right ideals

$$sS \subset x_1 \ldots x_iS \subset \ldots \subset x_1S.$$

By our assumption on the structure of the principal right ideals, this process cannot be continued indefinitely. It follows that $S = A^*G(S)$, where A^* is the submonoid of S generated by A. In fact, a stronger result is true.

Proposition 2.3 *Each element of S can be written* uniquely *as a product of an element of A^* followed by an element of $G(S)$, and the monoid A^* is free.*

Proof Suppose that

$$1 = x_1 \ldots x_m$$

where $m \geq 1$ and $x_i \in A$. Then $S = x_1 \ldots x_mS \subseteq x_1S$ and so x_1 is invertible, which is a contradiction. Now suppose that

$$x_1 \ldots x_m = y_1 \ldots y_n$$

where $x_i, y_j \in A$. By our result above, we can assume that $m, n \geq 1$. Now $x_1 \ldots x_mS = y_1 \ldots y_nS \subseteq x_1S, y_1S$. It follows that $x_1 = y_1$ and so, by left cancellation, $x_2 \dots x_m = y_2 \dots y_n$. This process can be repeated and because either $m < n$ or $n < m$ would lead to a contradiction, namely that an element of A is invertible, we must have that $m = n$ and $x_i = y_i$. Thus A^* is the free monoid on A.

Suppose that $xg = yh$ where $x, y \in A^*$, $g, h \in G(S)$, and $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$ where $x_i, y_j \in A$. Arguing as before, $xS = yS \subseteq x_1S, y_1S$ and so $x_1 = y_1$. By left cancellation $x_2 \dots x_m g = y_2 \dots y_n h$. If $m = n$ then we can repeat this argument to get $x = y$ and so $g = h$, by left cancellation. If $m < n$, then we can easily deduce that y_{m+1} is invertible, which is a contradiction. A similar argument shows that we cannot have $n < m$. ■

What we have done so far directly generalises Rees and was also broadly found by Nivat and Perrot [8]. We can say that Rees considered the case when the free monoid had only one generator whilst we are considering the case where the free monoid has n generators.

The next step in our analysis uses the uniqueness of the decomposition obtained in Proposition 2.2. Rees uses this uniqueness applied to the associativity law to prove that α is a group endomorphism. We shall use the same approach but in our more complex situation. By Proposition 2.2, for each $x \in A^*$ and $g \in G(S)$ there is a unique $x' \in A^*$ and a unique $g' \in G(S)$ such that $gx = x'g'$. We denote x' by $g \cdot x$ and $g' = g|_x$. We call $g \cdot x$ 'action' and $g|_x$ 'restriction'. The following proposition lists their properties: in fact, these are simply special cases of results known from the theory of Zappa-Szép products [1].

Proposition 2.4 *The following properties hold where $g, h \in G(S)$ and $x, y \in A^*$:*

(SS1) $1 \cdot x = x$.

(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$.

(SS3) $g \cdot 1 = 1$.

(SS4) $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

(SS5) $g|_1 = g$.

(SS6) $g|_{xy} = (g|_x)|_y$.

(SS7) $1|_x = 1$.

(SS8) $(gh)|_x = g|_{h \cdot x} h|_x$.

Proof The proof follows by considering properties of the identity element and different cases of the associativity law. From $1x = x$, we deduce both (SS1) and (SS7). From $g1 = g$, we deduce both (SS3) and (SS5). From $(gh)x = g(hx)$, we deduce both (SS2) and (SS8). Finally, from $(gx)y = g(xy)$, we deduce both (SS4) and (SS6). ■

We therefore have, in particular, an action of $G(S)$ on A^*. This action has some extra important properties. The length of a string x is denoted by $|x|$. An action of a group G on a free monoid A^* is *length preserving* if $|g \cdot x| = |x|$ for all $x \in A^*$. The *prefix order* on A^* is defined by $x \leq y$ iff $x = yz$ for some string z. The action is *prefix preserving* if $x = yz$ in A^* implies that $g \cdot x = (g \cdot y)z'$ for some string z'. This means precisely that if $x \leq y$ then $g \cdot x \leq g \cdot y$.

Lemma 2.5 *The action of $G(S)$ on A^* is length preserving and prefix preserving.*

Proof Prefix preserving follows from (SS4). We now prove that the action is length preserving. Observe first that by (SS3), if x is the empty string so too is $g \cdot x$. Conversely, if $g \cdot x = 1$ then $x = g^{-1} \cdot 1 = 1$ by (SS3). Thus $g \cdot x$ is the empty string iff x is. Let $x \in X$. Suppose that $g \cdot x = yz$ where y is a letter and z is a string, possibly empty. Then by (SS4), we have that

$$x = (g^{-1} \cdot y)(g^{-1}|_y \cdot z).$$

We know that $g^{-1} \cdot y$ cannot be empty and so has length at least one. Since the leftthand side has length one and lengths add, we deduce that $(g^{-1}|_y \cdot z)$ has length zero. Thus z is the empty string. It follows that letters are mapped to letters. The result now follows by (SS4) and induction. ■

The action of $G(S)$ on A^* is closely connected to the greatest right normal divisor $M(S)$ of S.

Proposition 2.6 *The action of $G(S)$ on A^* is faithful if and only if S is fundamental.*

Proof Suppose the action is faithful. Let $g \in M(S)$. Then $gs \in sG$ for all $s \in S$. Let $x \in A^*$. Then $gx = xg'$, but is also equal to $(g \cdot x)g|_x$. It follows that $g \cdot x = x$ for all $x \in A^*$. But by assumption, the action is faithful and so $g = 1$ and S is fundamental.

Conversely, suppose that S is fundamental and that $g \cdot x = x$ for all $x \in A^*$. Let $s \in S$ where $s = xh$. Then

$$gs = gxh = (g \cdot x)g|_x h = xg_x h = xh(h^{-1}g|_x h) \in sG.$$

It follows that $g \in M(S)$ and so $g = 1$ which proves that the action is faithful.■

The mistake that Nivat and Perrot made in [9] was to assume that this action is always trivial. It *is* trivial in the case of a free monoid with one generator because of the length preserving property of the action, but this is no longer true in the case of free monoids of more than one generator. In the case where the action *is* trivial, the conditions (SS1)–(SS8) say precisely that there is a homomorphism from A^* into the group of endomorphisms of G given by $\alpha(x)(g) = g|_x$ just as in the case considered by Rees. Our next example shows that the action of the group need not be trivial.

Example 2.7 Centre the Sierpinski gasket at the origin, and consider the monoid S of all similarities of the plane that map the gasket into itself. The group of units of this monoid is just the six element group of symmetries of the equilateral triangle. I shall now pick out certain important elements of S: a clockwise rotation by $\frac{2\pi}{3}$ denoted by ρ; a reflection in the vertical denoted by σ; and three similarities denoted T, L and R which halve the size of the gasket and then translate it to the top, left and right parts of the original gasket. It is not hard to see that the monoid generated by these similarities is S and that the submonoid of S generated by T, L and R is the free monoid on three generators. Simple calculations show that

$$\rho T = R\rho, \quad \rho L = T\rho, \quad \rho R = L\rho$$

and

$$\sigma T = T\sigma, \quad \sigma L = R\sigma, \quad \sigma R = L\sigma.$$

Thus every element of S can be written as a product of an element of a free monoid and a group element. This representation is unique: if $xg = yh$ where $g, h \in G(S)$ and $x, y \in \{T, L, R\}^*$ then $x = yhg^{-1}$. However elements of $\{T, L, R\}^*$ do not change the orientation of a triangle whereas non-identity elements of $G(S)$ do. Thus $g = h$ and so $x = y$. It is now easy to check that S is a monoid of the type considered in this section. In particular, the relations above show that we have defined a non-trivial action of the group $G(S)$ on the free monoid on three generators generated by T, L and R that satisfies the properties of Proposition 2.4. ■

We can now make the link with self-similar group actions. Before giving the formal definition, I shall motiviate it by considering the behaviour of an arbitrary automorphism θ of $(A^*, \leq)$, the free monoid regarded as a poset with respect to its prefix ordering [9]. The automorphism θ is prefix preserving, by assumption, and evidently length preserving. For each $a \in A$, we see that $ax \leq a$ implies $\theta(ax) \leq \theta(a)$ and so θ induces an order isomorphism θ' from the principal order ideal aA^* to the principal order ideal $\theta(a)A^*$. There are order isomorphisms $\lambda_a \colon A^* \to aA^*$ and $\lambda_{\theta(a)} \colon A^* \to \theta(a)A^*$ which arise from the fact that $(A^*, \leq)$ is a *uniform* poset. Define $\phi = \lambda_{\theta(a)}^{-1}\theta'\lambda_a$, an order automorphism of A^*. Then

$$\theta(ax) = \theta(a)\phi(x),$$

where ϕ is uniquely determined by a and θ. This feature of automorphisms of $(A^*, \leq)$ is used as the basis for the following definition.

A faithful group action of a group G on a free monoid X^* is said to be *self-similar* if for each $g \in G$ and $x \in X$ there exists a unique $y \in X$ and $h \in G$ such that $g \cdot (xw) = y(h \cdot w)$ for all $w \in X^*$. The proof of the following is now immediate.

Theorem 2.8 *Let S be a fundamental left cancellative monoid whose poset of principal right ideals is order isomorphic to the poset of principal right ideals of a free monoid A^*. Then there is a faithful action of $G(S)$ on A^* which is* *self-similar.* ■

In fact, the converse is true and so there is a correspondence between fundamental monoids of the type we are considering and self-similar group actions. This is proved in [3]. Rees' structure theorem (Theorem 3.3 [10]) is then generalised by the use of Zappa-Szép products. Specifically, the left cancellative monoids of the type we are considering are isomorphic to monoids of the form $A^* \times G(S)$ where the product is given by

$$(x,g)(y,h) = (x(g \cdot y), g|_y h).$$

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