IN McALISTER'S FOOTSTEPS: A RANDOM RAMBLE AROUND THE P-THEOREM.

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The work of Don McAlister has been an inspiration to all of us interested in semigroups. In 1974, Don published two papers which had a decisive impact on the subsequent development of semigroup theory. In these papers, two major theorems were proved: the 'covering theorem' and the 'P-theorem'. In this paper, we shall take the latter as the starting point for some excursions through our own and others' work.

1. A primer on categories and inverse semigroups

In this section, we shall review the basic definitions and results about categories and inverse semigroups we shall need, and indicate one way in which inverse semigroups give rise to categories.

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There are two equivalent definitions of 'category' in the literature. We shall give the 'arrows only' definition first, and then briefly indicate how the other definition works. Let C be a set equipped with a partially defined binary operation. An *identity* is any element e such that if ea is defined then $ea = a$, and if ae is defined then $ae = a$. If e is an identity and ae is defined then e is called a *right identity of* a, and if ea is defined then e is called a *left identity of* a. A *category* C is a set equipped with a partially defined multiplication such that each element has a unique left identity, called its *source*, and a unique right identity, called its *target*; the product ab is defined iff the source of a is equal to the target of b; the product is associative when it is defined; and finally, the source of ab is equal to the source of b, and the target of ab is equal to the target of a.

The other definition of a category is the 'objects and arrows' one. This starts with a directed graph whose vertices are called *objects* and whose directed edges are called *arrows*. In addition, for each object v there is a unique arrow 1_v which forms a loop at v. We now require a partial multplication to be defined on the arrows in which the loops of the form 1_v are the identities and the above axioms for a category hold.

Apart from these two variations on the definition of a category, the reader should also be aware that the product of two arrows ab is sometimes instead defined iff the target of a is equal to the source of b. Our first definition of multiplication models composition of functions when the arguments are written on the right, whereas our second definition models composition of functions when the arguments are written on the left.

In a category, the set of all arrows with source and target equal to a given identity e is a monoid called the the *local monoid at* e. A category with a single identity is therefore just a monoid. For us categories are algebraic structures in the usual way generalising monoids. The morphisms between categories are called *functors*.

A *groupoid* is a category in which for each element a there is an element b such that ab and ba are identities. Thus a groupoid with one identity is a group. A category is said to be *connected* if for each pair of identities e and f there is an arrow with source e and target f. Connected groupoids can be described in terms of groups. If G is a group and I is a set then $I \times G \times I$ becomes a connected groupoid when the product of triples is defined by $(i, g, j)(j, h, k) = (i, gh, k)$ and undefined otherwise. The identities in this case are the elements of the form $(i, 1, i)$. Groupoids constructed in this way are often known as *Brandt groupoids*. Given a connected groupoid, choose an identity e and fix it, and denote the set of identities of the groupoid by I.

For each identity f, pick an arrow a_f from e to f, which is possible because the groupoid is connected. The local monoid at e is in fact a group, which we shall denote by G. Define a function from the connected groupoid to the Brandt groupoid $I \times G \times I$ by g maps to $(j, a_j^{-1} g a_i, i)$ if g has source i and target j. It is straightforward to check that we have proved that every connected groupoid is isomorphic to a Brandt groupoid. We say that the isomorphism is defined by 'co-ordinatising' the original groupoid.

A semigroup S is said to be *inverse* if for each element s there exists a unique element t such that the following two equations hold:

$$s = sts \text{ and } t = tst.$$

The uniquely defined element t is called the *inverse* of s and is denoted s^{-1}. This is a generalisation of the definition of inverse used in group theory. However, the elements $s^{-1}s$ and ss^{-1} are not identites in general, even if the semigroup is a monoid, but retain one feature possessed of identities in that they are idempotents, where an *idempotent* is an element e such that $e^2 = e$. Remarkably, it can be proved that the product of two idempotents is an idempotent, and so the set of idempotents, $E(S)$, forms a subsemigroup, which is also commutative.

On every inverse semigroup, we can define a relation $\leq$ by

$$s \leq t \text{ iff } s = te \text{ for some idempotent } e.$$

This relation is a partial order, called the *natural partial order*. It intertwines nicely with the algebraic structure of the inverse semigroup in the sense that if $s \leq t$ then $s^{-1} \leq t^{-1}$, and if $s \leq t$ and $s' \leq t'$ then $ss' \leq tt'$. With respect to the natural partial order the set of idempotents is a meet semilattice when we define $e \wedge f = ef$. For this reason, we often refer to the *semilattice of idempotents* of an inverse semigroup.

The natural partial order can be used to define an important congruence on every inverse semigroup. Define the relation σ by

$$s \,\sigma\, s' \Leftrightarrow t \leq s, s' \text{ for some } t.$$

Then σ is a congruence, S/σ is a group, and σ is the smallest group congruence. In this paper, $G(S) = S/\sigma$ will be called the *universal group* of S. The natural map from S to $G(S)$ is denoted by γ.

The natural partial order plays an important role in the structure of inverse semigroups. Define a partial operation $\circ$ on S as follows:

$$s \circ t \text{ is defined iff } s^{-1}s = tt^{-1}$$

in which case set $s \circ t = st$. The structure $(S, \circ)$ is a groupoid. The groupoid product and the natural partial order together determine the semigroup product since

$$st = (se) \circ (et),$$

where $e = s^{-1}stt^{-1}$, and $se \leq s$ and $et \leq t$.

The representation of the multiplication of an inverse semigroup in terms of a groupoid multiplication is the first clue that categories may well play a role in studying inverse semigroups — they do in a number of different ways as we shall see throughout this paper.

Finally, we define one further relation on an inverse semigroup. The *compatibility relation* $\sim$ is defined by

$$s \sim t \text{ iff } s^{-1}t, st^{-1} \text{ are both idempotents.}$$

This relation is reflexive and symmetric, but not transitive in general. More on inverse semigroups can be found in [17].

A good example of an inverse semigroup is the *symmetric inverse monoid* on the set X, denoted $I(X)$, which consists of all bijections between subsets of X (*partial bijections*) with the operation of composition of partial functions. In this inverse semigroup, the idempotents are the identity functions on subsets, the natural partial order is the usual ordering of partial functions, the groupoid product of elements is only defined when the domain of the first matches exactly the image of the second, and a pair of partial bijections are compatible iff their union is another partial bijection. Every inverse semigroup can be embedded in a symmetric inverse semigroup, a result known as the *Vagner-Preston representation theorem*.

2. The *P*-theorem

The P-theorem centres on the class of E-unitary inverse semigroups. These semigroups were introduced by Saito in 1965 [52] who called them *proper inverse semigroups*. The 'p' of 'proper' explains the 'p' in 'P-theorem'. As we shall see, there are many, equivalent definitions of E-unitary inverse semigroups, each giving a different way of thinking about them. Perhaps the simplest is the following. An inverse semigroup is *E-unitary* if an element above an idempotent, in the natural partial order, is also an idempotent.

Why is this a good class of semigroups to study? Well, mainly because we can find so many interesting examples of them. For example, free inverse monoids are E-unitary, so we are off to a flying start. But here are some other reasons to study them:

(1) The inverse monoid generated by the Möbius transformations on the complex plane is E-unitary. Its universal group is the Möbius group [16].
(2) The inverse monoid of all right ideal isomorphisms between essential finitely generated right ideals of the free monoid on two generators is E-unitary. Its universal group is the Thompson group V [1].
(3) The linear clause monoids over operator domains having a single operation are E-unitary [27]. These monoids are closely related to the group theory of Patrick Dehornoy.
(4) A further connection between groups and E-unitary inverse semigroups is the M&M expansion of X-generated groups to X-generated E-unitary inverse semigroups [30].
(5) Kellendonk's topological groupoid constructed from an inverse semigroup is T_1 in general, but Hausdorff when the inverse semigroup is E-unitary [17].
(6) For every inverse semigroup S there is an E-unitary inverse semigroup T and a surjective homomorphism $\theta\colon T \to S$ which induces an isomorphism between $E(T)$ and $E(S)$. This is the 'covering theorem' [32]. There is sadly no space to say more about this theorem. We can do no more than refer you to McAlister's papers [36, 37, 39, 40, 41, 42] and those of Lawson [8, 10, 11, 12, 13].
(7) One *disadvantage* of E-unitary inverse semigroups is that if they have a zero then they are necessarily semilattices, because every element lies above the zero. This rather disappointing result is rectified in the definition of E^*-unitary and strongly E^*-unitary inverse semigroups. For more on these see [24].

One striking feature of E-unitary inverse semigroups is that they can be characterised in a wide variety of ways. Proofs and references can be found in [17].

Theorem 2.1. *The following are equivalent for an inverse semigroup S.*

(i) S is E-unitary.
(ii) The homomorphism $\gamma\colon S \to G(S)$ is idempotent pure *meaning that $\gamma(s) = 1$ implies that s is an idempotent.*
(iii) The homomorphism $\gamma\colon S \to G(S)$ is $\mathcal{L}$-injective*, meaning that γ restricted to each $\mathcal{L}$-class is injective.*
(iv) The function from S to $E(S) \times G(S)$ that maps s to $(s^{-1}s, \sigma(s))$ is injective.

(v) $\sigma(e) = E(S)$ *for each idempotent* e.
(vi) The compatibility relation is transitive.
(vii) The compatibility relation is equal to σ.

Both groups and semilattices are E-unitary. So the idea arises that maybe groups and semilattices could be used as building blocks for constructing all E-unitary inverse semigroups. A very nice heuristic which leads to the P-theorem is described in [42].[a] Essentially, one observes that semidirect products of semilattices by groups are E-unitary; that inverse subsemigroups of semidirect products of semilattices by groups needn't be semidirect products of semilattices by groups; and that inverse subsemigroups of E-unitary inverse semigroups are also E-unitary. If you have very good intuitions about inverse semigroups, you then come up with the following construction. Note that partially ordered sets will be abbreviated to *posets*. An *order ideal* in a poset is a subset that contains all elements beneath each element of the subset. A *McAlister triple* (G, X, Y) consists of a group G, a poset X, and an order ideal Y of X that is a meet semilattice under the induced order, such that G acts on X by order automorphisms satisfying the following two conditions:

(MT1) $G \cdot Y = X$.
(MT2) $g \cdot Y \cap Y \neq \emptyset$ for each $g \in G$.

Put

$$P = P(G, X, Y) = \{(y, g) \in Y \times G\colon\ g^{-1} \cdot y \in Y\}.$$

Define a binary operation on P by

$$(y, g)(y', g') = (y \wedge g \cdot y', gg')$$

where the meet always exists, is defined in the poset X, and belongs to Y. It can be checked that P is an E-unitary inverse semigroup with semilattice of idempotents isomorphic to Y and universal group G. Semigroups of the form $P(G, X, Y)$ are called *P-semigroups*. If $Y = X$ then we get back semidirect products of semilattices by groups. But the construction, although it looks like a semidirect product, isn't. The partially ordered set X is a crucial ingredient even though it seems to stand aloof from the proceedings. It also turns out that it cannot in general be replaced by a semilattice. The 'P-theorem' can now be stated.

[a]A paper which is neither random, nor rambling, but *is* about inverse semigroups.

Theorem 2.2. *Each E-unitary inverse semigroup is isomorphic to a P-semigroup.*

There are many different proofs of this theorem. For example:

(1) Don's own proof 1974 [33]. See Section 6.
(2) Boris Schein's proof 1975 [53]. See Section 3.
(3) Norman Reilly and Douglas Munn's proof 1976 (using free inverse semigroups) [51].
(4) Douglas Munn's proof 1976 [49]. See Section 3.
(5) The maverick alternative: 'the Q-theorem' by Mario Petrich and Norman Reilly 1979 [50]. See Section 3.
(6) Stuart Margolis and Jean-Eric Pin's proof 1987 (using the derived category) [29]. See Section 5.
(7) Mark V Lawson's proof of 1990 [6]. See Section 4.
(8) Helen James and Mark V Lawson's proof of 1999 [19]. See Section 6.
(9) Ben Steinberg's proof 2003 (using Schützenberger graphs) [56].

In subsequent sections, we shall look at the proof of the P-theorem from a number of different points of view each of which will provide a partial answer to the following question: what does the P-theorem mean?

3. Partial group actions

We shall begin with the most concrete way of thinking about E-unitary inverse semigroups. By the Vagner-Preston representation theorem, every inverse semigroup is isomorphic to an inverse semigroup of partial bijections. Accordingly, let S be an E-unitary inverse subsemigroup of a symmetric inverse monoid $I(X)$. Looking at our list of characterisations of a semigroup being E-unitary in Theorem 2.1, there are three that we want to highlight now: (v), (vi) and (vii). An element g of the universal group $G(S)$ is a σ-class and so by (vii) a set of pairwise compatible elements of S. We may therefore form the union, f_g, within $I(X)$, of the elements of g to obtain a well-defined partial bijection of X. If $1 \in G$ is the identity element then f_1 is an idempotent by (v). The idempotents in $I(X)$ are the identity functions defined on subsets of X. It is no loss in generality to assume that the domain of definition of f_1 is the whole of X, since if it isn't we can embed S in $I(X')$ where X' is the domain of f_1. If $g \in G$ and $x \in X$ then the element x need not belong to the set $\mathrm{dom}(f_g)$, but if it does then we define

$$g \cdot x = f_g(x).$$

We therefore have a partial function $G \times X \to X$ that defines what we call a 'partial action' of G on X. We now make this precise. A partial function $G \times X \to X$ which maps (g, x) to $g \cdot x$ is said to define a *partial group action* [25] if the following three axioms hold.

(PGA1) $1 \cdot x$ is always defined and equals x.
(PGA2) If $g \cdot x$ is defined then $g^{-1} \cdot (g \cdot x)$ is defined and equals x.
(PGA3) If $g \cdot (h \cdot x)$ is defined then $(gh) \cdot x$ is defined and they are equal.

Observe that (PGA3) is the crucial difference with 'global' group actions. If S is an E-unitary inverse semigroup of partial bijections of the set X then $G(S)$ acts partially on X. This leads us to think that there may be some connection between the structure of E-unitary inverse semigroups and partial group actions.

Partial group actions can easily be constructed. Let $G \times X \to X$ be a global group action and $X' \subseteq X$. Then G acts partially on X'. In fact, every partial group action arises in this way. The following theorem, known as the 'Globalisation Theorem', although formally proved by Johannes Kellendonk and Mark V Lawson, has been around in one form or another for a long time.

Theorem 3.1. *Let G act partially on the set X. Then there is a set $\bar{X}$, essentially unique, such that G acts on $\bar{X}$, $\bar{X}$ contains X, the restriction of the action of G to X is equal to the original partial action, and $G \cdot X = \bar{X}$.*

We say that the action of G on $\bar{X}$ is the *globalisation* of the partial action of G on X. We already see parallels with the definition of a McAlister triple: indeed, if (G, X, Y) is a McAlister triple then G acts partially on Y and since $G \cdot Y = X$ it follows that the action of G on X is the globalisation of the partial action of G on Y. The only difference is that the sets and actions have extra structure. Can we use the Globalisation Theorem to prove the P-theorem? The answer is 'yes' in two different ways.

Schein's proof Let S be an E-unitary inverse semigroup. Then by Theorem 2.1(iv), there is an embedding $\kappa \colon S \to E(S) \times G(S)$ which maps s to $(s^{-1}s, \sigma(s))$. Let A be the image of κ. The semigroup S is isomorphic to an inverse subsemigroup of $I(A)$ by the Vagner-Preston representation theorem. The globalisation of the partial action of $G(S)$ on A turns out to be $B = E(S) \times G(S)$, where $G(S)$ acts on B by left multiplication on the second component. In other words, the globalisation can be explicitly described and has nice properties. The ingredients for the McAlister triple corresponding to S can now be read off from the globalisation we have constructed.

Munn's proof and the 'Q-theorem' Let S be an E-unitary inverse smeigroup. Then by Theorem 2.1(iii), $\gamma\colon\ S \to G(S)$ is $\mathcal{L}$-injective. Thus given $g \in G(S)$ and $e \in E(S)$ there is *at most one* element $s \in S$ such that $s^{-1}s = e$ and $\gamma(s) = g$. Define a partial action of $G = G(S)$ on the set $E = E(S)$ by

$$g \cdot e = ss^{-1}$$

if $s^{-1}s = e$ and $\gamma(s) = g$. The set of elements of E on which G acts forms a non-empty order ideal of E. The partial action is order preserving, when this makes sense. The data of the partial action of G on E is equivalent to the semigroup S (this is the substance of the 'Q-theorem'). The construction of the globalisation will be found to be a poset; the details are exactly Munn's proof of the P-theorem.

Is the P-theorem equivalent to the globalisation theorem? Almost. We shall clarify this later when we talk about 'enlargements'. Globalisations of partial group actions are interesting and widespread in mathematics. For example, the globalisation in the case of the Möbius group is the Riemann sphere [25]. Ben Steinberg has looked at partial group actions on cell complexes and poses a number of interesting questions about their applications [55]. In particular, he suggests that Bass-Serre theory might be developed from the point of view of partial group actions.

4. Ordered groupoids

There are three ways of proving the P-theorem using category theory: Margolis and Pin's proof [29], Lawson's proof [6], and the proof of James and Lawson [19]. In this section, we shall discuss Lawson's proof, in Section 5 the proof of Margolis and Pin, and in the final section the proof due to James and Lawson.

In the 1950's, Charles Ehresmann developed the theory of ordered groupoids motivated by questions in differential geometry. Since inverse semigroups can be regarded as ordered groupoids the question arose of the implications of Ehresmann's work for inverse semigroup theory. The P-theorem provided the key for understanding these implications. The papers [6, 7, 9, 14, 15] developed the ordered groupoid approach to inverse semigroups motivated by the desire to combine Ehresmann's ideas [2] with those of McAlister. In this section, we shall explain how the proof of the P-theorem looks when viewed from the perspective of ordered groupoids. We shall also say something about the connection between ordered groupoids and partial group actions. We shall finish off by sketching out how the P-theorem can be viewed from the vantage-point of a sort of generalised homotopy theory.

An *ordered groupoid* is a groupoid whose set of arrows is also a poset satisfying a number of additional conditions: the order intertwines nicely with respect to taking inverses and with respect to products, just as the natural partial order for inverse semigroups; if e is an identity less than the source of an arrow g then there exists a unique element, denoted $(g|e)$, which is less than g and whose source is e — the element $(g|e)$ is called the *restriction of g to e*. Functors between ordered groupoids are ordered preserving functors and are termed *ordered functors*. The set of identities of an ordered groupoid is a poset. If this poset is also a meet semilattice then the ordered groupoid is called, for historical reasons, an *inductive groupoid*. Inverse semigroups can be regarded as inductive groupoids when one considers them with respect to their groupoid product and their natural partial order. Groupoids are ordered groupoids when ordered by equality, and posets are ordered groupoids when viewed as groupoids of idempotents. The fact that posets can be viewed as ordered groupoids is promising when thinking about the mysterious poset X in the P-theorem.

The starting point for understanding how ordered groupoids can shed light on the P-theorem is Theorem 4.12 of [37]:

Theorem 4.1. *An inverse semigroup is isomorphic to a semidirect product of a semilattice by a group iff the homomorphism γ to its universal group is $\mathcal{L}$-bijective.*

Henceforth, we shall call $\mathcal{L}$-injective maps *immersions*, and $\mathcal{L}$-bijective maps *coverings*. The difference between arbitrary E-unitary inverse semigroups, and those which are semidirect products of semilattices by groups now resides in the difference between immersions and coverings.

Can we convert our immersion to a covering? Well, yes and no. It can be extended to a covering but the covering will be from an ordered groupoid. There are two ideas on which the ordered groupoid proof of the P-theorem is based.

The first idea requires us to think about group actions in terms of groupoids. Let G be a group acting on the set X. This action can be regarded as a groupoid: in fact when we draw pictures of group actions with points representing the elements of X and arrows showing us how elements of G move the points of X around, we are precisely thinking of the action as a groupoid. Now let G be a group acting by order automorphisms on the poset X. Our groupoid becomes an ordered groupoid. Thus given an action of the group G on the poset X by order automorphisms, we define the semidirect product of X by G, denoted $P(G, X)$, to be the ordered groupoid whose arrows are pairs (x, g) which start at $(g^{-1} \cdot x, 1)$ and terminate at $(x, 1)$. We define $(x, g) \leq (x', g')$ iff $g = g'$ and $x \leq x'$. The projection from $P(G, X)$ to G is an ordered covering functor.

Theorem 4.1 can be generalised from inverse semigroups to arbitrary ordered groupoids.

Theorem 4.2. *An ordered groupoid is isomorphic to a semidirect product of a poset by a group iff it admits a surjective ordered covering functor to a group.*

The second idea we need is that of an 'enlargement'. An enlargement is a particular kind of relationship between an inverse semigroup and an inverse subsemigroup or, more generally, between an ordered groupoid and an ordered subgroupoid. The idea, which can be justified, is that an inverse semigroup (or ordered groupoid) and its enlargement are very similar in structure: the enlargement being a sort of expanded version of its substructure. The definition arose by combining ideas to be found in both Ehresmann and McAlister. In particular, in the case of inverse semigroups, it is McAlister's notion of a 'heavy' inverse subsemigroup combined with an extra notion mentioned in a remark on page 208 of [37]. It also recurs in McAlister's work on the local structure of regular semigroups [43, 44, 45, 46, 47, 48]. Enlargements play an important role in [15, 20, 21, 22].

Let G be an ordered subgroupoid of an ordered groupoid H. We say that H is an *enlargement* if the following three conditions hold.

(E1) G is an order ideal of H.
(E2) If the source and target of an arrow of H belong to G then the arrow belongs to G.
(E3) Each idempotent of H is connected by an arrow to an idempotent of G.

The ordered groupoid version of the P-theorem can now be stated.

Theorem 4.3. *The P-theorem is equivalent to the following statement. Let S be an E-unitary inverse semigroup. Then the immersion $\gamma\colon S \to G(S)$*

can be factorised $\theta = \iota\Theta$ where $\iota\colon S \to \bar{S}$ is an inclusion of S into an ordered groupoids $\bar{S}$ which is an enlargement of S, and $\Theta\colon \bar{S} \to G$ is a covering.

We shall call this process the 'enlargement of an immersion to a covering'.

This proof of the P-theorem looks quite different from the partial group action proofs described in the previous section. However, as we shall now show, they are really just different sides of the same coin. Recall that a group G acting on a set X can be repackaged as a groupoid equipped with a covering to G. It is easy to show that a group G acting *partially* on a set X can be repackaged as a groupoid equipped with an *immersion* to G. Theorem 4.3 above can be generalised: any surjective immersion from an ordered groupoid to a group can be enlarged to a covering. The above result is equivalent to the globalisation theorem for groups acting partially by order automorphisms on posets.

We therefore arrive at one answer to our question about the meaning of the P-theorem:

> *"Globalisations and the P-theorem are both aspects of one and the same problem."*

The theory of enlargements above has wide-ranging generalisations; more information can be found in [17]. But this is not quite the end of the story. Ben Steinberg looked afresh at the work on the maximum enlargement theorem through Rhodes-tinted spectacles [54]. Steinberg's work in turn led to a fully-fledged homotopy theory of ordered groupoids (and so of inverse semigroups) [23]. If you put all this together, you get another, quite-different looking interpretation of the P-theorem.

> *"The P-theorem is an analogue (in some sense) of the well-known result in topology that states that every continuous function can be factorised into a homotopy equivalence followed by a fibration."*

5. Extensions of semilattices by groups, the derived category and global semigroup theory

In this section, we take a look at the next categorical proof of the P-theorem due to Margolis and Pin [29], chronologically the first using categorical methods. This had ramifications beyond inverse semigroup theory in that it led to the introduction of methods from the algebraic theory of categories into semigroup theory. This has had a major influence on subsequent developments in semigroup theory and also provides a strong connection between the work of Don McAlister and John Rhodes' notion of Global Semigroup Theory.

The starting point for this approach is the important role played in both group theory and semigroup theory by wreath products. We mention just two examples. First, the theorem of Krasner and Kaloujnine [5] states that if $f : G \to H$ is a morphism of groups, then G embeds into the wreath product $N \wr H$ where $N = Ker(f)$. Thus every finite group G embeds into the wreath product of its Jordan-Hölder factors, arising from a composition series for G. Second, the Krohn-Rhodes Theorem [3] shows that every finite semigroup S divides[b] a wreath product of subgroups of S and the three element monoid consisting of two right zeroes and an identity element. By the Krasner and Kaloujnine Theorem we can further decompose the groups into their simple divisors.

How do these ideas relate to the P-theorem? Characterisation (ii) of an E-unitary inverse semigroup in Theorem 2.1, the one characterisation we have yet to use, says that an inverse semigroup S is E-unitary if and only if it is an extension of its semilattice $E(S)$ by its universal group $G(S)$. Suppose we could find a generalisation of the Krasner-Kaloujnine Theorem that would, given a homomorphism $f : S \to T$, obtain S from T via the wreath product and something playing the role of $Ker(f)$. Then we could apply such a construction to the case where S is an E-unitary inverse semigroup and T is its universal group to try to obtain a proof of the P-Theorem as well as to problems arising from the Krohn- Rhodes Theorem.

For this program to work, an appropriate notion of the kernel of a semigroup morphism $f : S \to T$ has to be found. We know that the image of a morphism is the quotient of S by the congruence relation associated to f. The first approach to the problem was given by Bret Tilson [58] who defined the *derived semigroup* of a morphism to be essentially a partial action of S on the collection of congruences classes of f. The innovation was that the derived semigroup was only locally inside S and not a subsemigroup as in the case of groups. This was successful enough to help prove (along with the notion of the Rhodes expansion) the difficult and fundamental theorem of Rhodes complexity theory which states that if S and T are finite semigroups and f is injective on subgroups of S, then the group

[b]That is, is a homomorphic image of a subsemigroup.

complexity of S is equal to that of T [58].

However, the derived semigroup fails to do the job precisely in cases like that of the morphism between an E-unitary semigroup and its universal group. Even a cursory look at the definition of the derived semigroup [58] shows that it has a category like multiplication with an externally adjoined zero to take care of products that are not defined. In the case of the morphism between an E-unitary semigroup and its universal group, the derived semigroup is an inverse semigroup, but its semilattice is a 0-disjoint union of $|G(S)|$ copies of the semilattice of S, whereas we want the "kernel" to have semilattice $E(S)$.

This problem was solved in [29] by discarding the offending zero and treating the derived object as a category. This was motivated by corresponding uses of groupoids in the theory of groups as exposed by Philip Higgins [4] and others. In this way, the *derived category* of the morphism between an E-unitary semigroup and its universal group is a category $D(\gamma)$ such that each local monoid is a semilattice. The following gives an outline of the use of the derived category to prove the P-Theorem. It is convenient to assume that S is a monoid. This presents no problem, as an inverse semigroup S is E-unitary if and only if the monoid S^1 obtained by adjoining an identity to S is E-unitary. Let $\gamma\colon\ S \to G(S)$ be the map from the E-unitary monoid S to its universal group $G(S)$. The derived category $D(\gamma)$ has objects $G(S)$ and for each $g, h \in G(S)$, the set of morphisms from g to h is $\{(g, m, h) | g(\gamma(m)) = h\}$. There is an obvious Brandt-groupoid-like composition that turns this into a category. The local monoid at any object g is a semilattice isomorphic to $E(S)$. The group G(S) acts on the category $D(\gamma)$ by left multiplication. This action is transitive on the objects. Thus the quotient $D(\gamma)/G(S)$ is a one object category, better known as a monoid! The monoid $D(\gamma)/G(S)$ is canonically isomorphic to S and the map from $D(\gamma)$ to $D(\gamma)/G(S)$ is a covering of categories [4]. The P-structure of S can be recovered from $D(\gamma)$. As expected, the group in the McAlister triple is $G(S)$ and the semilattice is $E(S)$. Pleasantly, the "mysterious" partially ordered set turns out to be the partially ordered set of $\mathcal{J}$-classes of $D(\gamma)$. Here we view $D(\gamma)$ as a partial associative structure and then Green's relations have the same definition and analogous properties as in the case of semigroups. In fact, they are the restriction of Green's relations to $D(\gamma)$ on the semigroup obtained by adding an external zero to $D(\gamma)$. As mentioned above, this semigroup is what Tilson defined to be the derived semigroup of γ.

This was just the beginning of the use of the derived category and related algebraic categories in the decomposition theory of semigroups and its applications to various disciplines. In fact, the idea goes back to what is called the Grothendieck construction. Bret Tilson made fundamental and deep contributions to this area and put it at the centre of semigroup theory. In particular, he showed that the construction is, in a precise sense, adjoint to the construction of semidirect products: another connection to the P-Theorem. See his paper [59] and the later joint work with Benjamin Steinberg [57].

Finally we give a connection between the McAlister Theorems and Rhodes' notion of global semigroup theory. We would like to study an arbitrary inverse semigroup S as an extension of its universal group $G(S)$. If S has a zero element, then except for semilattices this approach is doomed from the start, since $G(S)$ is easily seen to be the trivial group. In global semigroup theory, one looks for an "expansion" of S to remove the obstruction that 0 presents. An expansion is intuitively a semigroup that maps onto S, is close to S in its structure and has nicer properties; in our case, it would be E-unitary and thus we could build the expansion from its universal group and its semilattice via the P-Theorem. McAlister's Covering Theorem gives exactly this. Every inverse semigroup is an idempotent separating image of an E-unitary inverse semigroup (finiteness can be preserved). This means that the covering semigroup has the same semilattice as S in particular. Thus the two McAlister Theorems can be considered to be an early example of the methods and philosophy of Global Semigroup Theory.

6. Cancellative categories

In this section, we shall describe the remaining categorical proof of the P-theorem due to James and Lawson [19]. It is appropriate that we describe it last of all because it in fact takes us back to Don's original proof of the P-theorem, as we shall see.

The starting point is the class of bisimple inverse monoids. An inverse semigroup is bisimple if it consists of a single $\mathcal{D}$-class. It is a classical theorem of Clifford that bisimple inverse monoids are determined by the $\mathcal{R}$-class containing the identity: this is a right cancellative monoid in which the intersection of any two principal left ideals is again a principal left ideal. We shall call such monoids *division monoids* for short. Furthermore, from each (abstract) division monoid we can construct a bisimple inverse monoid. Recall that a (right) cancellative monoid is *right reversible* if for all elements s and t in the monoid, elements p and q can be found such that $ps = qt$. It is a classical theorem of Ore, that each right reversible cancellative monoid can be embedded in a group in such a way that each element of the group is of the form $a^{-1}b$ where a and b are elements of the monoid. It is interesting that Rees proved Ore's embedding theorem using, once again, E-unitary inverse semigroups. You can find his proof on pages 68 and 69 of [17]. Don McAlister and Bob McFadden proved the following result in Section 3 of [35].

Theorem 6.1. *Let S be a bisimple inverse monoid. Then S is E-unitary if and only if its associated division monoid is cancellative. In the E-unitary case, the McAlister triple describing S can be recovered easily from the embedding of the division monoid into its group of fractions.*

In [19], this argument is generalised to an arbitrary E-unitary inverse monoid. The role of the $\mathcal{R}$-class containing the identity is taken by the division category $C(S)$ of S in the sense of Leech [28]. This is a right cancellative category. The existence of least common left multiples is generalised to the condition that each pair of arrows with a common source has a pushout. It's proved that an inverse monoid S is E-unitary if and only if its division category is cancellative. When S is E-unitary the division catgeory $C(S)$ can be embedded in a connected groupoid, its groupoid of fractions, in a way that directly generalises the Ore embedding theorem. The McAlister triple describing S can be recovered easily from this embedding.

McAlister's original proof of the P-theorem can be obtained from the above proof by 'choosing coordinates'. The starting point is the result we mentioned in Section 1: connected groupoids are isomorphic to Brandt groupoids, the isomorphism being defined by 'co-ordinatising' the groupoid. If this result is applied to the proof above then we get *exactly* the first ever proof of the P-theorem.

Again, this is not the end of the story. Leech's categorical description applies to inverse monoids. How can it be generalised to inverse semigroups? McAlister's papers [31, 34, 38] provided the clues, and the theory was worked out in [18, 26].

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