INVERSE SEMIGROUP ENLARGEMENTS OF INVERSE MONOIDS

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Abstract

Let S be a fixed inverse monoid. We show how to construct, up to isomorphism, all inverse semigroups T containing an idempotent e such that S is isomorphic to eTe and T = TeT. We also show how a class of such semigroups T gives rise to a class of E-unitary covers of S over semilattices. Finally, we interpret our construction in terms of categories and establish connections with work of Grandis, and Morita equivalence of categories.

1. Introduction

We shall assume that the reader is familiar with the basic definitions of inverse semigroup theory. In general, we shall follow the notation and definitions of Howie [3] and Petrich [9]. We simply highlight here one or two pieces of notation. We denote the set of idempotents of a semigroup S by E(S). If S is regular and $x \in S$, then V(x) denotes the set of all inverses of x in S, in the von Neumann sense. In an inverse semigroup, the unique inverse of an element x will be denoted by x^{-1}. The natural partial order in an inverse semigroup will be denoted by ≤. If (P,≤) is a poset then a subset Q of P will be called an *order ideal* if $p \leq q \in Q$ implies that $p \in Q$.

This paper concerns a class of extensions of an inverse semigroup S defined as follows. Let S be an inverse subsemigroup of an inverse semigroup T. Then T is said to be an *enlargement* of S if the following conditions hold:

(E1) S is an order ideal of T.

(E2) If $x \in T$ and $x^{-1}x, xx^{-1} \in E(S)$ then $x \in S$.

(E3) For each $e \in E(T)$ there exists $f \in E(S)$ such that $e\mathcal{D}f$.

Of course, we could make the definition more general and define an enlargement in terms of an embedding of S in an inverse semigroup T. We prefer, however, to avoid unnecessary notational complications. The following was proved in [4].

Proposition 1.1. *Let* S *be an inverse monoid. Then* T *is an enlargement of* S *if, and only if, there exists an idempotent* $e \in T$ *such that* T = TeT *and* S = eTe.■

The situation represented by the above result, namely that a semigroup T has an idempotent e such that T = TeT, is rather common in semigroup theory. Our interest in this is motivated by the connection between the theory of E-unitary covers over semilattices of S and enlargements of S. This connection is discussed in more detail in Section 3. The aim of this paper is now readily stated. Let S be a fixed inverse monoid with identity e. We shall show in this and the following section how to construct all enlargements of S. Almost all the tools needed to solve this problem were developed by McAlister in [5],[6] and [7]. In Section 4, we examine the results of this paper from a categorical point of view. In the remainder of this section, we prove some auxillary results.

An important role in the solution to our problem is played by a special class of homomorphisms. A homomorphism $\theta:S \to T$ is said to be a *local isomorphism* if the restrictions $(\theta|eSe)$ are injective for all $e \in E(S)$. The following was proved as Lemma 1.3 of [6].

Proposition 1.2. *Let* $\theta:S \to T$ *be a local isomorphism between regular semigroups. Then for all* $x,y \in S$, *the restrictions* $(\theta|xSy)$ *are injective.*■

Although we are only interested in inverse semigroups, we shall also need to work with a rather more general class of regular semigroups. A regular semigroup is said to be *orthodox* if the idempotents form a subsemigroup. We refer the reader to Chapter VI of [3] for a discussion of orthodox, regular semigroups. Let S be a regular semigroup. Then the *minimum inverse semigroup congruence* is denoted by γ. The following is proved as Theorem VI.1.12 of [3].

Proposition 1.3. *Let* S *be an orthodox semigroup. Then* $(x,y) \in \gamma$ *if, and only if,* V(x) = V(y).■

A regular semigroup S is said to be *locally inverse* if eSe is inverse for all idempotents $e \in S$. The following is proved as Proposition 1.4 of [6].

Proposition 1.4. *Let* S *be a regular semigroup. Then* $\gamma^{\natural}:S \to S/\gamma$ *is a local isomorphism if, and only if,* S *is locally inverse and orthodox.*■

Locally inverse orthodox semigroups are often called *generalised inverse semigroups*. Orthodox semigroups which admit local isomorphisms onto inverse semigroups are generalised inverse semigroups.

The following simple results are of fundamental importance for this paper.

Proposition 1.5. *Let* S *be regular,* T *inverse and* $\theta: S \to T$ *a surjective local isomorphism.*

(i) *For all* $x \in S$ *and* $x' \in V(x)$*, we have that* $\theta(x') = \theta(x)^{-1}$.

(ii) *For all* $x,y \in S$, $\theta(xyx) = \theta(x)$ *implies that* $xyx = x$.

(iii) *For all* $e \in E(T)$, $\theta^{-1}(e) \subseteq E(S)$.

Proof. (i) If $x' \in V(x)$, then $xx'x = x$ and $x'xx' = x'$, and so

$$\theta(x)\theta(x')\theta(x) = \theta(x) \text{ and } \theta(x')\theta(x)\theta(x') = \theta(x').$$

But T is inverse. Thus $\theta(x') = \theta(x)^{-1}$.

(ii) Let $x' \in V(x)$. Then

$$xx'(xyx)x'x = xyx \quad \text{and} \quad xx'.x.x'x = x,$$

so that $xyx, x \in xx'Sx'x$. But $\theta(xyx) = \theta(x)$. Thus by Proposition 1.2, we have that $xyx = x$.

(iii) Let $\theta(x) = e$. Then $\theta(x^2) = \theta(x)^2 = e^2 = e$. Let $x' \in V(x)$. Then

$$\theta(x) = \theta(x^2) = \theta(x(x'x)x).$$

THus by (ii) above, we have that $x^2 = x$.■

Theorem 1.6. *Let* S *be orthodox,* T *inverse, and* $\theta: S \to T$ *a surjective local isomorphism. Then* $\ker\theta = \gamma$.

Proof. Since γ is the minimum inverse semigroup congruence we have that $\gamma \subseteq \ker\theta$. We now show that $\ker\theta \subseteq \gamma$. Let $(x,y) \in \ker\theta$. Thus $\theta(x) = \theta(y)$. By Proposition 1.3, we need to show that $V(x) = V(y)$. Let $x' \in V(x)$. Then $\theta(yx'y) = \theta(y)\theta(x')\theta(y)$. But by Proposition 1.5(i), $\theta(x') = \theta(x)^{-1}$. Thus

$$\theta(yx'y) = \theta(y)\theta(x)^{-1}\theta(y) = \theta(y)\theta(y)^{-1}\theta(y),$$

since $\theta(x) = \theta(y)$. It follows that $\theta(yx'y) = \theta(y)$, and so by Proposition 1.5(ii), we have that $yx'y = y$. Similarly, $x'yx' = x'$. Thus $x' \in V(y)$ and so $V(x) \subseteq V(y)$. By symmetry, $V(y) \subseteq V(x)$. Hence $V(x) = V(y)$.■

The main result of this section is the following.

Theorem 1.7. *Let* S *be a locally inverse, orthodox semigroup. Then, up to isomorphism,* S *has a unique locally isomorphic inverse image.*

Proof. By Proposition 1.4, $\gamma^{\natural}: S \to S/\gamma$ is a locally isomorphic inverse image. On the other hand, if $\theta: S \to T$ is a surjective local isomorphism onto an inverse semigroup T, then $\ker\theta = \gamma$ by Theorem 1.6, and so T is isomorphic to S/γ.■

The following result will be needed in Section 3.

Proposition 1.8. *Let* $\theta: S \to T$ *be a surjective local isomorphism where* S *is orthodox and* T *is inverse. Then*

(i) *Let* $e \in E(S)$ *and* $\theta(e)\mathcal{L}t$. *Then there exists* $s \in S$ *such that* $s\mathcal{L}e$ *and* $\theta(s) = t$.

(ii) *Let* $e \in E(S)$ *and* $\theta(e)\mathcal{R}t$. *Then there exists* $s \in S$ *such that* $s\mathcal{R}e$ *and* $\theta(s) = t$.

Proof. We shall prove (i), the proof of (ii) is similar. Since θ is surjective, there exists $u \in S$ such that $\theta(u) = t$. Put $s = ue$. Then

$$\theta(s) = \theta(ue) = \theta(u)\theta(e) = t\theta(e) = t,$$

since $\theta(e)\mathcal{L}t$. Let $u' \in V(u)$. By Theorem 1.1(B), of Chapter VI of [3], we have that $eu' \in V(s)$. Thus $s\mathcal{L}eu'ue$. But

$$\theta(eu'ue) = \theta(e)\theta(u)^{-1}\theta(u)\theta(e) = \theta(e)t^{-1}t\theta(e) = \theta(e)\theta(e)\theta(e) = \theta(e).$$

By Proposition 1.5(ii), we have that $eu'ue = e$. Thus $s\mathcal{L}e$ and $\theta(s) = t$ as required.■

2. Normalised Rees matrix semigroups

Our main tool for constructing enlargements will be Rees matrix semigroups of a rather special kind. We begin by reviewing the classical definitions and properties we shall need. Let S be an (inverse) semigroup. Let I be a set and let $p: I \times I \to S$ be a function; as usual we write $p(i,j) = p_{i,j}$. Put $M = M(S,I,p) = I \times S \times I$ equipped with the multiplication given by

$$(i,s,j)(k,t,l) = (i,sp_{j,k}t,l).$$

Then M is called a *Rees matrix semigroup over* S. The function p is called the *sandwich function.* We put $R = RM(S,I,p)$ the set of all regular elements of M. The following is proved in [5].

Proposition 2.1. *Let* S *be an inverse semigroup. Then for any sandwich function* RM(S,I,p) *is a regular, locally inverse semigroup.*■

Let S be an inverse monoid with identity e. A sandwich function is said to be *normalised* if the following conditions hold:

(S1) $p_{i,i} \in E(S)$ for all $i \in I$.

(S2) For some $1 \in I$, $p_{1,1} = e$.

(S3) $p_{i,j} = (p_{j,i})^{-1}$ for all $i,j \in I$.

(S4) $p_{i,j}p_{j,k} \leq p_{i,k}$ for all $i,j,k \in I$.

The above conditions are special cases of conditions appearing in Lemma 2.2

of [6]. A Rees matrix semigroup $M = M(S,I,p)$ over an inverse monoid S with identity e is said to be *normalised* if p is a normal sandwich function.

Proposition 2.2. *Let* $M = M(S,I,p)$ *be a normalised Rees matrix semigroup. Then*

(i) *For all* $i,j \in I$, *we have that* $p_{i,i}p_{i,j} = p_{i,j}$ and $p_{i,j}p_{j,j} = p_{i,j}$.

(ii) $(i,s,j) \in E(M)$ *if, and only if,* $s \le p_{i,j}$.

(iii) R *is orthodox.*

(iv) $(i,s,j) \in R$ *if, and only if,* $s^{-1}s \le p_{j,j}$ *and* $ss^{-1} \le p_{i,i}$.

(v) *If* $(i,s,j) \in R$ *then*

$$V(i,s,j) = \{(k,p_{k,j}s^{-1}p_{i,l},l): ss^{-1} \le p_{i,l}p_{l,i} \text{ and } s^{-1}s \le p_{j,k}p_{k,j}\}$$

(vi) *If* $(i,s,j),(k,t,l) \in R$ *then*

$$(i,s,j)\mathcal{R}(k,t,l) \text{ if, and only if, } i = k \text{ and } s\mathcal{R}t.$$

(vii) *If* $(i,s,j),(k,t,l) \in R$ *then*

$$(i,s,j)\mathcal{L}(k,t,l) \text{ if, and only if, } j = l \text{ and } s\mathcal{L}t.$$

Proof. (i) We prove the first equality, the proof of the second is similar. By (S3) and (S4) we have that

$$p_{i,j}(p_{i,j})^{-1} = p_{i,j}p_{j,i} \le p_{i,i}.$$

The result is now immediate.

(ii) (i,s,j) is an idempotent if, and only if, $s = sp_{j,i}s$. But the latter implies that $ss^{-1} = sp_{j,i}ss^{-1}$. However $sp_{j,i}$ is an idempotent, and so we have that (i,s,j) is an idempotent if, and only if, $ss^{-1} = sp_{j,i}$. Thus $s^{-1} = s^{-1}sp_{j,i}$ and so $s = p_{i,j}s^{-1}s$, by (S3). But this is the case if, and only if, $s \le p_{i,j}$.

(iii) Let $(i,s,j),(k,t,l) \in E(M)$. Then by (ii) above,

$$s \le p_{i,j} \text{ and } t \le p_{k,l}.$$

Thus by (S4), we obtain $sp_{j,k}t \le p_{i,l}$. But then by (ii) again this implies that $(i,s,j)(k,t,l) = (i,sp_{j,k}t,l)$ is an idempotent.

(iv) Suppose that (i,s,j) is regular. Then there exists $(k,t,l) \in V(i,s,j)$. This implies that $s = sp_{j,k}tp_{l,i}s$. Now,

$$p_{j,j}s^{-1}s = p_{j,j}s^{-1}sp_{j,k}tp_{l,i}s = s^{-1}sp_{j,j}p_{j,k}tp_{l,i}s$$

since $p_{j,j}$ is an idempotent by (S1). But $p_{j,j}p_{j,k} = p_{j,k}$ by (i) above. Thus

$$p_{j,j}s^{-1}s = s^{-1}sp_{j,k}tp_{l,i}s = s^{-1}s,$$

and so $s^{-1}s \le p_{j,j}$. Similarly, $ss^{-1} \le p_{i,i}$. Conversely, suppose that (i,s,j) is such that $s^{-1}s \le p_{j,j}$ and $ss^{-1} \le p_{i,i}$. Consider the element (j,s^{-1},i). Then

$$(i,s,j)(j,s^{-1},i)(i,s,j) = (i,sp_{j,j}s^{-1}p_{i,i}s,j).$$

But $sp_{j,j}s^{-1}p_{i,i}s = s$ since $s^{-1}s \le p_{j,j}$ and $ss^{-1} \le p_{i,i}$. Thus (i,s,j) is regular.

(v) Let $(k,t,l) \in V(i,s,j)$. Then $s = sp_{j,k}tp_{l,i}s$ and $t = tp_{l,i}sp_{j,k}t$. Thus

$$sp_{j,k} = (sp_{j,k})(tp_{l,i})(sp_{j,k}) \text{ and } tp_{l,i} = (tp_{l,i})(sp_{j,k})(tp_{l,i}),$$

and so $tp_{l,i} \in V(sp_{j,k})$. From $s = sp_{j,k}tp_{l,i}s$ we obtain that

$$s^{-1}s \le p_{j,k}p_{k,j} \text{ and } ss^{-1} \le p_{i,l}p_{l,i}.$$

Similarly,

$$t^{-1}t \le p_{l,i}p_{i,l}.$$

Thus from $tp_{l,i} = (sp_{j,k})^{-1}$ we obtain $tp_{l,i}(p_{l,i})^{-1} = (p_{j,k})^{-1}s^{-1}(p_{l,i})^{-1}$, and so $t = (p_{j,k})^{-1}s^{-1}(p_{l,i})^{-1}$. The converse is easy to check.

(vi) Let $(i,s,j),(k,t,l) \in R$. Suppose that $i = k$ and that $s\mathcal{R}t$. By definition there exist elements a and b such that $sa = t$ and $tb = s$. By (iv) above, we have that $s^{-1}s \le p_{j,j}$ and $t^{-1}t \le p_{l,l}$. Thus

$$sp_{j,j}a = t \text{ and } tp_{l,l}b = s.$$

But then

$$(i,s,j)(j,p_{j,j}ap_{l,l},l) = (i,sp_{j,j}p_{j,j}ap_{l,l},l) = (i,t,l).$$

Similarly,

$$(i,t,l)(l,p_{l,l}bp_{j,j},j) = (i,s,j).$$

Furthermore, by (iv), both $(j,p_{j,j}ap_{l,l},l)$ and $(l,p_{l,l}bp_{j,j},j)$ are regular. The converse is straightforward to check.

(vii) Similar to the proof of (vi) above.■

The above result shows that semigroups of the form $RM(S,I,p)$ are locally inverse orthodox semigroups.

Proposition 2.3. *Let* $R = RM(S,I,p)$ *be a normalised Rees matrix semigroup over an inverse semigroup* S *with identity* e. *Put* $e = (1,e,1)$. *Then*

(i) $R = ReR$.

(ii) S *is isomorphic to* eRe.

Proof. (i) Observe that if $(i,s,j) \in R$ then $(i,s,1),(1,s^{-1}s,j) \in R$, and

$$(i,s,j) = (i,s,1)(1,e,1)(1,s^{-1}s,j).$$

(ii) Define a function $\theta:S \to eRe$ by $\theta(s) = (1,s,1)$. It is easy to check that θ is an isomorphism.■

Proposition 2.4. *Let* R *be a locally inverse, orthodox semigroup, and let* e *be an idempotent of* R. *Suppose that* $R = ReR$ *and* $S = eRe$. *Then* S *is isomorphic to* $\gamma(e)(R/\gamma)\gamma(e)$ *and* $R/\gamma = (R/\gamma)\gamma(e)(R/\gamma)$.

Proof. Observe first, that if $\theta:R \to T$ is any surjective homomorphism, then $T = T\theta(e)T$ and $\theta(eRe) = \theta(e)T\theta(e)$. By Proposition 1.4, $\gamma^{\natural}$ is a local isomorphism, since S is orthodox and locally inverse. This implies that eRe and $\gamma(e)(R/\gamma)\gamma(e)$ are isomorphic. The result is now immediate.■

By Proposition 2.3 and Proposition 2.4 we now have the following way of constructing a class of inverse enlargements of S.

Theorem 2.5. *Let* S *be an inverse semigroup with identity* e *and let* $R = RM(S,I,p)$ *be a normalised Rees matrix semigroup over* S. *Put* $e = (1,e,1)$. *Then* S *is isomorphic to* $\gamma(e)(R/\gamma)\gamma(e)$ *and* $(R/\gamma) = (R/\gamma)\gamma(e)(R/\gamma)$. *Thus* R/γ *is an inverse enlargement of a local submonoid isomorphic to* S.■

That every inverse enlargement of S is obtained as above is the substance of the following result.

Theorem 2.6. *Let* S *be an inverse monoid with identity* e *and let* T *be an inverse enlargement of* S *such that* $S = eTe$. *Then there exists a normalised Rees matrix semigroup* $R = RM(S,I,p)$ *over* S *such that* T *is isomorphic to* R/γ.

Proof. Put $I = E(T)$. For each idempotent $i \in E(T)$, pick an element $x_i \in T$ such that $(x_i)^{-1}x_i \leq e$ and $x_i(x_i)^{-1} = i$. If $i = e$, then we choose $x_i = e$. If $i,j \in I$, then define

$$p_{i,j} = (x_i)^{-1}x_j.$$

It is clear that $p_{i,j} \in S$. It is easy to check that p is a normalised sandwich function. We now prove the remainder of the theorem. By Theorem 2.4 [5], T is a locally isomorphic image of $R = RM(S,I,p)$. By Proposition 2.1 and Proposition 2.2, R is orthodox and locally inverse. Thus by Theorem 1.7, R has a unique locally isomorphic inverse image, and so by Theorem 1.6, T is isomorphic to R/γ.■

From Theorems 2.5 and 2.6, it is evident that in order to construct all inverse enlargements of an inverse monoid S it is necessary to construct all normalised sandwich functions $p:I \times I \to S$. We discuss such functions further at the end of Section 4.

3. Almost factorisable enlargements

In this section, we show how a class of E-unitary covers over semilattices of an inverse semigroup S can be constructed by means of normalised Rees matrix semigroups. We first recall some definitions.

Elements x and y in an inverse semigroup S are said to be *compatible*, written $x \sim y$, if xy^{-1} and $x^{-1}y$ are both idempotents. A subset A of S is said to be *compatible* if any two elements are compatible. A compatible

order ideal of S is said to be *permissible*. The set of all permissible subsets of S is denoted by C(S). It is well-known (see [9]) that C(S) is an inverse monoid under the usual product of subsets. The group of units of C(S) is denoted $\Sigma(S)$. An inverse semigroup S is said to be *almost factorisable* if for each $s \in S$ there exists $A \in \Sigma(S)$ such that $s \in A$ [4].

The *minimum group congruence* σ on an inverse semigroup is defined by $(x,y) \in \sigma$ if, and only if, $ex = ey$ for some idempotent e. An inverse semigroup is said to be *F-inverse* if each σ-class contains a maximum element. F-inverse semigroups are automatically monoids. An F-inverse semigroup P is said to be an *F-inverse cover* of an inverse monoid S if there exists a surjective, idempotent separating homomorphism $\theta:P \to S$. Although every inverse monoid has an F-inverse cover, it is not known whether every finite inverse monoid has a finite F-inverse cover. The importance of almost factorisable semigroups stems from the following result, proved in [4].

Proposition 3.1. *A finite inverse monoid* S *has a finite F-inverse cover if, and only if,* S *has a finite almost factorisable enlargement.*■

In view of the results of the previous section, it is natural to ask which normalised sandwich functions into S yield almost factorisable enlargements. In the remainder of this section, we obtain some results which may help in resolving this question.

Let R be an orthodox, locally inverse semgroup. We say that two elements $x,y \in R$ are *compatible*, written $x \sim y$, if, and only if, for all $x' \in V(x)$ and $y' \in V(y)$ both $x'y$ and xy' are idempotents. A subset A of R is said to be *compatible* if any pair of elements are compatible. A *permissible* subset is a compatible order ideal, where the order is the usual Nambooripad order on a regular semigroup [8]. A *global permissible* subset is a permissible subset A such that for each $e \in E(R)$ there exists $a,b \in A$ such that for some $a' \in V(a)$ and $b' \in V(b)$ we have that $e = a'a = bb'$. R is said to be *almost factorisable* if each element of R is contained in a global, permissible subset.

Proposition 3.2. *Let* $\theta: R \to S$ *be a surjective, local isomorphism, where* R *is orthodox and locally inverse, and* S *is inverse. Then*
(i) $x \sim y$ *in* R *if, and only if,* $\theta(x) \sim \theta(y)$ *in* S.
(ii) $x \sim y$ *if, and only if, for some* $x' \in V(x)$ *and for some* $y' \in V(y)$, $x'y$ *and* xy' *are idempotents.*
(iii) *If* A *is a global permissible subset of* R *then* $\theta(A) \in \Sigma(S)$.
(iv) *If* $B \in \Sigma(S)$ *then* $\theta^{-1}(B)$ *is a global permissible subset of* R.
(v) R *is almost factorisable if, and only if,* S *is almost factorisable.*

Proof. (i) Suppose that $x \sim y$ in R. Then $x'y$ and xy' are idempotents for any inverses x' and y'. Thus $\theta(x'y)$ and $\theta(xy')$ are idempotents. But by Proposition 1.5(i), $\theta(x'y) = \theta(x)^{-1}\theta(y)$ and $\theta(xy') = \theta(x)\theta(y)^{-1}$. Hence $\theta(x) \sim \theta(y)$. Conversely, suppose that $\theta(x) \sim \theta(y)$ and that $x' \in V(x)$ and $y' \in V(y)$. Then

$$\theta(x'y) = \theta(x)^{-1}\theta(y) \text{ and } \theta(xy') = \theta(x)\theta(y)^{-1},$$

are both idempotents. But then by Proposition 1.5(iii), both $x'y$ and xy' are idempotents.
(ii) Suppose $x'y$ and xy' are idempotents for some $x' \in V(x)$ and $y' \in V(y)$. Then $\theta(x) \sim \theta(y)$ and so by (i), $x \sim y$.
(iii) We show first that $\theta(A)$ is an order ideal of S. Let $s \leq \theta(r)$, where $r \in A$. By Theorem 1.8 of [8], there exists $t \in R$ such that $t \leq r$ and $\theta(t) = s$. But A is an order ideal, and so $t \in A$. Thus $s \in \theta(A)$. Next, we show that $\theta(A)$ is a compatible subset of S. Let $\theta(a),\theta(b) \in \theta(A)$ where $a,b \in A$. Since a and b are compatible, it follows from (i) that $\theta(a)$ and $\theta(b)$ are compatible. Finally, we show that $\theta(A)$ is global in S. Let $e \in E(S)$. By Lallement's Lemma (Lemma II.4.6 of [3]), there exists $f \in E(R)$ such that $\theta(f) = e$. But A is global in R and so there exist $a,b \in A$ and $a' \in V(a)$ and $b' \in V(b)$ such that $aa' = e = b'b$. But then, by Proposition 1.5(i), this implies that $\theta(A)$ is global.
(iv) We have to show that $A = \theta^{-1}(B)$ is a global, permissible subset of R. Firstly, A is an order ideal. Suppose that $c \leq b \in A$. Then $\theta(c) \leq \theta(b) \in B$. But B is an order ideal, and so $\theta(c) \in B$, thus $c \in A$. Next, A is a compatible subset of R. Let $a,b \in A$. Since $\theta(a),\theta(b) \in B$ they are compatible, and so by (i), $a \sim b$. Finally, A is global. Let $e \in E(R)$. Then $\theta(e) \in E(S)$. But B is global in S and so there exist $u,v \in B$ such that

$$u^{-1}u = \theta(e) = vv^{-1}.$$

By Proposition 1.8(i), there exists $r \in R$ such that $r \mathcal{L} e$ and $\theta(r) = u$. Similarly, there exists $t \in R$ such that $t \mathcal{R} e$ and $\theta(t) = v$. Thus $r,t \in A$ and there exists $r' \in V(r)$ and $t' \in V(t)$ such that $r'r = e = tt'$.
(v) Suppose that R is almost factorisable and $s \in S$. Then $\theta(t) = s$ for some $t \in R$. But, by assumption, there exists a global permissble subset A of R such that $t \in A$. But by (iii) above, $\theta(A) \in \Sigma(S)$ and $s \in \theta(A)$. Thus A is almost factorisable. Conversely, suppose that S is almost factorisable and almost factorisable. Conversely, suppose that S is almost factorisable, there exists $t \in R$. Then $\theta(t) \in S$ and, since S is almost factorisable, there exists $B \in \Sigma(S)$ such that $\theta(t) \in B$. By (iv) above, $\theta^{-1}(B)$ is a global permissible subset of R containing t. Thus R is almost factorisable.■

The proofs of the following are now immediate.

Theorem 3.4. *Let* S *be an inverse monoid.*
(i) *The inverse almost factorisable enlargements of* S *are isomorphic to the semigroups* $RM(S,I,p)/\gamma$, *where* $RM(S,I,p)$ *are almost factorisable, normalised, Rees matrix semigroups.*
(ii) *A finite inverse monoid* S *has a finite F-inverse cover if, and only if, there is a normalised sandwich function* $p: I \times I \to S$ *where* I *is finite and* $RM(S,I,p)$ *is almost factorisable.*■

4. A paper of Grandis

The aim of this section is to establish connections between the work of Section 2 and a paper by Grandis [2] on the axiomatics of local structures in differential geometry, as well as with Morita equivalence of categories [1]. We do not as yet understand the significance of these connections. We begin with some definitions from category theory.

In this paper, we are only interested in categories as algebraic structures, so that all our categories will be small. Furthermore, we shall treat categories as one-sorted structures, where the role of objects is played by the identities. We denote the set of identities of a category C by C_o. If $x \in C$ then $\mathbf{d}(x)$ is the right identity of x and $\mathbf{r}(x)$ is the left identity of x. The product xy is defined iff $\mathbf{d}(x) = \mathbf{r}(y)$. The function $C \to C_o \times C_o$ defined by $x \to [\mathbf{r}(x),\mathbf{d}(x)]$ is called the *anchor function.* If e and f are identities in a category C then the sets

$$\text{hom}(e,f) = \{x \in C: \mathbf{d}(x) = e \text{ and } \mathbf{r}(x) = f\},$$

are called *hom-sets* and the sets $\text{end}(e) = \text{hom}(e,e)$ are called *end-sets.* A category C is said to be *strongly connected* if all hom-sets are non-empty. Strongly connected categories give rise to semigroups in a natural way: let C be strongly connected and let $p: C_o \times C_o \to C$ be a cross-section of the anchor function. Define a product $\circ$ on C by

$$x \circ y = xp(\mathbf{d}(x),\mathbf{r}(y))y.$$

Then $(C,\circ)$ is a semigroup obtained by *consolidation* from the category C.

A category is said to be *regular* if for every morphism x there exists a morphism y, called an *inverse*, such that xyx and yxy are defined and x = xyx and y = yxy. A regular category in which each element has a unique inverse is said to be *inverse*. We denote the unique inverse of an element x in an inverse category by x′. The hom-sets of an inverse category come equipped with a natural partial order: if x and y belong to the same hom-set then $x \le y$ if, and only if, x = yx′x. Inverse categories enjoy similar properties to those of inverse monoids.

Proposition 4.1. *Let* M(S,I,p) *be a normalised Rees matrix semigroup over the inverse monoid* S. *Define a partial product on* R = RM(S,I,p) *by* (i,s,j).(k,t,l) = (i,st,l) *if* j = k. *Then this product is a restriction of the usual multiplication in* R, *and* (R,.) *is a strongly connected inverse category with set of identities the elements of the form* $(i,p_{i,i},i)$. *In addition, there exists an identity* $(1,p_{1,1},1)$ *in the category* R *whose endomorphism monoid is isomorphic to* S, *such that for each element* (i,s,j) *in* R *there exist elements* (i,u,1) *and* (1,v,j) *satisfying*

$$(i,s,j) = (i,u,1)(1,v,j).$$

Proof. The product $(i,s,j)(j,t,l) = (i,sp_{j,j}t,l)$ in the semigroup R. But by Proposition 2.2(iv), we have that $sp_{j,j} = s$. Thus the product '.' is the restriction of the semigroup product. Next we show that the elements of the form $(i,p_{i,i},i)$ are the identities of the product '.'. The product $(i,p_{i,i},i)(i,s,j)$ (resp. $(i,s,j)(j,p_{j,j},j)$) is just (i,s,j) by Proposition 2.2(iv). Thus the elements $(i,p_{i,i},i)$ are certainly identities, and it easy to see that all identities are of this form. It is now clear that (R,.) is a category. If $(i,s,j) \in R$ then $(j,s^{-1},i) \in R$ and

$$(i,s,j) = (i,s,j)(j,s^{-1},i)(i,s,j),$$

and

$$(j,s^{-1},i) = (j,s^{-1},i)(i,s,j)(j,s^{-1},i).$$

Thus (j,s^{-1},i) is an inverse of (i,s,j). Furthermore, it is easy to check that (j,s^{-1},i) is the unique inverse of (i,s,j). Thus (R,.) is an inverse category. It is strongly connected because of the existence of the elements $(i,p_{i,j},j)$. It is clear that the identity $(1,p_{1,1},1)$ has endomorphism monoid isomorphic to S. The last assertion follows from Proposition 2.3.■

We now follow Grandis [2] and define what is meant by a 'manifold' in an inverse category. Let $\mathcal{I}$ be an inverse category, I a set and $p^*: I \times I \to \mathcal{I}$ a function satisfying the following three conditions:

(M1) $p^*(i,i)$ is an identity of $\mathcal{I}$ for all $i \in I$.
(M2) $p^*(i,j)p^*(j,k) \le p^*(i,k)$ for all $i,j,k \in I$.
(M3) $p^*(i,j)' = p^*(j,i)$ for all $i,j \in I$.
Then p* is called a *manifold* in $\mathcal{I}$.

Proposition 4.2. *Let* R = RM(S,I,p) *be regarded as an inverse category. Define a function* $p^*: R_o \times R_o \to R$ *by*

$$p^*((i,p_{i,i},i),(j,p_{j,j},j)) = (i,p_{i,j},j).$$

Then p* *is a manifold and a cross section of the anchor function.*
Proof. It is clear from Proposition 4.1 that (M1) holds. To show that (M2) holds. Observe that

$$p^*(i,j)p^*(j,k) = (i,p_{i,j},j)(j,p_{j,k},k) = (i,p_{i,j}p_{j,k},k).$$

By (S4), we have that $p_{i,j}p_{j,k} \le p_{i,k}$. Thus

$$p_{i,j}p_{j,k} = p_{i,k}p_{k,j}p_{j,i}p_{i,j}p_{j,k}.$$

Now

$$(i,p_{i,k},k)(i,p_{i,j}p_{j,k},k)'(i,p_{i,j}p_{j,k},k) = (i,p_{i,k}p_{k,j}p_{j,i}p_{i,j}p_{j,k},k).$$

Thus

$$(i,p_{i,k},k)(i,p_{i,j}p_{j,k},k)'(i,p_{i,j}p_{j,k},k) = (i,p_{i,j}p_{j,k},k).$$

Hence

$$p^*(i,j)p^*(j,k) \le p^*(i,k).$$

Since $(i,p_{i,j},j)' = (j,p_{j,i},i)$ by (S3), it follows that (M3) holds.■

The next result will enable us to connect our work with Morita equivalence of categories.

Proposition 4.3. *Let* $\mathcal{I}$ *be an inverse category with distinguished identity* e. *Then the following two properties are equivalent*
(i) *For every identity* f *in* $\mathcal{I}$ *there exist elements* u *and* v *such that* f = uv *and* **d**(u) = **r**(v) = e.
(ii) *For every* x *in* $\mathcal{I}$ *there exist elements* u *and* v *such that* **d**(u) = **r**(v) = e *and* x = uv.
If (i) *or* (ii) *holds then* $\mathcal{I}$ *is strongly connected.*
Proof. It is clear that (ii) implies (i). Suppose that (i) holds. Let $x \in \mathcal{I}$, and suppose that **d**(x) = f and **r**(x) = f′. Then by (i), there exist elements u,v such that f = uv and **d**(u) = **r**(v) = e and elements w,z such that f′ = wz and **d**(w) = **r**(z) = e. Now x = f′xf = (wz)x(uv) = (wzxu)v where **d**(wzxu) = **r**(v) = e. Thus (ii) holds. Finally, to show that $\mathcal{I}$ is strongly connected. Let f,f′ be any identities in $\mathcal{I}$. By (i), there exist elements u,v,w and z such that f = uv and **d**(u) = **r**(v) = e and f′ = wz and

$d(w) = r(z) = e$. Thus wv is defined and $d(wv) = f$ and $r(wv) = f'$. Hence $hom(f,f')$ is non-empty.∎

We refer the reader to [1] for the definition of *Morita equivalence* of categories. It follows from the penultimate paragraph of [1] that a category $\mathcal{I}$ is morita equivalent to an endomorphism monoid end(e) if, and only if, condition (ii) (or equivalently (i)) in Proposition 4.3 holds.

We can now prove that all enlargements of an inverse monoid S are obtained by embedding S as an endomorphism monoid of suitable inverse categories.

Theorem 4.4. *Let $\mathcal{I}$ be an inverse category which is Morita equivalent to* $end(e) = S$ *for some identity* e. *Let* $p^*:\mathcal{I}_0 \times \mathcal{I}_0 \to \mathcal{I}$ *be a cross-section which is also a manifold in $\mathcal{I}$. Then $(\mathcal{I},\circ)$, the consolidation of $\mathcal{I}$ with respect to the cross-section p^*, is a locally inverse, orthodox regular semigroup such that* $S = e\circ\mathcal{I}\circ e$ *and* $\mathcal{I} = \mathcal{I}\circ e\circ\mathcal{I}$. *Hence $(\mathcal{I},\circ)/\gamma$ is an enlargement of* S, *and every enlargement of* S *is obtained in this way.*

Proof. Observe first that x is an idempotent in $(\mathcal{I},\circ)$ if, and only if, $x \le p^*(r(x),d(x))$. To see this, suppose that $x\circ x = x$. Then

$$xp^*(d(x),r(x))x = x.$$

Observe that $xp^*(d(x),r(x))$ is an idempotent in $end(r(x))$. We have that

$$x' = x'(xp^*(d(x),r(x)))xx',$$

and

$$xp^*(d(x),r(x)))xx' = xx'xp^*(d(x),r(x)) = xp^*(d(x),r(x)).$$

Thus $x' = x'xp^*(d(x),r(x))$ and so $x = p^*(d(x),r(x))'x'x$. Hence

$$x \le p^*(d(x),r(x))'.$$

The converse is straightforward to check. Now let x and y be idempotents in $(\mathcal{I},\circ)$. Then

$$x \le p^*(r(x),d(x)) \text{ and } y \le p^*(r(y),d(y)).$$

Thus

$$x\circ y = xp^*(d(x),r(y))y \le p^*(r(x),d(x))p^*(d(x),r(y))p^*(r(y),d(y)).$$

But

$$p^*(r(x),d(x))p^*(d(x),r(y))p^*(r(y),d(y)) \le p^*(r(x),d(y)).$$

Thus $x\circ y$ is an idempotent in $(\mathcal{I},\circ)$. $(\mathcal{I},\circ)$ is regular, for $x = xx'x = x\circ x'\circ x$. It is easy to check that $S = e\circ\mathcal{I}\circ e$. To show that $\mathcal{I} = \mathcal{I}\circ e\circ\mathcal{I}$, let $x \in \mathcal{I}$. Then $x = uev$ for some $u,v \in \mathcal{I}$. Thus $x = u\circ e\circ v$. It follows from this that $(\mathcal{I},\circ)$ is locally inverse, for any regular semigroup T, having an idempotent f such that fTf is inverse and satisfying T = TfT, must be locally inverse. By Proposition 2.4, we have that $(\mathcal{I},\circ)/\gamma$ is an

enlargement of S. The converse follows from Theorems 2.5 and 2.6 and Propositions 4.1 and 4.2.∎

The above result suggests that there is a connection between enlargements of an inverse monoid and Morita equivalence of categories. In particular, the theory of F-inverse covers may ultimately be related to Morita equivalence of categories.

To conclude this paper we show that normalised sandwich functions are in correspondence with a class of manifolds in a suitable category. Let S be an inverse category. The *idempotent completion* Sp(S) of S is the category with elements of the form (e,x,f) where $e,f \in E(S)$ and $exf = x$ and left and right identities defined by

$$r(e,x,f) = (e,e,e) \text{ and } d(e,x,f) = (f,f,f).$$

It is an inverse category in which $(e,x,f)' = (f,x^{-1},e)$.

Theorem 4.5. *Let* S *be an inverse monoid with identity* e.
(i) *Let* $p:I \times I \to S$ *be a normalised sandwich function. Define a function* $p^*:I \times I \to Sp(S)$ *by* $p^*(i,j) = (p_{i,i},p_{i,j},p_{j,j})$. *Then* p^* *is a manifold in* Sp(S) *such that* $p^*(1,1) = (e,e,e)$ *for some* $1 \in I$.
(ii) *Let* $p^*:I \times I \to Sp(S)$ *be a manifold in* Sp(S) *such that* $p^*(1,1) = (e,e,e)$ *for some* $1 \in I$. *Define a function* $p:I \times I \to Sp(S)$ *by* $p(i,j) = \pi_2 p^*(i,j)$, *where* π_2 *is the projection onto the second component. Then* p *is a normalised sandwich function. Furthermore, the function taking* p *to* p^* *is a bijection.*

Proof. (i) Observe first that p^* is a well-defined function by Proposition 2.2(i). (M1) holds by (S1). To show that (M2) holds we have to show that in the inverse category Sp(S) we have that

$$(p_{i,i},p_{i,j}p_{j,k},p_{k,k}) \le (p_{i,i},p_{i,k},p_{k,k}).$$

But

$$(p_{i,i},p_{i,k},p_{k,k})(p_{i,i},p_{i,j}p_{j,k},p_{k,k})'(p_{i,i},p_{i,j}p_{j,k},p_{k,k})$$

is just

$$(p_{i,i},p_{i,k}p_{k,j}p_{j,i}p_{i,j}p_{j,k},p_{k,k}).$$

But from the proof of Proposition 4.2, this is $(p_{i,i},p_{i,j}p_{j,k},p_{k,k})$. (M3) holds by (S3). (S2) implies that $p^*(1,1) = (e,e,e)$.
(ii) (S1) holds: $p(i,i) = \pi_2 p^*(i,i)$. But $p^*(i,i)$ is an idempotent in Sp(S) and so is of the form (f,f,f) where f is an idempotent in S. Thus p(i,i) is an idempotent in S. (S2) holds: by assumption, $p^*(1,1) = (e,e,e)$ and so $p(1,1) = e$. (S3) holds: $p(i,j) = \pi_2 p^*(i,j)$ and $p^*(i,j) = p^*(j,i)'$. But then from the form taken by inverses in Sp(S) we have that $p(i,j) = p(j,i)^{-1}$.

Finally, observe that if $(i,x,j) \leq (i,y,j)$ in Sp(S) then $x \leq y$ in S. Thus (S4) holds. The final part of the theorem is clear.∎

The above result suggests that we may obtain a better understanding of enlargements of inverse monoids by investigating the application of the categorical ideas on manifolds contained in [2].

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