Non-Commutative Stone duality

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October, 2011

Joint work with Daniel Lenz

Survey

Jean Renault, A groupoid approach to C^*-algebras, LNM 793, 1980.

"The notions of topological and of Lie groupoid were introduced by Ehresmann for applications to differential topology and geometry."
(from the Introduction)

" Together with the notion of a groupoid, the notion of inverse semi-group (sic) plays an important role in this work." (p9)

This books begins to examine the relationships between

this talk   future research

inverse semigroups $\longleftrightarrow$ topological groupoids $\longleftrightarrow$ C^*-algebras

Let G be a groupoid with set of identities G_0. A subset $A \subseteq G$ is called a <u>local bisection</u> if $A^{-1}A, AA^{-1} \subseteq G_0$. The local bisections form an inverse semigroup under subset multiplication.

<u>Example</u> Let X be a finite non-empty set. Let $X \times X$ be the universal groupoid on X. Identify the inverse semigroup of local bisections of $X \times X$. [New professors only]

If G has extra structure (topological) then one can restrict attention to suitable local bisections (compact-open).

On p 142, Renault effectively asks for the exact relationship between inverse semigroups and special kinds of topological groupoids (now called étale).

He notes that the inverse semigroups that arise have extra structure of a Boolean nature.

A.L.T. Paterson, Groupoids, inverse semigroups, and their operator algebras, Birkhäuser, 1999.

Brings Renault up to date.

Constructs the "universal groupoid of an inverse semigroup" by functional analytic means.

When the inverse semigroup is E-unitary the groupoid is Hausdorff.

Gives much greater weight to inverse semigroups.

Johannes Kellendonk, The local structure of tilings and their integer groups of CoInvariants, <u>Comm. Math. Phys.</u> <u>187</u> (1997), 115-157.

By accident, a seminal paper in the development of inverse semigroup theory.

"Tilings furnish (discrete) models of solids. These solids do not have to be periodic, as it is the case, for instance, for quasi-crystals."

He showed how to construct an inverse semigroup from a tiling — partial translational symmetry semigroup — then a topological groupoid and then a C^*-algebra, with a view to doing physics.

Daniel H. Lenz, On an order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.

Generalizes both Paterson and Kellendonk.
Dispenses with functional analysis.

Key idea reworked by Lawson, Margolis, Steinberg, The étale groupoid of an inverse semigroup as a groupoid of filters, arXiv.

$$\begin{tikzcd} \text{inverse semigroups} \arrow[r, bend left, "\text{filters}"] & \text{topological groupoids} \arrow[l, bend left, "\text{local bisections}"] \end{tikzcd}$$

Other work

Lawson, Steinberg, Ordered groupoids and étendues, Cahiers de Top. XLV (2004), 82-108. Ehresmann topologies on inverse semigroups as analogues of sites on category.

Steinberg, A groupoid approach to discrete inverse semigroup algebras, Adv. Math. 223 (2010), 689-727. "To study ample groupoids it is convenient to discuss generalized boolean algebra and Stone duality". [My emphasis]

Exel, Inverse semigroups and combinatorial C^*-algebras, Bull. Braz. Math. Soc. 39 (2008), 191-313.

"Tight representations", "tight spectrum".

Lawson, The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, <u>Comms in Algebra</u> <u>35</u> (2007), 4068–4087. Constructs <u>Completions</u> of the polycyclic monoids in which identities such as $p\bar{p}^{-1} + q\bar{q}^{-1} = 1$ are supposed to hold. Somehow related to Exel's tight representations.

How, if at all, is this other work related to the preceding work?

There is one further piece of evidence.

What is a space?

Take points as basic and you get a topological space.

Take open sets as basic and you get a frame.

A frame is a complete lattice that satisfies the infinite distributivity law.

There is an adjunction

$$\underline{\mathrm{Space}} \underset{\Phi}{\overset{\Omega}{\rightleftarrows}} \underline{\mathrm{Frame}}^{op}$$

$\Omega(X)$ the frame of open sets of a topological space

$\Phi(X)$ the topological space whose points are completely prime filters.

This adjunction can be used to deduce some classic dualities.

A Boolean algebra is not a frame but can be completed to one. A Stone space is a Hausdorff space with a basis of compact-open sets (= clopen).

Stone duality (1937)

The category of unital Boolean algebras is dually equivalent to the category of Stone spaces.

Johnstone ("Stone spaces" CUP) (xiv)

"... his main work had been in functional analysis ... [he was led] to the consideration of algebras of commuting projections in Hilbert space."

Where did frames come from

. . .

(p 76) " It was C. Ehresmann [1957] and his student J. Bénabou [1958] who first took the decisive step of regarding complete Heyting algebras as 'generalized topological spaces' in their own right".

Ch. Ehresmann, Gattungen von lokalen Strukturen, <u>Jber. Deutsch. Math.-Verein</u> <u>60</u> (1957), 59–77.

" This article is very important, since it contains most of the ideas which are developed in subsequent papers..."

Oeuvres Complètes, Partie II-1, p 358.

All of the preceding work can be unified by the following idea.

An abstract pseudogroup is an inverse monoid with all compatible joins and is infinitely distributive when this makes sense. Its semilattice of idempotents is a frame.

Non-commutative frame theory is the theory of abstract pseudogroups. It provides a setting for all of the work I have mentioned so far.

- Abstract pseudogroups were first defined by Ehresmann in his "Gattung" paper. The frame theorists 'forgot' about the inverse semigroups and only remembered their idempotents.

- Schein proved that every inverse semigroup could be completed to an abstract pseudogroup: Completions, translational hulls and ideal extensions of inverse semigroups, *Czech. J. Maths.* **23** (1973), 575-610.

- Resende has studied them, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147-209.

He goes in a different, but interesting, direction.

Sample theorem

A <u>Boolean inverse semigroup</u> is an inverse $\wedge$-semigroup which is distributive and whose semilattice of idempotents is a (generalized) Boolean algebra.

A <u>Boolean groupoid</u> is a hausdorff étale topological groupoid with a basis of compact-open local bisections whose space of identities is a boolean space.

<u>Étale topological groupoid</u> is a topological groupoid whose frame of open sets forms a monoid under subset multiplication.

Non-commutative Stone duality

The category of Boolean inverse semigroups is dual to the category of Boolean groupoids (for proper definitions of morphisms).

This theorem provides the setting, for example, to understand the relationship between the polycyclic inverse monoids, the Cuntz C^*-algebras and the Thompson group.

[We can do group theory as well!!]

In conclusion

How to win a Nobel prize

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BBC NEWS

SCIENCE & ENVIRONMENT

5 October 2011 Last updated at 15:42

Nobel win for crystal discovery

By Jennifer Carpenter
Science reporter, BBC News

The Nobel prize for chemistry has gone to a single researcher for his discovery of the structure of quasicrystals.

The new structural form was previously thought to be impossible and provoked controversy.

Daniel Shechtman, from Technion - Israel Institute of Technology in Haifa, will receive the entire 10m Swedish krona (£940,000) prize.

The Nobel prize in chemistry caps this year's science awards.

Professor David Phillips, president of the Royal Society of Chemistry, called quasicrystals "quite beautiful".

He added: "Quasicrystals are a fascinating aspect of chemical and material science - crystals that break all the rules of being a crystal at all."

Dr Shechtman had to fight a fierce battle against established science to convince others of what he had first seen in his lab at the National Institute of Standards and Technology in Washington - formerly called the National Bureau of Standards - on an April morning in 1982.

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From "Inverse semigroups"
World Scientific, 1998

Preface

Introduction

An appreciation of symmetry appears to be a feature of the human mind: even the earliest human artifacts were shaped in ways which transcended the purely utilitarian. The mathematical theory of symmetry grew out of investigations into the solutions of polynomial equations. Developing ideas of Lagrange and Abel, Galois discovered that the nature of these solutions could be determined by studying certain symmetries of the equation. It was from this work that the concept of a group as a mathematical device for measuring symmetry gradually evolved.

The theory of groups is one of the most successful branches of algebra with applications which range from error–correction in communication systems to the existence of elementary particles. But although group theory is certainly concerned with symmetry, it is by no means the case that the converse is true. This was highlighted in spectacular fashion in 1984, when Shechtman et al announced the discovery of a metallic phase whose diffraction pattern showed icosahedral symmetry, something which had long been assumed impossible on the basis of a result from group theory known as the 'crystallographic restriction'. The fault lay not in group theory, but in the assumptions underlying the translation of the intuitive idea of symmetry into a mathematical one. As in all translations, something was lost in the process.

In this book, we concentrate on just one aspect of our intuitive idea of symmetry which fails to be captured by groups: namely, the relationship between the parts and the whole. To see that group theory does not capture this relationship in a satisfactory way, one need only consider those natural forms, modelled by fractals, which are not highly symmetrical in the classical sense, but do possess self-similarity properties in which the whole is repeated at smaller scales. We instinctively feel that such forms possess symmetry but, due to their global irregularity, not of a kind which can be detected by groups. Self-similarities are examples of what we term partial symmetries, by which we mean symmetries between the parts of a structure. Partial symmetries are

The theory described in this talk
was motivated by quasi-crystals
and, in particular, the partial
translational symmetry inverse semigroups
— "tiling semigroups" — introduced
by Kellendonk in the 1990's.