THE ALGEBRAIC THEORY OF PSEUDOGROUPS AND ATLASES

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Pseudogroups

> "The notion of pseudogroups arose during a long process of trying to understand the foundations of geometry."

B. L. Reinhart *Differential geometry of foliations*, Springer-Verlag, 1983.

A *pseudogroup* Γ is a set of partial homeomorphisms of a topological space closed under composition of partial functions and inverses.

They were introduced by Lie in the 1880's.

Veblen and Whitehead in 1932 recognized in such structures suitable algebraic vehicles for generalizing the Erlanger Programme to the developing theory of differential manifolds.

Pseudogroups generalize groups of transformations: bijections are replaced by partial bijections — *symmetries* are replaced by *partial symmetries*.

Groups are axiomatizations of groups of transformations. Can the notion of pseudogroup be similarly axiomatized?

The answer is yes, in two different ways:

1. As special kinds of semigroups called *inverse semigroups*.

2. As special kinds of ordered groupoids called *inductive groupoids*.

Inverse semigroups

Replace the symmetric group $S(X)$ on X by the symmetric inverse monoid $I(X)$, the semigroup of all partial bijections on the set X.

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

We have the following Cayley-type theorem.

Theorem 1 [Wagner-Preston] Every inverse semigroup is isomorphic to an inverse subsemigroup of a symmetric inverse monoid.

Inverse semigroups not only have an algebraic character but also an order-theoretic one:

- Define $a \leq b$ iff $a = ba^{-1}a$. This gives an order on S that is a reflection of the inclusion order of partial bijections.

- The set of idempotents $E(S)$ of an inverse semigroup is always a meet-semilattice.

Inductive groupoids

Let $(G, \cdot)$ be a groupoid in the category theory sense. Elements are arrows

$$\mathbf{r}(x) \xleftarrow{x} \mathbf{d}(x).$$

Set of identities is G_o.

We say that a groupoid equipped with an order on the set of arrows is an *ordered groupoid* if the following axioms hold:

(OG1) If $x \leq y$ and $x' \leq y'$ and $\exists x \cdot x'$ and $\exists y \cdot y'$ then $x \cdot x' \leq y \cdot y'$.

(OG2) If $x \leq y$ then $x^{-1} \leq y^{-1}$.

(OG3) If $e \leq \mathbf{d}(x)$ then there exists a unique element $(x \mid e)$, the *restriction*, such that $(x \mid e) \leq x$ and $\mathbf{d}(x \mid e) = e$.

(OG4) If $e \leq \mathbf{r}(x)$ then there exists a unique element $(e \mid x)$, the *corestriction*, such that $(e \mid x) \leq x$ and $\mathbf{r}(e \mid x) = e$.

If $(G_o, \leq)$ is a meet-semilattice then the ordered groupoid is called an *inductive groupoid*.

These definitions are due to Ehresmann.

Let $(G, \cdot, \leq)$. Define $\otimes$ on G by

$$x \otimes y = (x \mid e) \cdot (e \mid y)$$

where

$$e = \mathbf{d}(x) \wedge \mathbf{r}(y).$$

Then $(G, \circ)$ is an inverse semigroup.

Theorem 2 [Ehresmann-Schein-Nambooripad] The category of inverse semigroups and their homomorphisms is isomorphic to the category of inductive groupoids and their morphisms.

Both Ehresmann and Wagner were differential geometers.

Differential manifolds

Pseudogroups are used with atlases to define differential manifolds (and other structures in differential geometry).

An *atlas* $\mathcal{A}$ from X to Y is, in particular, a set of partial bijections from X to Y.

Can the notion of atlas be axiomatized?

Yes — carried out by Wagner.

Requires not a *binary* operation as for pseudogroups but a *ternary* operation.

Generalized heaps

Let A be a set equipped with a ternary operation $\{-,-,-\}\colon A^3 \to A$. Then $(A, \{-,-,-\})$ is a *generalized heap* if the following axioms hold:

(GH1) $\{xxx\} = x$.

(GH2)

$$\begin{aligned} \{\{x_1 x_2 x_3\} x_4 x_5\} &= \{x_1 \{x_4 x_3 x_2\} x_5\} \\ &= \{x_1 x_2 \{x_3 x_4 x_5\}\}. \end{aligned}$$

(GH3) $\{xx\{yyz\}\} = \{yy\{xxz\}\}$.

(GH4) $\{\{zxx\}yy\} = \{\{zyy\}xx\}$.

Remarks

- If S is an inverse semigroup it becomes a generalized heap when one defines $\{xyz\} = xy^{-1}z$.

- In the group case, non-empty subsets closed under this ternary operation are precisely cosets. Work by Baer and Prüfer *Schar*, Certaine *abstract coset*, Kock *pregroups*.

- Related to later work by Kock in synthetic differential geometry: *pregroupoids, principal fibre bundles, torsors*.

- Theory of generalized heaps developed almost solely in the former USSR as a generalization of inverse semigroups. Wagner proved a Cayley-type theorem for them.

- Notion of generalized heap being studied again recently in the theory of algebras: Bertram and Kinyon.

- My goal is to show that generalized heaps arise naturally from the Morita theory of inverse semigroups.

Morita theory of inverse semigroups: motivation

- Inverse semigroups closely related to C^*-algebras: the partial isometries of any C^*-algebra form an ordered groupoid.

- Rieffel formulated the notion of an equivalence bimodule for C^*-algebras for defining strong Morita equivalence of C^*-algebras.

- Steinberg defined an analogous notion for inverse semigroups.

- Funk, Lawson and Steinberg developed the Morita theory of inverse semigroups: close connections with the theory of étendues

Morita theory of inverse semigroups: definition

Let S and T be inverse semigroups. A set X is an (S,T)-*biset* if

- There are actions $S \times X \to X$ and $X \times T \to X$.
- The actions are compatible.
- $SX = X$ and $XT = X$.

An *equivalence biset* from S and T consists of an (S,T)-biset equipped with surjective functions

$$\langle -,- \rangle \colon X \times X \to S \text{ and } [-,-] \colon X \times X \to T$$

such that the following axioms hold, where $x, y, z \in X$ and $s \in S$ and $t \in T$:

(MC1) $\langle sx, y\rangle = s\langle x, y\rangle$.

(MC2) $\langle y, x\rangle = \langle x, y\rangle^{-1}$.

(MC3) $\langle x, x\rangle x = x$.

(MC4) $[x, yt] = [x, y]t$.

(MC5) $[x, y] = [y, x]^{-1}$.

(MC6) $x[x, x] = x$.

(MC7) $\langle x, y\rangle z = x[y, z]$.

- If there is an equivalence biset from S to T then S and T are said to be *Morita equivalent*.

- It can be shown that this is the correct definition of Morita equivalence for inverse semigroups: in particular, it dovetails well with the Morita theory of topological groupoids and C^*-algebras.

- **Key observation**: given an equivalence biset, the definition on the set X of the ternary operation

$$\{xyz\} = \langle x, y \rangle z$$

gives rise to a generalized heap.

From generalized heaps to equivalence bisets: the problem

Given a generalized heap $(X, \{-,-,-\})$ we have to do the following:

- Manufacture two inverse semigroups S and T.
- Show that X is an (S,T)-biset.
- Construct suitable maps $\langle -,- \rangle \colon X \times X \to S$ and $[-,-] \colon X \times X \to T$.

The most difficult part turns out to be constructing the inverse semigroups S and T.

I shall therefore focus on how this problem was solved.

The inverse semigroups $X^{-1}X$ and XX^{-1}

Given a generalized heap $(X, \{-,-,-\})$, I shall explain how to construct inverse semigroups denoted by $XX^{-1} = S$ and $X^{-1}X = T$.

A semigroup B is said to be a *right normal band* if it satisfies the following two conditions:

(RNB1) $a^2 = a$ for all $a \in B$.

(RNB2) $abc = bac$.

Left normal bands are defined dually.

Lemma [Wagner]

1. Define the binary operation $\circ$ on X by
$$x \circ y = \{xxy\}.$$
Then $(X, \circ)$ is a right normal band.

2. Define the binary operation $\bullet$ on X by
$$x \bullet y = \{xyy\}.$$
Then $(X, \bullet)$ is a left normal band.

On any semigroup Green's relation $\mathcal{R}$ is the equivalence relation defined on pairs of elements that generate the same principal right ideal. The relation $\mathcal{L}$ is defined dually.

Lemma

1. On right normal bands the relation $\mathcal{R}$ is the minimum semilattice congruence.

2. On left normal bands the relation $\mathcal{L}$ is the minimum semilattice congruence.

We put $E = X^\circ/\mathcal{R}$ and $p\colon X^\circ \to E$, and $F = X^\bullet/\mathcal{L}$ and $q\colon X^\bullet \to F$ where E and F are semilattices.

Regard the set of ordered pairs in $\mathcal{R}$ as defining a groupoid.

On $\mathcal{R}$ define the following relation

$(x, y) \preceq (u, v) \Leftrightarrow x = x \bullet u,\ y = y \bullet v,\ y = \{xuv\}$.

This is a preorder.

Denote the equivalence class containing the pair (x, y) by $x^{-1}y$. Put $X^{-1}X$ equal to the set of equivalence classes.

Theorem [MVL] The set $X^{-1}X$ is an inverse semigroup with semilattice of idempotents isomorphic to the semilattice F when the product is defined by

$$x^{-1}y \otimes u^{-1}v = \{\{yuu\}yx\}^{-1}\{yuv\}.$$

□

Idea of proof

- An equivalence relation regarded as a groupoid is equipped with a suitable preorder.

- The equivalence relation corresponding to the preorder is used to construct a quotient groupoid.

- The quotient groupoid automatically has an order and becomes an ordered groupoid.

- This ordered groupoid is in fact inductive.

- One then uses the E-S-N Theorem to get an inverse semigroup.

- The full proof uses Kock's notion of a *pre-groupoid*.

Concluding remarks: Wagner was right

Pseudogroups of transformations $\longrightarrow$ inverse semigroups

Atlases $\longrightarrow$ Morita equivalence of inverse semigroups