Boolean inverse monoids and MV-algebra

Mark V Lawson
Heriot-Watt University, Edinburgh
July 2021
m.v.lawson@hw.ac.uk

In collaboration with Phil Scott (Ottawa)

My goal today

The goal of this talk is to show how two different generalizations of Boolean algebras are related to each other:

Boolean inverse monoids and MV-algebras.

MV stands for *multivalued logic*.

Terminology

I shall assume familiarity with basic (inverse) semigroup theory.

Define a and b to be *compatible*, denoted by $a \sim b$, if and only if $a^{-1}b$ and ab^{-1} are idempotents.

Recall that in an inverse semigroup S the elements a and b having an upper bound implies that a and b are compatible.

Being compatible is therefore a necessary condition for a pair of elements to have a join.

I shall use the following relations defined on any inverse semigroup.

Let e and f be idempotents. Then $e \mathscr{D} f$ if and only if there exists a such that $a^{-1}a = e$ and $aa^{-1} = f$.

$a \mathscr{J} b$ if and only if $SaS = SbS$. So, a and b generate the same principal ideal.

Let e and f be idempotents. Define $e \preceq f$ if and only if $e \mathscr{D} e' \leq f$ for some idempotent e'.

An inverse semigroup is said to be *Dedekind finite* if $e \mathscr{D} f \leq e$ implies that $e = f$. (In semigroup theory, the term 'completely semisimple' is usually employed; I avoid it because the term 'semisimple' is overused). In a Dedekind finite inverse semigroup, we have that $\mathscr{D} = \mathscr{J}$.

1. Boolean inverse monoids

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

This leads us to think of inverse semigroup theory from a lattice-theoretic perspective.

The following is for clarity. An inverse monoid is *Boolean* if it satisfies the following three conditions:

1. Compatible pairs of elements have all binary compatible joins: we write $a \vee b$ for the join of a and b.

2. Multiplication distributes over such joins: so $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$.

3. The set of idempotents $\mathsf{E}(S)$ of S forms a Boolean algebra with respect to the usual order.

Examples

1. Any group with an adjoined zero. We write these as G^0.

2. The symmetric inverse monoids $\mathcal{I}(X)$ are Boolean inverse monoids. If X has n letters, we usually write $\mathcal{I}_n$.

Boolean inverse monoids arise naturally as soon as you are interested in embedding inverse monoids in rings.

Theorem [Lawson-Paterson] *Let S be an inverse submonoid with zero of the multiplicative monoid of a ring R. Then there is a Boolean inverse submonoid S'' such that $S \subseteq S'' \subseteq R$.*

Every inverse monoid with zero gives rise to a Boolean inverse monoid.

Theorem [Booleanization/Lawson] *Let S be an inverse monoid with zero. Then there is a Boolean inverse monoid* $\mathsf{B}(S)$ *and an embedding $\beta \colon S \to \mathsf{B}(S)$ such that if $\theta \colon S \to T$ is an homomorphism to a Boolean inverse monoid then there is a unique morphism $\gamma \colon \mathsf{B}(S) \to T$ such that $\theta = \beta\gamma$.*

Recall that an *atom* in a partially ordered set (with zero) is an element a such that $b \leq a$ implies that $b = 0$ or $b = a$.

Theorem *Let S be a* finite *Boolean inverse monoid.*

1. *The set of atoms G of S forms a groupoid under the restricted product and $S \cong \mathsf{K}(G)$, the local bisections of G.*

2. *S is isomorphic to a finite direct product of finite symmetric inverse monoids if and only if S is fundamental. We call such monoids* matricial.

2. MV-algebras

MV-algebras are another generalization of Boolean algebras arising from multivalued logic.

See the book [CDM]: R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Springer, 2000.

An *MV-algebra* is a set A equipped with a binary operation $\oplus$, a unary operation $\neg$ and two constants 0 and 1 such that the following axioms hold:

1. $\oplus$ is associative.

2. $\oplus$ is commutative.

3. The identity is 0.

4. $\neg\neg x = x$.

5. $\neg 0 = 1$.

6. 1 is the zero.

7. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. This mysterious identity actually says that $x \vee y = y \vee x$. See below.

Examples

1. All Boolean algebras are MV-algebras.

2. The closed unit interval $[0,1]$ is an MV-algebra when we define $\neg x = 1 - x$ and $x \oplus y = \max\{1, x + y\}$.

3. For each $n \geq 1$, define

$$Ł_n = \left\{0, \frac{1}{n}, \ldots, \frac{(n-1)}{n}, 1\right\}$$

with the above operations. Then $Ł_n$ is a finite MV-algebra.

An element x of an MV-algebra is called an *idempotent* if and only if $x \oplus x = x$.

The set of idempotents in an MV-algebra forms a Boolean algebra (Corollary 1.5.4 of [CDM]).

The following was proved as Corollary 1.5.5 of [CDM].

Theorem *An MV-algebra is a Boolean algebra if and only if every element is idempotent.*

Thus, MV-algebras are non-idempotent generalizations of Boolean algebras, though they are still commutative. There is a very rich theory of MV-algebras.

The following generalizes the structural description of finite Boolean algebras as finite direct products of $\mathbb{B}$, the two-element Boolean algebra.

It was proved as Proposition 3.6.5 of [CDM]).

Theorem *Every finite MV-algebra is a finite direct product of MV-algebras of the form* Ł_n.

Define $x \leq y$ if and only if $x \oplus z = y$ for some z. Then $\leq$ is a partial order (Lemma 1.1.2 of [CDM]). The folllowing was proved as Proposition 1.1.5 and Proposition 1.5.1 of [CDM].

Proposition *Every MV-algebra is a distributive lattice with respect to the partial order $\leq$.*

It can be proved that

$$x \vee y = \neg(\neg x \oplus y) \oplus y$$

and

$$x \wedge y = \neg(\neg x \vee \neg y)$$

by Proposition 1.1.5 of [CDM].

The following will be useful later. It can also be proved that

$$x \oplus y = x \oplus (\neg x \wedge y)$$

where $x \leq \neg(\neg x \wedge y)$.

The basic question

- Boolean inverse monoids generalize Boolean algebras.
- MV-algebras generalize Boolean algebras.
- *This raises the question of how Boolean inverse monoids and MV-algebras are related.*

We shall show that the theory of MV-algebras is subservient to the theory of Boolean inverse monoids.

We shall refer to our paper

[LS]: M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, *Journal of Pure and Applied Algebra* **221** (2017), 45–74.

See also [W]: F. Wehrung, *Refinement monoids, equidivisibility types, and Boolean inverse semigroups*, Springer, lecture Notes in Mathematics 2188, 2017.

3. The partial interval monoid

How can we study Boolean inverse monoids?

We need invariants — abelian groups are usual but we shall be happy with abelian monoids.

The first such invariant was introduced in

[KLLR]: G. Kudryavtseva, M. V. Lawson, D. H. Lenz, P. Resende, Invariant means on Boolean inverse monoids, *Semigroup Forum* **92** (2016), 77–101.

(Aside: Other invariants are the homology groups associated with the Stone groupoid of the Boolean inverse monoid.)

Let S be a Boolean inverse monoid.

Denote the idempotent $\mathscr{D}$-class containing the idempotent e by $[e]$. Define $\mathsf{Int}(S)$ to be the set $\frac{\mathsf{E}(S)}{\mathscr{D}}$ equipped with the following *partially defined* operation

$$[e] + [f] = [e' \vee f']$$

if $e \mathscr{D} e'$ and $f \mathscr{D} f'$ and $e'f' = 0$ and undefined otherwise. We write $\exists [e] + [f]$ if the sum is defined. Put $\mathbf{0} = [0]$ and $\mathbf{1} = [1]$.

We call $(\mathsf{Int}(S), +, \mathbf{0}, \mathbf{1})$ the *partial interval monoid* associated with S. This terminology will now be explained.

A structure $(E, +, 0)$ is called a *partial commutative monoid* if it satisfies the axioms (E1), (E2) and (E3) below.

(E1) $\exists a + b$ if and only if $\exists b + a$ and when they exist they are equal.

(E2) $\exists (a+b)+c$ if and only if $\exists a+(b+c)$ and when they exist they are equal.

(E3) For all $a \in E$, $\exists a + 0$ and $a + 0 = a$.

Proposition *Let S be a Boolean inverse monoid. Then*

$$(\mathsf{Int}(S), +, \mathbf{0})$$

is a partial commutative monoid.

An *effect algebra* is a partial commutative monoid with an element 1 and a unary operation $a \mapsto a'$ that also satisfies axioms (E4) and (E5) below:

(E4) $\exists a + 1$ if and only if $a = 0$.

(E5) $\exists a + a'$ always, and $a + a' = 1$. We require that a' is the unique element such that $a + a' = 1$.

Observe that $0' = 1$.

(Effect algebras were developed independently by mathematical physicists studying quantum measurement theory and quantum effects.)

In an effect algebra, we may define a partial order by $a \leq b$ if and only if there exists c such that $a + c = b$.

If this partial order endows the effect algebra with a lattice structure, we speak of *lattice-ordered effect algebras*.

We need one further property that partial commutative monoids may possess:

(E6) *Refinement property*: if $a_1 + a_2 = b_1 + b_2$ then there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{21}$ and $a_2 = c_{12} + c_{22}$, and $b_1 = c_{11} + c_{12}$ and $b_2 = c_{21} + c_{22}$. (My thanks to JA for spotting the original typo).

Theorem *A lattice-ordered effect algebra that satisfies the refinement property is the same thing as an MV-algebra when we define*

$$a \oplus b = a + (a' \wedge b).$$

Proposition *Let S be a Boolean inverse monoid. Then*

$$(\mathsf{Int}(S), +, \mathbf{0}, \mathbf{1})$$

is a partial commutative monoid satisfying axiom (E4) and axiom (E6).

It is natural to ask when the partial interval monoid of S is an effect algebra.

An inverse monoid is said to be *factorizable* if each element is below an element in the group of units.

The following result shows how the inverse monoid structure and the Boolean algebra structure interact.

Lemma [Proposition 2.7 of [LS]] *Let S be a Boolean inverse monoid. Then S is factorizable if and only if $e \mathscr{D} f$ implies that $\bar{e} \mathscr{D} \bar{f}$.*

The following is now almost immediate.

Proposition *Let S be a Boolean inverse monoid. Then its partial interval monoid is an effect algebra if and only if S is factorizable where we define $[e]' = [\bar{e}]$.*

Lemma *Let S be a Boolean inverse monoid. Then $[e] \leq [f]$ if and only if $e \preceq f$ in S. If S is factorizable then $\leq$ is a partial order.*

4. Foulis inverse monoids

A *Foulis inverse monoid* is defined to be a factorizable Boolean inverse monoid S such that $S/\mathscr{J} = S/\mathscr{D}$ is a lattice under subset inclusion.

If S is a Foulis monoid, put Ł$(S) = \mathsf{Int}(S)$ equipped with the following everywhere defined operation $\oplus$:

$$[e] \oplus [f] = [e] + \Big([e]' \wedge [f]\Big)$$

where $[e] \wedge [f]$ is constructed using the lattice structure on $S/\mathscr{J}$.

The following is a nice introduction to our theory:

[WL]: Weiyun Lu, *Topics in many-valued and quantum algebraic logic*, MSc Thesis, University of Ottawa, 2016.

Theorem [Section 2.2 of [LS]] *Let S be a Foulis monoid. Then $(Ł(S), \oplus, ', \mathbf{0}, \mathbf{1})$ is an MV-algebra.*

An MV-algebra isomorphic to one of the form $Ł(S)$ where S is a Foulis monoid is said to be *coordinatizable*

The following result shows when our MV-algebra invariant is 'trivial'.

Lemma *Let S be a Foulis monoid. Then $Ł(S)$ is a Boolean algebra if and only if all idempotents in S are central.*

Recall that a Boolean inverse monoid is said to be *matricial* if it is isomorphic to a finite direct product of finite symmetric inverse monoids.

Matricial inverse monoids *are* Foulis monoids.

Theorem [Theorem 2.14 of [LS]] *The finite MV-algebra* $Ł_n$ *is coordinatized by* $\mathcal{I}_n$*, the finite symmetric inverse monoid on* n *letters. More generally, every finite MV-algebra is coordinatized by a matricial monoid.*

The following was first proved in [LS].

Theorem *Every countable MV-algebra is coordinatizable.*

The above theorem was generalized by Wehrung [W] as his Theorem 5.2.10 using different techniques.

Theorem *Every MV-algebra is coordinatizable.*

The above theorem therefore completely answers the question of the nature of the relationship between MV-algebras and Boolean inverse monoids.

5. AF inverse monoids

We shall sketch out our result on the coordinatization of countable MV-algebras.

We first need to be explicit about what kind of substructures we shall be dealing with.

Let S be a Boolean inverse monoid. Let T be an inverse submonoid of S which is a Boolean inverse monoid in its own right. We say that T is a *subalgebra* of S if $\mathsf{E}(T)$ is a Boolean subalgebra of $\mathsf{E}(S)$ and compatible joins in T are the same as compatible joins in S.

Let S be a countable Boolean inverse monoid. We say it is *AF (Approximately Finite)* if $S = \bigcup_{i=0}^{\infty} S_i$ where $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ and where each S_i is a matricial subalgebra of S. We assume that $S_0 = \mathbb{B}$, the two-element Boolean algebra. They are always factorizable.

This definition is modelled after the definition of *AF C^*-algebras* where finite direct products of finite symmetric inverse monoids replace finite dimensional C^*-algebras.

The theory of AF C^*-algebras was introduced in:

[B]: O. Bratteli, Inductive limits of finite dimensional C^*-algebras, *Transactions of the AMS* **171** (1972), 195–234.

The theory developed there relies a lot on matrix units which form inverse semigroups.

Bratteli diagrams give rise to AF inverse monoids and conversely.

In [DM]: D. Mundici, Logic of infinite quantum systems, *International Journal of Theoretical Physics* **32** (1993), 1941–1955, we have the following quote:

> ... AF C^*algebras should be regarded as sort of noncommutative Boolean algebras ...

We claim that the above quote actually applies more strongly to AF inverse monoids since commutative AF inverse monoids *are* Boolean algebras.

Theorem 3 from [DM] states that there is a bijection between AF C^*-algebras whose Murray-von Neumann order is a lattice and countable MV-algebras.

By modifying Mundici's proof we actually proved the following:

Theorem [LS] *Every countable MV-algebra can be coordinatized by an AF inverse monoid satisfying the lattice condition.*

END OF LECTURE