The polycyclic inverse monoids and the Thompson groups revisited

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1. Background

The starting point for this work were the following two papers:

- J.-C. Birget, The groups of Richard Thompson and complexity, *IJAC* **14** (2004), 569–626.

- M. V. Lawson, The polycyclic inverse monoids P_n and the Thompson groups $V_{n,1}$, *Comms. Alg.* **35** (2007), 4068–4087.

My paper has been substantially generalized in two steps. First, to 1-vertex higher rank graphs, and then to higher rank graphs with a finite number of vertices:

- M. V. Lawson, A. Vdovina, Higher dimensional generalizations of the Thompson groups, *AM* **369** (2020), 107191.

- M. V. Lawson, A. Sims, A. Vdovina, Higher dimensional generalizations of the Thompson groups, via higher rank graphs, In Preparation.

Accordingly, free monoids are generalized first to a class of cancellative monoids and then to a class of cancellative categories.

The goal of this talk is to return to my 2007 paper but develop the theory there in the light of my work with Sims and Vdovina.

This has been written up as the following paper:

The polycyclic inverse monoids and the Thompson groups revisited, arXiv:2006.15338.

2. Free monoids

Let $A_n = \{a_1, \ldots, a_n\}$ be a finite alphabet where $n \geq 2$. The set of all finite strings over A_n is denoted by A_n^*. This is a monoid under concatenation of strings with the empty string ε as the identity. In fact, A_n^* is the free monoid on A_n. The key property of the free monoid that we shall need is the following. This is an arithmétic property.

Lemma *The free monoid A_n^* is singly aligned. This means that for any strings x and y we have that $xA_n^* \cap yA_n^*$ is either empty or again a principal right ideal.*

Aside on prefix codes

Let u and v be strings. We say that they are prefix comparable if $uA_n^* \cap vA_n^* \neq \varnothing$, else they are prefix incomparable.

A finite subset X of A_n^* is called a prefix code if its elements are pairwise prefix incomparable.

A prefix code X is called a maximal prefix code if every element of A_n^* is prefix comparable with some element of X.

Important properties of the free monoid

- It is cancellative.

- It is conical, meaning that its group of units is trivial.

- The intersection of any two finitely generated right ideals is a finitely generated right ideal.

- Every finitely generated right ideal is generated by a prefix code.

3. The polycyclic inverse monoid P_n

Fix a free monoid A_n^*. Let x and y be any strings. Denote by xy^{-1} the partial bijection with domain yA_n^*, codomain xA_n^* which does the following: $yu \mapsto xu$.

Thus y^{-1} 'pops y', and x 'pushes x'.

The set of all such partial functions together with the empty partial function forms an inverse monoid called the polycyclic inverse monoid on n letters.

It was introduced by Nivat and Perrot in 1970.

4. The inverse monoid D_n

Let R_1 and R_2 be right ideals of A_n^*. A function $\theta \colon R_1 \to R_2$ is called a *morphism* if $\theta(rx) = \theta(r)x$ for all $r \in R_1$ and $x \in A_n^*$.

Define D_n to be the set of all bijective morphisms between the finitely generated right ideals of A_n^*.

Theorem *D_n is a distributive inverse $\wedge$-monoid and the distributive completion of P_n.*

We now unpack what this theorem says.

An inverse monoid is distributive if it has all binary compatible joins and multiplication distributes over such joins.

A morphism of distributive inverse monoids is a homomorphism that preserves compatible joins.

An inverse $\wedge$-monoid is an inverse monoid with all binary meets.

A distributive inverse monoid D is said to be the distributive completion of the inverse semigroup S if there is a monoid homomorphism $\delta\colon S \to D$ such that if $\alpha\colon S \to T$ is any homomorphism to a distributive inverse monoid T then there is a unique morphism $\beta\colon D \to T$ such that $\beta\delta = \alpha$.

5. How to get a group from D_n

Let S be any inverse monoid with zero.

A non-zero idempotent e in S is said to be essential if $ef \neq 0$ for all non-zero idempotents f.

An element a of S is said to be essential if both $a^{-1}a$ and aa^{-1} are essential.

Define S^e, the essential part of S, to consist of all essential elements of S. It is an inverse monoid (without zero).

Let T be any inverse semigroup. The congruence σ is defined on T by $a \, \sigma \, b$ if and only if there exists an element c such that $c \leq a, b$.

Then T/σ is a group and if ρ is any congruence on T such that T/ρ is a group then $\sigma \subseteq \rho$.

Thus σ is the *minimum group congruence* on T.

The following is now a reinterpretation of what Birget did.

Theorem *Let* $n \geq 2$. *Then* $G_n = D_n^e/\sigma$ *is the Thompson group* $G_{n,1}$ *or* $V_{n,1}$.

D_n^e is the set of all bijective morphisms between the finitely generated right ideals generated by maximal prefix codes.

The above construction can be generalized to finitely aligned small categories with only a finite number of identities. The issue is what one can say about the group. This requires extra structure, such as that provided by higher rank graphs.

6. The group G_n as a group of units

A distributive inverse monoid is said to be Boolean if its semilattice of idempotents is a Boolean algebra.

We shall now show how to realize the group G_n as a group of units of a Boolean inverse monoid.

This will enable us to connect the groups G_n with étale groupoids.

We shall need a definition from the following paper:

D. Lenz, An order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.

Let S be an inverse semigroup with zero. We shall define a congruence $\equiv$ on S which we call the Lenz congruence.

Define $\equiv$ on S as follows: $s \equiv t$ if and only if the following two conditions hold:

1. If $0 < x \leq s$ then there exists a non-zero element x' such that $x' \leq x, t$.

2. If $0 < y \leq t$ then there exists a non-zero element y' such that $y' \leq y, s$.

The relation $\equiv$ is a congruence and 0-restricted meaning that $a \equiv 0$ implies that $a = 0$.

How can we understand $\equiv$? Denote by V_s the set of ultrafilters in S that contain the element s.

Lemma *If* $\equiv$ *is idempotent-pure then* $s \equiv t$ *if and only if* $V_s = V_t$.

A congruence ρ is idempotent-pure if and only if $a \, \rho \, e$, where e is an idempotent, implies that a is an idempotent.

The above is rather abstract.

In the concrete example of the inverse monoid D_n, it can be proved that the congruence $\equiv$ is determined only by the fact that

$$a_1 a_1^{-1} \vee \ldots \vee a_n a_n^{-1} \equiv 1_{A_n^*}.$$

The idempotent on the lefthand side is the identity function on the set $A_n^* \setminus \{\varepsilon\}$.

The apparently tiny difference difference between the identity function defined on the set A_n^* and the identity function defined on the set $A_n^* \setminus \{\varepsilon\}$ drives the theory.

Our key theorem is the following.

Theorem *The inverse monoid $D_n/\equiv$ is a Boolean inverse monoid whose group of units is G_n.*

Put $C_n = D_n/\equiv$. We call it the Cuntz inverse monoid.

Under non-commutative Stone duality, the Boolean inverse monoid has a Stone groupoid $\mathsf{G}(C_n)$ whose topological full group is isomorphic to C_n.

From work of Matui, it is known that the group C_n and the étale groupoid $\mathsf{G}(C_n)$ determine each other up to isomorphism.

7. The group G_n as a group of automorphisms

We shall now use some ideas from the following papers:

- G. Higman, *Finitely presented infinite simple groups*, Notes on Pure Mathematics, No. 8 (1974). Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. vii+82 pp.

- R. Statman, Cartesian monoids, *Electronic Notes in Theoretical Computer Science* **265** (2010), 437–451.

We define a class of universal algebras.

An n-ary Cantor algebra is a structure $(X, \alpha_1, \ldots, \alpha_n, \lambda)$, where $\alpha_1, \ldots, \alpha_n$ are unary operations and λ is an n-ary operation, satisfying the following two laws:

CA1 $(x\alpha_1, \ldots, x\alpha_n)\lambda = x$ for all $x \in X$.

CA2 $(x_1, \ldots, x_n)\lambda\alpha_i = x_i$ where $1 \leq i \leq n$.

Given such an algebra, we may define a bijection $\beta \colon X \to X^n$ by $x\beta = (x\alpha_1, \ldots, x\alpha_n)$ for each $x \in X$. Conversely, every bijection from X to X^n defines an n-ary Cantor algebra.

Let M_n be the monoid (not inverse, but it is a restriction semigroup) constructed from the free monoid A_n^* consisting of all surjective morphisms between finitely generated right ideals together with the empty partial function. Thus $D_n \subseteq M_n$.

Each non-zero element of M_n can be regarded as a tree whose leaves are labelled by elements of the free monoid.

We may easily extend the definition of the Lenz congruence and we then define $\mathscr{C}_n = M_n / \equiv$.

Define $\mathscr{T}_n$ to be the submonoid of $\mathscr{C}_n$ consisting of total maps.

Each non-zero element of $\mathscr{T}_n$ can be represented by a tree which is a maximal prefix code whose leaves are labelled by elements of the free monoid.

Denote by ρ_a the function 'multiply on the right by a'. Denote by ρ_a^{-1} the partial function 'erase a on the right'.

On the set $\mathscr{T}_n$, define an n-ary operation λ by

$$(f_1, \ldots, f_n)\lambda = f_1\rho_{a_1}^{-1} \cup \ldots \cup f_n\rho_{a_n}^{-1}$$

and define n unary operations $\alpha_1, \ldots, \alpha_n$ by

$$(f)\alpha_i = f\rho_{a_i}.$$

Lemma *The universal algebra* $(\mathscr{T}_n, \alpha_1, \ldots, \alpha_n, \lambda)$ *is an* n*-ary Cantor algebra.*

Theorem *The* n*-ary Cantor algebra* $(\mathscr{T}_n, \alpha_1, \ldots, \alpha_n, \lambda)$ *is the free* n*-ary Cantor algebra on one generator.*

Theorem *The automorphism group of the* n*-ary Cantor algebra* $\mathscr{T}_n$ *is the Thompson group* G_n*.*

Our aim is to extend these results to a class of generalizations of the Thompson groups.

But the really interesting question is what is the connection between this work and logic?

Thank you!