AF inverse monoids

Mark V Lawson
Heriot-Watt University
and the
Maxwell Institute for Mathematical Sciences, UK

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In collaboration with Phil Scott (Ottawa)

The goal of this talk is to show how two different generalizations of Boolean algebras are related to each other:

Boolean inverse monoids and MV-algebras.

In this talk, we shall need to distinguish between *semigroups* and *monoids*, the latter being semigroups with an identity. Our semigroups will usually have zeros.

In a Boolean algebra, the complement of an element e will be denoted by $\bar{e}$.

1. Revision of inverse semigroups

Inverse semigroups originated as abstractions of pseudogroups of transformations.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that

$$a = aa^{-1}a \text{ and } a^{-1} = a^{-1}aa^{-1}.$$

Two key immediate examples:

1. Groups are the inverse semigroups with exactly one idempotent.

2. Meet semilattices are the inverse semigroups in which every element is idempotent.

Inverse semigroups come equipped with an internally defined order.

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation $\leq$ is a partial order on an inverse semigroup. In addition, if $a \leq b$ then $a^{-1} \leq b^{-1}$ and if also $c \leq d$ then $ac \leq bd$.*

This order is called the *natural partial order*.

Let S be an inverse semigroup. Elements of the form $a^{-1}a$ and aa^{-1} are idempotents.

Denote by $\mathsf{E}(S)$ the set of idempotents of S.

Remarks

1. $\mathsf{E}(S)$ is a commutative subsemigroup or *semilattice*.

2. $\mathsf{E}(S)$ is an order ideal of S.

Observation Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A subset is said to be *compatible* if each pair of distinct elements in the set are compatible.

Elements in inverse semigroups need to be compatible before they are even eligible to have a join.

An inverse semigroup is *fundamental* if the only elements that centralize all idempotents are themselves idempotents.

For more on general inverse semigroup theory, see my book
M. V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific 1998.

But if you are into that whole brevity thing, also see
M. V. Lawson, *Primer on inverse semigroups I*,
https://arxiv.org/abs/2006.01628.

2. Boolean inverse monoids

- An inverse semigroup is called a *meet-semigroup* if it has all binary meets.

- An inverse semigroup is said to have *finite joins* if each finite compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be a *Boolean* if it is distributive and its semilattice of idempotents is a Boolean algebra (with respect to the natural partial order).

Let X be a non-empty set. Denote by $\mathcal{I}(X)$ the set of all partial bijections of X. Then $\mathcal{I}(X)$ is a Boolean inverse monoid called *the symmetric inverse monoid*.

If X is finite with n elements, then we denote the corresponding symmetric inverse monoid by $\mathcal{I}_n$.

An inverse monoid isomorphic to a finite direct product of finite symmetric inverse monoids is said to be *matricial*. Matricial inverse monoids are Boolean.

Theorem [Paterson, Wehrung] *Let S be a submonoid of a ring with involution R such that S is an inverse monoid with respect to the involution. Then there is a Boolean inverse monoid T such that $S \subseteq T \subseteq R$.*

The above result is significant when viewing inverse monoids in relation to C^*-algebras.

Theorem [Lawson] *Every inverse monoid can be embedded in a universal Boolean inverse monoid.*

We view categories as 1-sorted structures: everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as 'groups with many identities'.

If G is a groupoid denote its set of identities by G_o.

A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Proposition *The set of all local bisections of a groupoid forms a Boolean inverse meet-monoid.*

Remarkably, the finite Boolean inverse monoids can be completely described.

Theorem [Lawson, Malandro]

1. *The finite Boolean inverse monoids are isomorphic to the inverse monoids of local bisections of finite discrete groupoids.*

   Compare with the structure theory of finite Boolean algebras: here, finite sets are replaced by finite groupoids.

2. *The finite fundamental Boolean inverse monoids are precisely the matricial inverse monoids.*

We regard Boolean inverse monoids as non-commutative Boolean algebras.

Stone's duality generalizes: Boolean inverse monoids are in duality with étale topological groupoids whose space of identities is a Boolean space. We call these *Boolean groupoids*.

Boolean inverse meet-monoids are in duality with Hausdorff Boolean groupoids.

For more on Boolean inverse monoids see: F. Wehrung, *Refinement monoids, equidivisibility types, and Boolean inverse semigroups*, Lecture Notes in Mathematics 2188, Springer, 2017.

3. MV-algebras

See the book [CDM] R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Springer, 2000.

MV-algebras are another generalization of Boolean algebras arising from many-valued logic.

An *MV-algebra* is a set A equipped with a binary operation $\oplus$, a unary operation $\neg$ and two constants 0 and 1 such that the following axioms hold:

1. $\oplus$ is associative.

2. $\oplus$ is commutative.

3. The identity is 0.

4. $\neg\neg x = x$.

5. $\neg 0 = 1$.

6. 1 is the zero.

7. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. This mysterious identity actually says that $x \vee y = y \vee x$. See below.

An element x of an MV-algebra is called an *idempotent* if and only if $x \oplus x = x$.

The set of idempotents in an MV-algebra forms a Boolean algebra (Corollary 1.5.4 of [CDM]).

Accordingly, an MV-algebra is a Boolean algebra if and only if every element is idempotent (Corollary 1.5.5 of [CDM]).

All Boolean algebras are MV-algebras.

The closed unit interval $[0,1]$ is an MV-algebra when we define $\neg x = 1 - x$ and $x \oplus y = \max\{1, x+y\}$.

For each $n \geq n$, define

$$Ł_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{(n-2)}{(n-1)}, 1\right\}$$

with the above operations. Then $Ł_n$ is a finite MV-algebra.

It is a theorem (Proposition 3.6.5 of [CDM]) that every finite MV-algebra is a finite direct product of MV-algebras of the form $Ł_n$.

Define $x \leq y$ if and only if $x \oplus z = y$ for some z. Then $\leq$ is a partial order (Lemma 1.1.2 of [CDM]) and gives the MV-algebra the structure of a distributive lattice (Proposition 1.1.5 and Proposition 1.5.1 of [CDM]).

It can be proved that

$$x \vee y = \neg(\neg x \oplus y) \oplus y$$

and

$$x \wedge y = \neg(\neg x \vee \neg y)$$

by Proposition 1.1.5 of [CDM].

It can also be proved that

$$x \oplus y = x \oplus (\neg x \wedge y)$$

where $x \leq \neg(\neg x \wedge y)$.

Boolean inverse monoids generalize Boolean algebras.

MV-algebras generalize Boolean algebras.

This raises the question of how Boolean inverse monoids and MV-algebras are related.

We shall show that the theory of MV-algebras is subservient to the theory of Boolean inverse monoids.

We shall refer to our paper [LS] M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, *Journal of Pure and Applied Algebra* **221** (2017), 45–74.

4. Foulis inverse monoids

An inverse monoid is said to be *factorizable* if each element is below an element in the group of units.

The following result shows how the inverse monoid structure and the Boolean algebra structure interact.

Lemma [Proposition 2.7 of [LS]] *Let S be a Boolean inverse monoid. Then S is factorizable if and only if $e \mathscr{D} f$ implies that $\bar{e} \, \mathscr{D} \, \bar{f}$.*

Factorizable Boolean inverse monoids are *Dedekind finite* or *completely semisimple* meaning that $e \, \mathscr{D} \, f$ and $e \leq f$ imply that $e = f$. In particular, $\mathscr{D} = \mathscr{J}$.

Let S be a factorizable Boolean inverse monoid. Denote an element of $\mathsf{E}(S)/\mathscr{D}$ by $[e]$. Put $\text{Ł}(S) = \mathsf{E}(S)/\mathscr{D}$.

Define a partial addition on $\text{Ł}(S)$ as follows: $[e] + [f] = [i \vee j]$ where $i \,\mathscr{D}\, e$ and $j \,\mathscr{D}\, f$ and $ij = 0$. Define $\neg[e] = [\bar{e}]$. Put $\mathbf{0} = [0]$ and $\mathbf{1} = [1]$.

Assume that S also satisfies the *lattice condition*: $S/\mathscr{J}$ is a (distributive) lattice. Define

$$[e] \oplus [f] = [e] + (\neg[e] \wedge [f])\,.$$

A *Foulis monoid* is a factorizable Boolean inverse monoid satisfying the lattice condition.

Theorem [Section 2.2 of [LS]] *Let S be a Foulis monoid. Then* $(Ł(S), \oplus, \neg, \mathbf{0}, \mathbf{1})$ *is an MV-algebra.*

An MV-algebra isomorphic to one of the form Ł(S) where S is a Foulis monoid is said to be *coordinatizable*.

Symmetric inverse monoids are factorizable if and only if they are finite.

Finite symmetric inverse monoids are Foulis monoids as are matricial inverse monoids.

Theorem [Theorem 2.14 of [LS]] *The finite MV-algebra* $Ł_{n+1}$ *is coordinatized by* $\mathcal{I}_n$. *More generally, every finite MV-algebra is coordinatized by a matricial monoid.*

The proof of the following is given after Theorem 4.10 of [LS].

Theorem *Every countable MV-algebra is coordinatizable.*

The above theorem was generalized by Wehrung as Theorem 5.2.10 using different techniques.

Theorem *Every MV-algebra is coordinatizable.*

The above theorem therefore completely answers the question of the nature of the relationship between MV-algebras and Boolean inverse monoids.

5. AF inverse monoids

We shall sketch out our result on the coordinatization of countable MV-algebras.

We first need to be explicit about what kind of substructures we shall be dealing with.

Let S be a Boolean inverse monoid. Let T be an inverse submonoid of S which is a Boolean inverse monoid in its own right. We say that T is a *subalgebra* of S if $\mathsf{E}(T)$ is a Boolean subalgebra of $\mathsf{E}(S)$ and compatible joins in T are the same as compatible joins in S.

Let S be a countable Boolean inverse monoid. We say it is *AF (Approximately Finite)* if $S = \bigcup_{i=1}^{\infty} S_i$ where $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$ and where each S_i is a matricial subalgebra of S. They are always factorizable.

This definition is modelled after the definition of *AF* C^**-algebras* where finite direct products of finite symmetric inverse monoids replace finite dimensional C^*-algebras.

The theory of AF C^*-algebras was introduced in: O. Bratteli, Inductive limits of finite dimensional C^*-algebras, *Transactions of the AMS* **171** (1972), 195–234.

The theory developed there relies a lot on matrix units which form inverse semigroups.

Bratteli diagrams give rise to AF inverse monoids and conversely.

In [DM] D. Mundici, Logic of infinite quantum systems, *International Journal of Theoretical Physics* **32** (1993), 1941–1955, we have the following quote:

> . . . AF C^*algebras should be regarded as sort of noncommutative Boolean algebras . . .

We claim that the above quote actually applies more strongly to AF inverse monoids since commutative AF inverse monoids *are* Boolean algebras.

Theorem 3 from [DM] states that there is a bijection between AF C^*-algebras whose Murray-von Neumann order is a lattice and countable MV-algebras.

By modifying Mundici's proof we actually proved the following:

Theorem [LS] *Every countable MV-algebra can be coordinatized by an AF inverse monoid satisfying the lattice condition.*

The following is a nice introduction to our theory: Weiyun Lu, *Topics in many-valued and quantum algebraic logic*, MSc Thesis, University of Ottawa, 2016.

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