NON-COMMUTATIVE STONE DUALITY

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March 2011

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0. Stone duality

We use the term *boolean algebra* to mean *generalized boolean algebra*.

If a boolean algebra has a top element we shall say that it is *unital*.

Marshall H. Stone's classical theorem is that the category of boolean algebras is dual to the category of *boolean spaces* — that is, hausdorff topological spaces with a basis of compact-open sets.

This theorem links algebra and order, in the guise of boolean algebras, with topology.

The aim of this talk is to show you how this theorem may be generalized and what that generalization may be used for.

1. Inverse semigroups

Inverse semigroups model partial symmetries in the same way that groups model symmetries.

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

Observe that $s^{-1}s$ and ss^{-1} are idempotents, and that $(s^{-1})^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1}$.

Set of idempotents of S denoted by $E(S)$ with the usual order $e \leq f$ iff $e = ef = fe$.

If e and f are idempotents then $e \, \mathcal{D} \, f$ if and only if there exists $a \in S$ such that $e = a^{-1}a$ and $f = aa^{-1}$. This is an equivalence relation which, to some extent, measures the complexity of S.

Examples

1. Groups are the inverse semigroups with a single idempotent.
2. Semilattices are the inverse semigroups in which every element is an idempotent.
3. The symmetric inverse monoid $I(X)$ of all partial bijections on a set X is the prototype for inverse semigroups.
4. Pseudogroups are special kinds of inverse semigroups, and motivated the definition.
5. A *bisection* A in a groupoid G is a subset $A \subseteq G$ such that $A^{-1}A, AA^{-1} \subseteq G_o$, the set of identities of G. The set of bisections of a groupoid forms an inverse semigroup under subset multiplication.

Inverse semigroups not only have an algebraic character but also an order-theoretic one.

- Define $a \leq b$ iff $a = ba^{-1}a$. This gives an order on S that is compatible with the multiplication. In addition, $s \leq t$ implies $s^{-1} \leq t^{-1}$ and the idempotents form an order ideal.

- The set of idempotents $E(S)$ of an inverse semigroup is always a meet-semilattice.

Think of the symmetric inverse monoid to understand what these definitions mean.

Inverse semigroups are algebraic versions of pseudogroups of transformations, and their semilattices of idempotents the last remnants of a topology.

Idea: Regard inverse semigroups as generalizations of semilattices.

2. Some history

Since the monograph of J. Renault, *A groupoid approach to C^*-algebras*, Springer-Verlag, 1980, it has been known that the following are closely related to each other

inverse semigroups, topological groupoids,
C^*-algebras.

Most attention has focussed on C^*-algebras and topological groupoids.

I want to focus instead on inverse semigroups and topological groupoids.

My interest in this question began with some papers of J. Kellendonk:

J. Kellendonk, The local structure of tilings and their integer group of coinvariants, *Commun. Math. Phys.* **187** (1997), 115–157.

J. Kellendonk, Topological equivalence of tilings, *J. Math. Phys.* **38** (1997), 1823–1842.

He showed how to associate an inverse semigroup with any tiling.

This inverse semigroup was the 'inverse semigroup of partial translational symmetries' of the tiling. Thus could say something about aperiodic tilings which have no global translational symmetries.

At that time, I was interested in the algebraic structure of these semigroups. They turned out to be prototypes of the class of *strongly E^*-unitary inverse semigroups*.

But there was a topological side: from an inverse semigroup one constructed a topological groupoid. What was the meaning of this construction? Buried under a thicket (!) of functional analysis it was hard to tell.

Renault's work was updated in A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, 1999. This had a much stronger algebraic flavour than Renault's book, but was still essentially functional analytic.

A breakthrough in understanding came with Daniel Lenz, An order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.

As a result of his work, Stuart Margolis and I realized that the topological groupoid associated with an inverse semigroup was constructed from filters.

This opened the way for the work to be described in this talk which is joint with Daniel Lenz.

It also adapts ideas due to Exel who has been pioneering the connections between inverse semigroups and C^*-algebras.

3. Motivations

1. How do we build groups? The usual answer to this question would be ‘by symmetries’, but this isn’t always the case. The Thompson groups $V_{n,r}$, for example, are built from combinatorial information, in the first instance.

Our answer to this question is: groups are built by glueing together partial bijections. This suggests that inverse semigroups could be used to build groups — if we knew what ‘glueing’ meant.

I will return to this idea towards the end.

2. What information do inverse semigroups contain? Information about the partial symmetries of a structure. But how can we extract information from these partial symmetries?

Idea: Transform the data contained in the inverse semigroup into topological form and then use the well-developed tools available there.

3. Why should C^*-algebras have anything to do with inverse semigroups? Let $PI(A)$ be the set of partial isometries of a C^*-algebra A. Define a partial binary operation $s \cdot t$ if and only if $s^*s = tt^*$. Then $(PI(A), \cdot)$ is a groupoid.

Define $s \leq t$ by $s = ss^*t$ and $ss^* = ss^*tt^*$. Then $(PI(A), \cdot, \leq)$ is an 'ordered groupoid'.

The ordering on the projections is given by $e \leq f$ if and only if $e = ef$. Suppose that this is a $\wedge$-semilattice. Define

$$s \circ t = (se)(et) \text{ where } e = s^*s \wedge tt^*.$$

Then $(PI(A), \circ)$ is an inverse semigroup.

Question What interesting examples of C^*-algebras are there where the order on the projections above is a $\wedge$-semilattice ordering?

4. Inverse semigroup definitions

An *inverse* $\wedge$*-semigroup* is an inverse semigroup S in which for all $s, t \in S$ we have that $s \wedge t$ exists. The theory described in this talk applies to this class of inverse semigroups — but it can be generalized.

Elements s and t in an inverse semigroup are said to be *compatible*, written $s \sim t$, if $s^{-1}t$ and st^{-1} are both idempotents.

In order that $s \vee t$ exist it is necessary that $s \sim t$.

If an inverse semigroup has joins of all finite compatible subsets it is said to be *finitely complete*.

An inverse semigroup is said to be *distributive* if the following holds. Let $\{a_1, \ldots, a_m\}$ be a finite subset of S and let $a \in S$ be any element. If $\bigvee_{i=1}^{m} a_i$ exists then both $\bigvee_{i=1}^{m} aa_i$ and $\bigvee_{i=1}^{m} a_i a$ exist and we have the following two equalities

$$a\left(\bigvee_{i=1}^{m} a_i\right) = \bigvee_{i=1}^{m} aa_i \text{ and } \left(\bigvee_{i=1}^{m} a_i\right)a = \bigvee_{i=1}^{m} a_i a.$$

An inverse $\wedge$-semigroup is said to be *boolean* if it is finitely complete and distributive and has a boolean algebra of idempotents.

Symmetric inverse monoids are boolean.

The following two definitions combine idea due to Lenz and Exel.

Let a be an element of an inverse semigroup S and $a^{\downarrow}$ the principal order ideal it generates. A subset $\{b_1, \ldots, b_m\} \subseteq a^{\downarrow}$ is said to be a *cover* of a if for each $0 \neq a' \leq a$ there exists b_i such that $a' \wedge b_i \neq 0$.

Intuitively, covers are replacements for joins.

A homomorphism $\theta \colon S \to T$ to a finitely complete distributive inverse semigroup is said to be a *cover-to-join* map if whenever $\{b_1, \ldots, b_m\}$ is a cover of a then $\theta(a) = \bigvee_{i=1}^{m} \theta(b_i)$.

5. The theorems

Completion theorem Let S be an inverse $\wedge$-semigroup. Then there is a finitely complete distributive inverse $\wedge$-semigroup $\mathsf{D}(S)$ and a homomorphism $\delta\colon S \to \mathsf{D}(S)$ which is a cover-to-join map which has the following universal property: for every cover-to-join map $\theta\colon S \to T$ to a finitely complete distributive inverse semigroup there is a unique join-preserving homomorphism $\bar{\theta}\colon \mathsf{D}(S) \to T$ such that $\bar{\theta}\delta = \theta$.

If $\mathsf{D}(S)$ is boolean then we say that S is *weakly boolean* and we say that $\mathsf{D}(S)$ is the *boolean completion* of S.

When an inverse semigroup is weakly boolean is determined by its idempotents.

Filters, prime filters and ultrafilters can all be suitably defined in inverse $\wedge$-semigroups.

Booleanization theorem Let E be a meet semilattice with zero. Then E is weakly boolean if and only if every prime filter of E is an ultrafilter.

If D(S) is a boolean *monoid* then S is said to be *compactable*.

We say that a groupoid is *boolean* if it is a hausdorff, étale, topological groupoid with a basis of compact-open bisections whose space of identities is a boolean space.

- With each boolean groupoid G, we may associate a boolean inverse semigroup $\mathsf{B}(G)$ of its compact-open bisections.
- With each weakly boolean inverse semigroup S, we may associate a boolean groupoid $\mathsf{G}(S)$ constructed from the ultrafilters of S.

Comparison theorem Let S be a weakly boolean inverse semigroup. Then the asssociated topological groupoid G(S) is boolean, and B(G(S)) is isomorphic to D(S).

Duality theorem [Generalization of Stone Duality] There is a duality between the category of boolean inverse semigroups and the category of boolean groupoids.

Summary

S weakly boolean inverse semigroup
D(S) boolean inverse semigroup
G(S) boolean groupoid
B compact-open bisections of
G ultrafilters of

$$\begin{tikzcd}
S \arrow[r] \arrow[d, dashed] & \mathsf{D}(S) \arrow[dl, shift left, "\mathsf{G}"] \\
\mathsf{G}(S) \arrow[ur, shift left, "\mathsf{B}"] &
\end{tikzcd}$$

6. Examples

(i) Basic example

The finite symmetric inverse monoid $I(X)$. This is already a boolean inverse monoid. Its associated groupoid is the set $X \times X$ with the usual groupoid multiplication equipped with the discrete topology.

(ii) The polycyclic inverse monoids

The *polycyclic monoid* P_n, where $n \geq 2$, is defined as a monoid with zero generated by the variables $a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$ subject to the relations

$$a_i^{-1} a_i = 1 \,\text{and}\, a_i^{-1} a_j = 0, i \neq j.$$

Every non-zero element of P_n is of the form yx^{-1} where x and y are elements of the *free monoid* on $\{a_1, \ldots, a_n\}$.

The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable in which case

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some } z \end{cases}$$

The polycyclic monoid P_n is a weakly boolean inverse monoid.

Theorem The boolean completion of P_n is called (here) the *Cuntz inverse monoid* C_n.

This monoid is congruence-free and its group of units is the Thompson group $V_{n,1}$.

It can be described as the freest inverse semigroup satisfying the following conditions:

- It is finitely complete and distributive.
- It contains a copy of P_n and every element of C_n is the join of a finite subset of P_n.
- $1 = \bigvee_{i=1}^{n} a_i a_i^{-1}$.

O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, (1999) is, in fact, a study of cover-to-join maps from P_n to $I(X)$.

(iii) Graph inverse semigroups
These generalize polycyclic inverse monoids.

The free monoid is replaced by the free category on a directed graph G (eventually with finiteness conditions imposed).

The inverse semigroup P_G which arises is called the *graph inverse semigroup*.

Let S be an inverse semigroup.

- Then S is *combinatorial* if all subgroups are trivial.

- It is *unambiguous* if for all idempotents e and f such that $ef \neq 0$ then either $e \leq f$ or $f \leq e$.

- It satisfies the *dedekind height property* if $e^{\uparrow} \cap E(S)$ is finite for all non-zero idempotents e.

- It is 0-*bisimple* if for any two non-zero idempotents e and f there is an element s such that $e = s^{-1}s$ and $f = ss^{-1}$.

Theorem (Jones and Lawson) Graph inverse semigroups are precisely the inverse semigroups which are combinatorial, unambiguous, satisfy the dedekind height condition, have maximal idempotents, and for which each $\mathcal{D}$-class contains a unique maximal idempotent.

The polycyclic inverse monoids are the graph inverse semigroups which are 0-bisimple.

Theorem The graph inverse semigroup P_G is weakly boolean when the graph G has the property that the in-degree of each vertex is finite and at least 2. The boolean completion of such a graph inverse semigroup is called (here) the *Cuntz-Krieger inverse semigroup* C_G.

When in addition, the graph G is strongly connected, the Cuntz-Krieger semigroup is congruence-free.

The Cuntz-Krieger semigroup is the freest inverse semigroup satisfying the following conditions:

- It is finitely complete and distributive.
- It contains a copy of P_G and every element of C_G is the join of a finite subset of P_G.
- $e = \bigvee_{f' \in \widehat{e}} f'$ for each maximal idempotent e of P_G where $\widehat{e}$ is the set of idempotents immediately below e.

Unambiguous inverse semigroups are closely connected to the theory of ultrametric spaces. The following papers take a topological groupoid and C^*-algebra point of view.

B. Hughes, Trees and ultrametric spaces: a categorical equivalence, *Adv. Math.* **189** (2004), 148–191.

B. Hughes, Trees, ultrametrics and noncommutative geometry, arXiv:math/0605131.

B. Hughes, Local similarities and the Haagerup property, *Groups Geom. Dyn.* **3** (2009).

Question The compactable case of the above theorem arises when the graph has a finite number of vertices. In that case, the Cuntz-Krieger inverse semigroup C_G is a monoid with a group of units. What are these groups? The Thompson groups $V_{n,1}$ are examples.

More generally, we see that compactable inverse semigroups give rise to boolean monoids with potentially interesting groups of units.

In this way, we construct groups from inverse semigroups by glueing together elements from an inverse semigroup.