PSEUDOGROUPS

Mark V Lawson
Heriot-Watt University, Edinburgh
July 2018

Celebrating the LXVth birthday of
Mária Szendrei

With the collaboration of

Peter Hines
Ganna Kudryavtseva
Johannes Kellendonk
Daniel Lenz
Stuart Margolis
Pedro Resende
Phil Scott
Benjamin Steinberg
Alistair Wallis.

Pseudogroups of transformations

Let X be a topological space. A *pseudogroup of transformations on* X is a collection Γ of homeomorphisms between the open subsets of X (called *partial homeomorphisms*) such that

1. Γ is closed under composition.

2. Γ is closed under 'inverses'.

3. Γ contains all the identity functions on the open subsets.

4. Γ is closed under arbitrary non-empty unions when those unions are partial bijections.

Inverse semigroups arose by abstracting pseudogroups of transformations in the same way that groups arose by abstracting groups of transformations.

There were three independent approaches:

1. Charles Ehresmann (1905–1979) in France.

2. Gordon B. Preston (1925–2015) in the UK.

3. Viktor V. Wagner (1908–1981) in the USSR.

They all three converge on the definition of ‘inverse semigroup’.

An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

Definition. An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

Theorem [Schein completion] *Let S be an inverse semigroup. Then there is a pseudogroup $\Gamma(S)$ and a map $\gamma\colon S \to \Gamma(S)$ universal for maps to pseudogroups.*

BUT there was no impact on the theory of pseudogroups.

Possibly, this was due to the lack of an étale groupoid associated with a pseudogroup.

This has now been rectified.

See

P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

M. V. Lawson, D. H. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170.

This leads to the following perspective on inverse semigroup theory.

Let S be a pseudogroup. An element $a \in S$ is said to be *finite* if $a \leq \bigvee_{i \in I} a_i$ implies that $a \leq \bigvee_{i=1}^{n} a_i$ for some finite subset $\{1, \ldots, n\} \subseteq I$.

Denote by $\mathsf{K}(S)$ the set of finite elements of S.

An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

A pseudogroup S is said to be *coherent* if each element of S is a join of finite elements and the set of finite elements forms a distributive inverse semigroup.

Theorem *The category of distributive inverse semigroups is equivalent to the category of coherent pseudogroups.*

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

Commutative	**Non-commutative**
meet semilattice	inverse semigroup
frame	pseudogroup
distributive lattice	distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup

For the remainder of this talk, I will focus on surveying the theory of Boolean inverse semigroups, currently the most interesting class of pseudogroups.

Ring-like features

The binary join in a Boolean inverse semigroup is called *addition* (adjective: *additive*).

An *additive ideal* is an ideal that is closed under any compatible binary joins. A Boolean inverse semigroup with no, non-trivial additive ideals is said to be 0-*simplifying*.

A *simple* Boolean inverse semigroup is one that is 0-simplifying **and** fundamental.

- The *generalized rook matrix semigroup* $R_n(S)$ over a Boolean inverse semigroup S consists of all $n \times n$ matrices with a finite number of non-zero elements where elements in the same row have orthogonal ranges, and elements the same column have orthogonal domains.

- Tensor products of Boolean inverse semigroups can be defined.

- A Morita theory of pseudogroups, and so Boolean inverse semigroups, is being developed.

- There is a non-commutative Stone duality with étale groupoids whose identity spaces are locally compact Boolean; monoids correspond to the compact case; Hausdorff corresponds to the $\wedge$-semigroup case.

- There is also a non-commutative version of the Loomis-Sikorski theorem. See *Soficity and other dynamical aspects of groupoids and inverse semigroups* by Luiz Gustavo Cordeiro, PhD Thesis, University of Ottawa, 2018.

Finite Boolean inverse semigroups

Theorem *Let S be a finite Boolean inverse semigroup.*

1. *S is isomorphic to a finite direct product $R_{n_1}(G_1^0) \times \ldots \times R_{n_r}(G_r^0)$ of generalized rook matrix semigroups over finite groups with zero adjoined.*

2. *If S is fundamental then S is isomorphic to a finite direct product $I_{n_1} \times \ldots \times I_{n_r}$ of finite symmetric inverse monoids.*

3. *If S is simple then S is isomorphic to a finite symmetric inverse monoid.*

Group connections

For any element a, the idempotent $a^{-1}a \vee aa^{-1}$ is called the *extent* of a and is denoted by $\mathbf{e}(a)$.

A non-zero element a in an inverse semigroup is said to be an *infinitesimal* if $a^2 = 0$. The following result explains why infinitesimals are important.

Proposition *Let S be a Boolean inverse monoid and let a be an infinitesimal. Then*

$$a \vee a^{-1} \vee \overline{\mathbf{e}(a)}$$

is an involution.

A Boolean inverse monoid is *piecewise factorizable* if each element s can be written

$$s = \bigvee_{i=1}^{m} g_i e_i$$

where g_i are units and e_i idempotents.

Proposition *Every countably infinite atomless 0-simplifying Boolean inverse $\wedge$-monoid is piecewise factorizable.*

Theorem [Matui's spatial realization theorem] *Two countably infinite atomless simple Boolean inverse $\wedge$-monoids are isomorphic if and only if their groups of units are isomorphic.*

Constructing Boolean inverse semigroups

Theorem *Let S be an inverse semigroup with zero that is a subsemigroup of the multiplicative semigroup of a ring R. Then there is a Boolean inverse subsemigroup S'' of R such that $S \subseteq S''$.*

Theorem [Booleanization] *Let S be an inverse semigroup with zero. Then there is a Boolean inverse semigroup* $\mathsf{B}(S)$ *and an embedding $\beta \colon S \to \mathsf{B}(S)$ which is universal for maps to Boolean inverse semigroups.*

The above result is the algebraic version of Paterson's *universal groupoid* of an inverse semigroup.

Under certain circumstances, the *tight completion* of an inverse semigroup is a Boolean inverse semigroup: see individual papers by Lawson, Lenz and Exel.

Example The tight completion of the polycyclic inverse monoid P_n, where $n \geq 2$, is the *Cuntz inverse monoid* C_n. This is a congruence-free, Boolean inverse $\wedge$-monoid. Its group of units is the Thompson group V_n. When $n = 2$, we write $V = V_2$. Observe that the group of units of $C_2 \otimes C_2$ is the group $2V$, introduced by Matt Brin [Bleak].

The type monoid

This is our first *invariant* for studying Boolean inverse semigroups.

Let S be a Boolean inverse semigroup with set of idempotents $\mathsf{E}(S)$. Let $e, f \in \mathsf{E}(S)$. If $e \perp f$, denote their join by $e \oplus f$

Let M be a commutative monoid (whose binary operation we write as addition). A function $\beta \colon \mathsf{E}(S) \to M$ is called a *monoid valuation* if the following conditions hold:

(V1) $\beta(0) = 0$.

(V2) $\beta(e \oplus f) = \beta(e) + \beta(f)$ whenever $e \perp f$.

(V3) If $e \mathscr{D} f$ then $\beta(e) = \beta(f)$.

Put

$$\mathsf{T}^{\mathsf{part}}(S) = \mathsf{E}(S)/\mathscr{D}$$

with the $\mathscr{D}$-class containing e denoted by $[e]$.

On the set $\mathsf{T}^{\mathsf{part}}(S)$ define a partial binary operation $\oplus$ by $[e] \oplus [f] = [e' \vee f']$ whenever $e \mathscr{D} e'$ and $f \mathscr{D} f'$ and $e' \perp f'$ but undefined otherwise.

Proposition. *Let S be a Boolean inverse semigroup. Then* $(\mathsf{T}^{\mathsf{part}}(S), \oplus, [0])$ *is a conical partial refinement monoid and its construction is functorial.*

The universal monoid of $\mathsf{T}^{\mathsf{part}}(S)$ is denoted by $\mathsf{T}(S)$ and is called the *type monoid* of S. (We call $\mathsf{T}^{\mathsf{part}}(S)$ the *partial type monoid*.)

It is a conical, refinement monoid. The natural map $\tau\colon \mathsf{E}(S) \to \mathsf{T}(S)$ is called the *type function*.

Theorem. *Let S be a Boolean inverse semigroup. Then the type function $\tau\colon \mathsf{E}(S) \to \mathsf{T}(S)$ is the universal monoid valuation.*

MV-algebras

An *MV-algebra* is a structure $(A, \oplus, \neg, 0)$ such that $(A, \oplus, 0)$ is a commutative monoid where the following axioms hold:

1. $\neg\neg x = x$.

2. $x \oplus \neg x = \neg 0$.

3. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Example. The idempotent MV-algebras are precisely the Boolean algebras.

Theorem [Lawson & Scott; Wehrung] *Let S be a factorizable Boolean inverse monoid such that $S/\mathscr{J}$ is a lattice. Then the partial type monoid is, in fact, an MV-algebra and every MV-algebra arises in this way.*

For some worked examples, see *Topics in many-valued and quantum algebraic logics* by Weiyun Lu, MSc Thesis, University of Ottawa, 2016.

Semisimple Boolean inverse semigroups

A Boolean inverse semigroup S is said to be *semisimple* if the set $a^{\downarrow}$ is finite for every $a \in S$.

Theorem

1. *Each semisimple Boolean inverse semigroup is a restricted direct product of* 0*-simplifying semisimple Boolean inverse semigroups.*

2. *Each* 0*-simplifying semisimple Boolean inverse semigroup is isomorphic to a generalized rook matrix semigroup over a group with zero.*

Theorem *The Boolean inverse semigroups with type monoid* $\mathbb{N}$ *are precisely the* 0*-simplifying semisimple semigroups.*

Theorem *The Boolean inverse semigroups whose associated étale groupoids are discrete are precisely the semisimple ones.*

The theory of the type monoid is further developed in

F. Wehrung, *Refinement monoids, equidecomposability types, and Boolean inverse semigroups*, LNM 2188, Springer, 2017.

A research problem

Theorem [The dichotomy theorem] *Let S be a 0-simplifying Boolean inverse semigroup. Then exactly one of the following holds:*

1. *S is atomless.*

2. *S is semisimple.*

A countably infinite, atomless Boolean inverse monoid is called a *Tarski monoid*.

The idempotents of a Tarski monoid form the *Tarski algebra* whose Stone space is the Cantor space.

Classify the simple Tarski monoids.

To classify 0-simplifying Tarski monoids will require the theory of extensions of Boolean inverse monoids (and so cohomology).

Example In their inverse semigroup-theoretic reproving of a theorem of Feldman and Moore, Donsig, Fuller and Pitts in *Von Neumann algebras and extensions of inverse semigroups*, do exactly this in a special case.