FROM
PARTIAL SYMMETRIES
TO
NON-COMMUTATIVE STONE DUALITY

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The work I shall describe evolved, and continues to develop, in collaboration with
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Principal theme of this talk

Shechtman's work on quasi-crystals, for which he received the 2011 Nobel prize for chemistry, inspired both mathematicians and physicists to investigate more deeply the theory of aperiodic tilings.

However, it also raised more general questions about the nature of symmetry and how it can be formalized mathematically.

In this talk, I shall describe one way, inverse semigroups, in which the classical notion of group has been extended to deal with more exotic notions of symmetry.

Naming of parts

- We use the term *boolean algebra* to mean *generalized boolean algebra*. If a boolean algebra has a top element we shall say that it is *unital*.

- A *semigroup* is a set equipped with an everywhere defined associative binary operation. *Monoids* are semigroups with an identity. Semigroups with zero are called *semigroups with zero*.

- A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

- A *groupoid* G is a (for us, small) category with every arrow invertible. The set of identities (or objects) of G is denoted by G_o. The ‘o’ stands for ‘objects’.

Plan of talk

1. The big picture.

2. Non-commutative Stone duality.

3. Future work.

1. The big picture

From pseudogroups to inverse semigroups

Galois introduced groups as groups of symmetries of polynomials.

In the 1880's Lie, motivated by Galois, studied symmetries of differential equations which led to *infinite continuous groups*, which weren't groups at all, and came to be called

pseudogroups.

The definition varies but the following will do for today. A *pseudogroup* Γ on a topological space X is a collection of partial homeomorphisms closed under products and inverses satisfying the (sheaf-like) condition that if $\alpha_i \in \Gamma$ is a family of elements whose union α is a homeomorphism then $\alpha \in \Gamma$.

Veblen and Whitehead in 1932 recognized in such structures suitable algebraic vehicles for generalizing the Erlanger Programme to the developing theory of differential manifolds.

> "The notion of pseudogroups arose during a long process of trying to understand the foundations of geometry."

B. L. Reinhart *Differential geometry of foliations*, Springer-Verlag, 1983.

Pseudogroups generalize groups of transformations: bijections are replaced by partial bijections — *symmetries* are replaced by *partial symmetries*.

Groups are axiomatizations of groups of transformations. Can the notion of pseudogroup be similarly axiomatized? Yes.

- Wagner, 1952, USSR. Generalized groups.
- Preston, 1954, UK. Inverse semigroups.
- Ehresmann, 1957, F. Inductive groupoids.

Both Wagner and Ehresmann were differential geometers interested in the foundations of differential geometry.

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

Observe that $s^{-1}s$ and ss^{-1} are idempotents, and that $(s^{-1})^{-1} = s$ and $(st)^{-1} = t^{-1}s^{-1}$.

It can be proved that idempotents commute (Munn and Penrose).

Set of idempotents of S, denoted by $E(S)$, equipped with an order $e \leq f$ iff $e = ef = fe$ which makes $E(S)$ a *meet semilattice*.

Classical examples

- Groups are the inverse semigroups with a single idempotent.

- Meet semilattices are the inverse semigroups in which every element is an idempotent.

- Abelian inverse semigroups are presheaves of abelian groups over meet semilattices.

- Pseudogroups are special kinds of inverse semigroups.

Examples à la mode

- Tiling semigroups. Constructed as inverse semigroups of partial translational symmetries of tilings, especially aperiodic tilings. Kellendonk, 1997.

- Self-similar group actions. Discovered by J.-F. Perrot in his Thése, 1972 (not many people know this) and rediscovered by Grigorchuk in the 1980's.

- Graph inverse semigroups related to Cuntz-Krieger algebras.

The above examples illustrate the ability of inverse semigroups to describe non-classical symmetries.

Cayley resurgens

The set of all partial bijections $I(X)$ on the set X is an inverse semigroup, called the *symmetric inverse monoid*.

We have the following Cayley-type theorem.

Theorem [Wagner-Preston] *Every inverse semigroup is isomorphic to an inverse subsemigroup of a symmetric inverse semigroup.*

Thus inverse semigroups formalize pseudogroups in the same way that groups formalize groups of transformations.

The ghosts of departed quantities

The semilattice of idempotents of a pseudogroup is isomorphic to the lattice of open sets of a topological space.

A *frame* is a complete infinitely distributive lattice. The theory of frames can be viewed as an approach to spaces in which open sets, and not points, are taken as basic.

Peter Johnstone writes

> It was Ehresmann . . . and his student Bénabou . . . who first took the decisive step in regarding complete Heyting algebras as 'generalized topological spaces'.

Inverse semigroup theory can be regarded as non-commutative frame theory

Names for nameless things

An inverse semigroup S is equipped with two important relations:

- $s \leq t$ is defined if and only if $s = te$ for some idempotent e. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

- $s \sim t$ if and only if st^{-1} and $s^{-1}t$ both idempotents. *Compatibility relation*. Not in general an equivalence relation. Controls when pairs of elements are *eligible* to have a join.

2. Non-commutative Stone duality

The problem

Inverse semigroups, topological groupoids and C^*-algebras are somehow related.

J. Renault, *A groupoid approach to C^*-algebras*, Springer-Verlag, 1980.

A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, 1999.

The relationship between inverse semigroups and topological groupoids was clarified in Lawson and Lenz's *Pseudogroups and their étale groupoids*, arXiv:1107.5511v2.

An idea

Theorem [Marshall H. Stone] *The category of Boolean algebras is dual to the category of Boolean spaces — that is, hausdorff topological spaces with a basis of compact-open sets.*

This theorem links algebra and order, in the guise of boolean algebras, with topology.

We shall generalize this theorem by replacing *Boolean algebras* by *Boolean inverse semigroups* and *Boolean spaces* by *Boolean groupoids*.

An inverse semigroup S is said to be *Boolean* if it satisfies the following three conditions:

1. Each pair of compatible elements has a join.

2. Products distribute over joins where they exist.

3. The semilattice of idempotents of S is a Boolean algebra.

An inverse semigroup satisfying just (1) and (2) is said to be *distributive*.

Example Symmetric inverse monoids are Boolean.

A groupoid G is said to be *Boolean* if it satisfies the following

1. G is a topological groupoid.

2. G is étale — this means that the domain map is a local homeomorphism.

3. The space of identities G_o is Boolean.

We may now state our non-commutative generalization of Stone duality.

Theorem 1 [Lawson, Lenz] *The category of Boolean inverse semigroups is dual to the category of Boolean spaces.*

Proof

There are two proofs.

The first is a direct generalization of classical Stone duality.

The second works within non-commutative frame theory.

This is all well and good, but where are the Boolean inverse semigroups?

A completion theorem

Let $A \subseteq s^{\downarrow}$ be a finite subset of the elements beneath s. We say that A *covers* s and write $s \to A$ if for each $0 \neq t \leq s$ there exists $a \in A$ such that a and t have a non-zero lower bound. This condition says that s is *eligible* to be the join of A.

A homomorphism $\theta \colon S \to T$ to a distributive inverse semigroup is said to be *tight* if $a \to \{a_1, \ldots, a_m\}$ implies that $\theta(a) = \bigvee_{i=1}^{m} \theta(a_i)$.

Theorem 2 [Lawson, Lenz] *For each inverse semigroup S there is a distributive inverse semigroup* D(S) *and a tight homomorphism $\delta \colon S \to$* D(S) *which is universal for tight maps to distributive inverse semigroups.*

A definition of staggering importance

An inverse semigroup S is said to be *pre-Boolean* or satisfy the *compactness condition* if $\mathsf{D}(S)$ is Boolean.

If S is pre-Boolean we say that $\mathsf{D}(S)$ is its *Boolean completion*.

Really important examples Under suitable assumptions, graph inverse semigroups and tiling semigroups are pre-Boolean.

A simple example

The finite symmetric inverse monoid $I(X)$. This is already a boolean inverse monoid. Its associated groupoid is the set $X \times X$ with the usual groupoid multiplication equipped with the discrete topology.

An interesting example

The *polycyclic monoid* P_n, where $n \geq 2$, is defined as a monoid with zero generated by the variables $a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$ subject to the relations

$$a_i^{-1} a_i = 1 \text{ and } a_i^{-1} a_j = 0, i \neq j.$$

Every non-zero element of P_n is of the form yx^{-1} where x and y are elements of the *free monoid* on $\{a_1, \ldots, a_n\}$.

The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable in which case

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some } z \end{cases}$$

The polycyclic monoid P_n is a pre-Boolean inverse monoid.

Theorem The boolean completion of P_n is called (here) the *Cuntz inverse monoid* CI_n.

1. This monoid is congruence-free.

2. Its group of units is the Thompson group $V_{n,1}$.

3. Its associated groupoid is the groupoid also associated with the Cuntz C^*-algebra C_n.

So

O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, (1999) is, in fact, a study of tight maps from P_n to $I(X)$.

Although we set out to model exotic symmetries, this theory also leads to the construction of interesting groups. The groups are obtained by glueing together partial symmetries into global symmetries.

The group $V_{2,1}$ is a finitely presented infinite simple group.

3. Future work

1. Find natural conditions on an inverse semigroup that ensure that it is pre-Boolean. Tiling semigroups of 'nice' tilings and graph inverse semigroups of 'nice' graphs are pre-Boolean.

2. Develop a dimension theory (K-theory) for inverse semigroups.

3. Investigate the groups that arise as groups of units of Boolean inverse monoids.

4. Investigate the connections between inverse semigroups, topological groupoids and C^*-algebras.

Philosophische Bemerkung

Before beginning a Hunt, it is wise to ask someone what you are looking for before you begin looking for it.

Winnie the Pooh