K. S. S. Nambooripad Endowment Lecture
GROUPOIDS IN INVERSE SEMIGROUP THEORY

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Context

In my appreciation of the work of my PhD supervisor John Fountain, I wrote the following:

> "I was particularly in awe of Nambooripad . . . who with his theory of biordered sets appeared to be a mathematical magician."

See — V. A. R. Gould, M. V. Lawson, A Tribute to John B. Fountain, *Semigroup Forum* **99** (2019), 1–8.

When writing the above statement, I had in mind the following paper by Nambooripad (still unsurpassed):

K. S. S. Nambooripad, *Structure of regular semigroups I.*, Mem. Amer. Math. Soc. **22** (1979), no. 224, vii+119pp.

One of the things I learnt from Nambooripad was to apply category theory to the study of semigroups: where a category is regarded as a generalized monoid.

Specifically, to apply groupoid theory to the study of inverse semigroups.

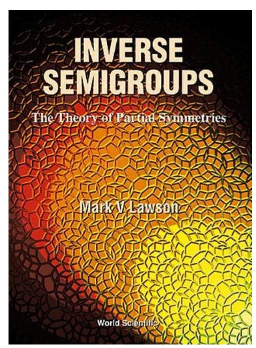
I have done this in two, quite distinct ways in my research:

1. *Inverse semigroups are regarded as special kinds of ordered groupoids.* This invites us to develop inverse semigroup within the framework of ordered groupoid theory. This led to a new perspective on the theory of E-unitary inverse semigroups.

2. *Inverse semigroups are viewed as non-commutative meet semilattices.* This leads us to generalize classical Stone duality to a non-commutative setting and to a connection with a class of topological groupoids.

In this talk, I will describe these two different applications of groupoid theory to the theory of inverse semigroups.

My book —– M. V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific, 1998 —– contains a discussion of the theory of ordered groupoids applied to the study of inverse semigroups.



The paper — M. V. Lawson, D. H. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170 — is a convenient starting point for the theory of non-commutative Stone duality.

Essential groupoid theory

Informally, a groupoid is a group with many identities and a partially defined multiplication. A groupoid with a single identity is a group.

Formally we define groupoids as follows; observe that for us everything is an arrow.

A *groupoid* G is a (for us, small) category in which every arrow is invertible.

The set of identities of G is denoted by G_o. The 'o' stands for 'objects'.

In a groupoid, there are structural maps $\mathbf{m}$, $\mathbf{d}$ and $\mathbf{r}$.

Define $\mathbf{d}(g) = g^{-1}g$ and $\mathbf{r}(g) = gg^{-1}$.

The product of g and h is defined precisely when $\mathbf{d}(g) = \mathbf{r}(h)$; in this case, we write $\mathbf{m}(g, h) = gh$.

Essential inverse semigroup theory

Groups are the abstract versions of groups of bijections whereas inverse semigroups are the abstract versions of inverse semigroups of partial bjiections.

A semigroup S is an inverse semigroup if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Inverse semigroups contain lots of idempotents since $a^{-1}a$ and aa^{-1} are idempotents. (An inverse semigroup with exactly one idempotent is a group).

The idempotents of an inverse semigroup commute with each other. If e is an idempotent so too is aea^{-1} for any $a \in S$.

Observe that $ae = a(a^{-1}ae) = a(ea^{-1}a) = (aea^{-1})a$. Thus idempotents can 'pass through elements' whilst still remaining idempotents.

Let S be an inverse semigroup. Let $a \in S$. It is useful to define the following idempotents

$$\mathbf{d}(a) = a^{-1}a \text{ and } \mathbf{r}(a) = aa^{-1}.$$

Keep in mind the following diagram

$$\mathbf{r}(a) \xleftarrow{a} \mathbf{d}(a).$$

Regard the elements of an inverse semigroup as being abstract partial bijections.

The natural partial order on an inverse semigroup

Let S be an inverse semigroup. Define $a \leq b$ iff $a = be$ for some idempotent e.

- $\leq$ is a partial order called the natural partial order.

- If $a \leq b$ then $a^{-1} \leq b^{-1}$.

- If $a \leq b$ and $c \leq d$ then $ac \leq bd$.

If e and f are idempotents then $e \leq f$ iff $e = ef$.

Observe that if $a, b \leq c$ then both $a^{-1}b$ and ab^{-1} are idempotents.

More generally, we say that a and b are compatible if both $a^{-1}b$ and ab^{-1} are idempotents.

It follows, that a necessary condition for a and b to have a join is that they be compatible.

1. Ordered groupoids and inverse semigroups

The goal of this part of the talk is to show that the theory of ordered groupoids sheds light on the theory of E-unitary inverse semigroups.

The groupoid associated with an inverse semigroups

Let S be an inverse semigroup. We can associate a groupoid with S as follows.

Restrict the multiplication of S to those pairs (a, b) where $\mathbf{d}(a) = \mathbf{r}(b)$.

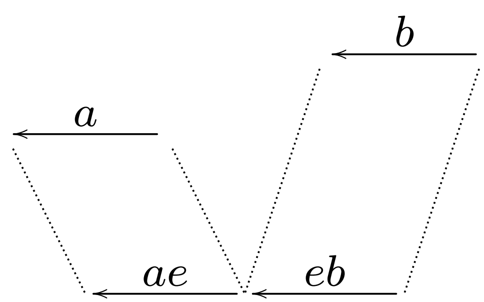
Lemma *With the above definitions,* $(S, \cdot)$ *is a groupoid.*

We lose information by restricting to the groupoid, but we can recapture S by using the natural partial order.

Let $a, b \in S$. Put $e = \mathbf{d}(a)\mathbf{r}(b)$. Then

$$ab = (ae) \cdot (eb).$$

The following diagram shows what is going on



We need to formalize the notion of a groupoid equipped with a partial order.

Ordered groupoids

This definition goes back to the work of Charles Ehresmann.

Let G be a groupoid equipped with a partial order. We say that it is an *ordered groupoid* if it satisfies the following axioms:

(OG1) $x \leq y$ implies that $x^{-1} \leq y^{-1}$.

(OG2) If $x \leq y$ and $x' \leq y'$ and $\exists xx'$ and $\exists yy'$ then $xx' \leq yy'$.

(OG3) Let e be an identity $e \leq \mathbf{d}(x)$. Then there exists a unique elemement $(x \mid e)$, called the *restriction of* x *to* e, such that $(x \mid e) \leq x$ and $\mathbf{d}(x \mid e) = e$.

(OG4) Let e be an identity $e \leq \mathbf{r}(x)$. Then there exists a unique elemement $(e \mid x)$, called the *corestriction of* x *to* e, such that $(e \mid x) \leq x$ and $\mathbf{r}(e \mid x) = e$.

The ESN-theorem

An ordered groupoid $(G, \cdot, \leq)$ is called an *inductive groupoid* if the set of identities of G with respect to $\leq$ is a meet semilattice.

Theorem

1. If S is an inverse semigroup then $(S, \cdot, \leq)$ is an inductive groupoid. Here, $\leq$ is the natural partial order.

2. If $(G, \cdot, \leq)$ is an inductive groupoid then we may define an inverse semigroup on G by defining $x \otimes y = (x \mid e) \cdot (e \mid y)$ where $e = \mathbf{d}(x) \wedge \mathbf{r}(y)$.

3. **Ehresmannn-Schein-Nambooripad**. The category of inductive groupoids and order-preserving functors is isomorphic to the category of inverse semigroups and prehomomorphisms, where $\theta \colon S \to T$ is a *prehomomorphism* if $\theta(st) \leq \theta(s)\theta(t)$.

The ESN-theorem generalizes the two ways of viewing meet semilattices.

But what is important is that it embeds the category of inverse semigroups in the larger category of ordered groupoids.

We can use the ‘extra space’ this affords to prove new theorems about inverse semigroups.

For example, it enables us to prove a ‘co-ordinate free’ version of the classical P-theorem describing the structure of E-unitary inverse semigroups.

The structure of E-unitary inverse semigroups: preparation

Let $\theta \colon G \to H$ be a functor between groupoids.

Then θ is *star-injective* if $\theta(g) = \theta(g')$ and $\mathbf{d}(g) = \mathbf{d}(g')$ implies that $g = g'$.

Then θ is *covering* if θ is star injective and if h is such that $\mathbf{d}(h) = e'$ where $\theta(e) = e'$ then there exists $g \in G$ such that $\mathbf{d}(g) = e$ and $\theta(g) = h$.

Groupoids from group actions

Let G be a group acting on the set X. We write $(g,x) \mapsto g \cdot x$. We can turn the set $X \times G$ into a groupoid $P(G,X)$ whose elements can be visualized as

$$x \xleftarrow{g} g^{-1} \cdot x.$$

The arrows are the ordered pairs (x,g) where $\mathbf{d}(x,g) = (g^{-1} \cdot x, 1)$ and $\mathbf{r}(x,g) = (x,1)$. There is a covering functor $\pi \colon P(G,X) \to G$ given by $\pi(x,g) = g$.

Theorem *Let Π be a groupoid, G a group and $\pi \colon \Pi \to G$ a surjective covering functor. Then G acts on Π_o, and Π is isomorphic to $P(G, \Pi_o)$.*

Enlargements

Let G be an ordered subgroupoid of the ordered groupoid H. We say that H is an *enlargement* of G if the following axioms hold:

(E1) G_o is an order ideal of H_o.

(E2) If $x \in H$ and $\mathbf{d}(x), \mathbf{r}(x) \in G$ then $x \in G$.

(E3) If $e \in H_o$ then there exists $x \in H$ such that $\mathbf{r}(x) = e$ and $\mathbf{d}(x) \in G$.

Ordered groupoids from group actions on posets

Theorem

1. *Let G be a group acting by order automorphisms on the poset X. Then $\Pi(X,G)$ is an ordered groupoid and $\pi : \Pi(X,G) \to G$ is a surjective ordered covering functor.*

2. *If Π is an ordered groupoid that admits a surjective ordered covering functor onto the group G then Π is isomorphic to the ordered groupoid obtained from the action of G by order automorphisms on the poset Π_o.*

The structure of E-unitary inverse semigroups

Let S be an inverse semigroup. It is said to be E-*unitary* if $e \leq a$, where e is an idempotent, implies that a is an idempotent.

The following is a co-ordinate free version of what is termed the 'P-theorem'. It is a special case of a construction due to Charles Ehresmann.

Theorem *Let S be an E-unitary semigroup and let S/σ be its maximum group image. Then, regarded as an inductive groupoid, there is an embedding $\iota \colon S \to \Pi$ into an ordered groupoid such that $\iota(S) \subseteq \Pi$ is an enlargement and $\pi \colon \Pi \to S/\sigma$ is a surjective ordered covering functor.*

For the general result, please see Section 8.3 of my book.

2. Non-commutative Stone duality

I shall focus on one part of this duality theory — that relating to Boolean inverse monoids — for simplicity, but it can be generalized.

Basic definitions

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

Pseudogroups are the correct abstractions of pseudogroups of transformations.

This leads us to think of inverse semigroup theory from a lattice-theoretic perspective.

An inverse semigroup is a *meet-semigroup* if has has all binary meets.

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

Summary

Commutative	**Non-commutative**
Meet semilattice	Inverse semigroup
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup
	Boolean inverse meet-semigroup

Commutative (= Classical) Stone duality

This was developed by Marshall Stone in 1936. He showed that Boolean algebras could be described in topological terms.

A topological space is said to be 0-*dimensional* if it has a base of clopen sets. A compact Hausdorff space which is 0-dimensional is called a *Boolean space*.

If X is a Boolean space, we denote the Boolean algebra of clopen subsets of B by $\mathsf{B}(X)$.

Let B be a Boolean algebra. Define $\mathsf{X}(B)$ to be the set of ultrafilters on B. If $a \in B$ denote by V_a the set of ultrafilters containing a. Define a topology σ on $\mathsf{X}(B)$ whose open sets are unions of the sets of the form V_a. We call $\mathsf{X}(B)$ the *Stone space* of B.

Theorem [Classical Stone duality]

1. *Let* B *be a Boolean algebra. Then* $B \cong \mathsf{BX}(B)$.

2. *Let* X *be a Boolean space. Then* $X \cong \mathsf{XB}(X)$.

Examples of classical Stone duality

1. Let B be a finite Boolean algebra. Then each ultrafilter is determined by an atom. The Stone space of B is then simply the finite set of atoms equipped with the discrete topology.

2. Tarski proved that any two atomless, countably infinite Boolean algebras are isomorphic. We call any atomless, countably infinite Boolean algebra a *Tarski algebra*. The Stone space of the Tarski algebra is a second-countable, 0-dimensional, compact Hausdorff space with no isolated points; such a space is homeomorphic to the *Cantor space*.

Our programme

We shall generalize the above to a non-commutative setting:

Boolean algebras $\rightarrow$ Boolean inverse monoids

Topological spaces $\rightarrow$ topological groupoids

One can (but I won't here) replace monoids by semigroups (which means that compact is replaced by locally compact) and analogous results can be proved for distributive inverse semigroups.

The correct setting for all of the above is a dual adjunction linking pseudogroups and étale groupoids.

Etale groupoids

A groupoid is said to be *topological* if it equipped with a topology and all maps associated with the groupoid are continuous.

The most important class of topological groupoids are the *étale groupoids*.

These are the topological groupoids in which $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms.

Resende's characterization of étale groupoids below explains why they are so important: their topology forms a monoid. They therefore have algebraic alter egos.

Proposition *A topological groupoid G is étale if and only if G_o is an open set and the product of any two open sets in G is an open set.*

Local bisections

To build inverse semigroups from groupoids, we shall need the following

Let G be a groupoid. A subset $A \subseteq G$ is called a *local bisection* if $g, h \in A$ and $\mathbf{d}(g) = \mathbf{d}(h)$ (respectively, $\mathbf{r}(g) = \mathbf{r}(h)$) then $g = h$.

Boolean groupoids

An étale groupoid G is called *Boolean* if its identity space G_o is a Boolean space.

Passing from Boolean groupoids to Boolean inverse monoids is easy.

Proposition *Let G be a Boolean groupoid. Denote by* $\mathsf{KB}(G)$ *the set of all compact-open local bisections of G. Then* $\mathsf{KB}(G)$ *is a Boolean inverse monoid.*

Stone groupoids

Passing from Boolean inverse monoids to Boolean groupoids is a little trickier and I shall omit the details here.

Let S be a Boolean inverse monoid. Denote the set of ultafilters containing a by V_a. Denote by $\mathsf{G}(S)$ the set of all ultrafilters on S. Let σ be the topology on $\mathsf{G}(S)$ with basis the set V_a where $a \in S$.

Theorem *Let S be a Boolean inverse monoid. Then $\mathsf{G}(S)$ is a Boolean groupoid.*

We call $\mathsf{G}(S)$ the *Stone groupoid* of S.

Theorem [Non-commutative Stone duality]

1. *Let S be a Boolean inverse monoid. Then $S \cong \mathsf{KB}(\mathsf{G}(S))$.*

2. *Let G be a Boolean groupoid. Then $G \cong \mathsf{G}(\mathsf{KB}(G))$.*

Refinements

I do not have time to go into details, so I shall simply summarize the most important results in the following table. Think of it as a dictionary between algebra and topology:

Boolean inverse monoid	**Boolean groupoid**
Meet-monoid	Hausdorff
Fundamental	Effective
Tarski algebra of idempotents	Cantor space of identities
0-simplifying	Minimal
0-simple	Minimal and purely infinite
Group of units	Topological full group
Finite	Discrete
Basic inverse meet-monoids	Hausdorff principal
Countable	Second countable

Non-commutative Stone duality plays an important role in the theory of C^*-algebras where inverse semigroups arise naturally.