# The universal Boolean inverse semigroup presented by the abstract Cuntz-Krieger relations

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# I. COMMUTATIVE STONE DUALITY

## 1. Commutative (= classical) Stone duality

Classical Boolean algebras will be called *unital Boolean algebras.*

Generalized Boolean algebras will be called *Boolean algebras*.

A *Boolean space* is a 0-dimensional locally compact Hausdorff space; we are often interested in the *compact* case.

# Stone's theorem

**Theorem** [Commutative Stone duality, Stone 1937 and Doctor 1964] *The category of Boolean algebras (resp. unital Boolean algebras) and proper homomorphisms (resp. homomorphisms ) is dually equivalent to the category of Boolean spaces (resp. compact Boolean spaces) and proper continuous functions (resp. continuous functions).*

The locally compact Boolean space corresponding to a Boolean algebra is constructed by topologizing the set of ultrafilters. We call the topological space constructed in this way its *Stone space*.

The Boolean algebra corresponding to a locally compact Boolean space is constructed by taking the set of all compact-open subsets.

# II. NON-COMMUTATIVE STONE DUALITY

## 2. Motivation

We generalize classical Stone duality to a non-commutative setting in the following way:

- Boolean *spaces* are generalized to Boolean *groupoids.*

- Boolean *algebras* are generalized to Boolean *inverse semigroups.*

# 3. Etale topological groupoids

For background on étale groupoids, read Section I.1 of Resende's lecture notes.

We view categories as 1-sorted structures: everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a (small) category in which every arrow is invertible. We regard groupoids as *groups with many identities*. The set of identities of the groupoid $G$ is denoted by $G_o$.

A subset $A \subseteq G$ of a groupoid $G$ is called a *local bisection* if $g, h \in A$ and $g^{-1}g = h^{-1}h \in A$ (resp. $gg^{-1} = hh^{-1}$) implies $g = h$.

# Boolean groupoids

We work with *topological groupoids*.

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

**Theorem** [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets with the identity of the monoid being the space of identities.*

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| A *Boolean groupoid* is an étale topological groupoid whose space of identities is a Boolean space. |

# Aside on étale groupoids

Etale groupoids are intrinsically interesting; they play a pivotal rôle in the study of étendues (a class of toposes).

But more general kinds of groupoids are also important in the theory of operator algebras: such as open groupoids. See, for example,

Dana P. Williams, *A Tool kit for groupoid $C^*$-algebras*, AMS, 2019.

*This is all we need to know about groupoids.*

# 4. Boolean inverse semigroups

For background on inverse semigroups, see my Primer.

A semigroup $S$ is said to be *inverse* if for each $a \in S$ there exists a unique element $a^{-1}$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. Elements of the form $a^{-1}a$ and $aa^{-1}$ are idempotents.

Let $X$ be a non-empty set. Denote by $\mathscr{I}(X)$ the set of all partial bijections of $X$ to itself equipped with the usual product of partial functions. This is called the *symmetric inverse monoid* on $X$.

Idempotents in symmetric inverse monoids are the identity functions on subsets.

# The natural partial order

Inverse semigroups come equipped with an internally defined order. Let $S$ be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

**Proposition** *The relation $\leq$ is a partial order. In addition, if $a \leq b$ then $a^{-1} \leq b^{-1}$ and if also $c \leq d$ then $ac \leq bd$.*

This order is called the *natural partial order*.

In the symmetric inverse monoid, the natural partial order is subset-inclusion of partial functions.

# The compatibility relation

To talk about joins in inverse semigroups, we have to be careful. Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for $a$ and $b$ to have an upper bound is that $a^{-1}b$ and $ab^{-1}$ be idempotent. This leads to the following relation.

Define $a \sim b$ if $a^{-1}b$ and $ab^{-1}$ are idempotent. This is the *compatibility relation*.

A subset is said to be *compatible* if each pair of distinct elements in the set is compatible.

In order that a pair of elements is *eligible* to have a join, they must be compatible.

# Boolean inverse semigroups

An inverse semigroup is said to be *distributive* if it has finite compatible joins and multiplication distributes over such joins.

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| A *Boolean inverse semigroup* is a distributive inverse semigroup whose set of idempotents forms a Boolean algebra. |

Boolean algebra operations may be generalized to Boolean inverse semigroups. Let $b \leq a$ in a Boolean inverse. Define $a \setminus b = a(a^{-1}a \setminus b^{-1}b)$.

*Morphisms* of Boolean inverse semigroups preserve compatible joins.

# Finite Boolean inverse monoids

The following theorem should be compared with the structure of finite Boolean algebras.

**Theorem** [Lawson, Malandro] *The finite Boolean inverse monoids are isomorphic to the inverse monoids of all local bisections of finite discrete groupoids.*

The finite symmetric inverse monoids $\mathscr{I}(X)$ are Boolean inverse monoids. The finite discrete groupoids in this case are $X \times X$.

## 5. Non-commutative Stone duality

A subset $\mathscr{A} \subseteq S$ of a Boolean inverse monoid is called a *filter* if $a, b \in \mathscr{A}$ implies that there is a $c \in \mathscr{A}$ such that $c \leq a, b$, and if $a \in \mathscr{A}$ and $a \leq b$ then $b \in \mathscr{A}$. It is said to be *proper* if $0 \notin \mathscr{A}$. A subset $\mathscr{A} \subseteq S$ of a Boolean inverse semigroup is called an *ultrafilter* if it is a maximal proper filter.

If $S$ is a Boolean inverse semigroup denote by $\mathsf{G}(S)$ the set of ultrafilters of $S$; this can be made into a groupoid.

If $G$ is a Boolean groupoid denote by $\mathsf{KB}(G)$ the set of all compact-open local bisections.

# Theorem I

**Theorem** [Non-commutative Stone duality I, Lawson & Lenz]

1. *If $S$ is a Boolean inverse semigroup then* $\mathsf{G}(S)$ *is a Boolean groupoid, called the* Stone groupoid *of $S$. If $S$ is a monoid then the identity space of* $\mathsf{G}(S)$ *is compact.*

2. *If $G$ is a Boolean groupoid then* $\mathsf{KB}(G)$ *is a Boolean inverse semigroup. If the identity space of $G$ is compact then* $\mathsf{KB}(G)$ *is a monoid.*

3. *If $S$ is a Boolean inverse semigroup then* $S \cong \mathsf{KB}(\mathsf{G}(S))$*, and if $G$ is a Boolean groupoid then* $G \cong \mathsf{G}(\mathsf{KB}(G))$.

# Additive ideals

Let $S$ be a Boolean inverse semigroup. A subset $I$ of $S$ is called an *additive ideal* if it is a semigroup ideal closed under finite compatible joins.

Let $\theta\colon S \to T$ be a homomorphism between Boolean inverse semigroups. Then the kernel of $\theta$ is an additive ideal.

Let $I$ be an additive ideal in a Boolean inverse semigroup $S$. Then we may define a congruence $\equiv_I$ on $S$ by $a \equiv_I b$ if and only if there exists $c \leq a, b$ such that $a \setminus c$, $b \setminus c$ are in $I$. Denote $S/\equiv_I$ by $S/I$. The natural map $S \to S/I$ is a morphism of Boolean inverse semigroups. The morphisms of Boolean inverse semigroups of this form are precisely those which are *weakly-meet-preserving* meaning that if $t \leq \theta(a), \theta(b)$ then there exists $c \leq a, b$ such that $t \leq \theta(c)$; this result is due to Ganna Kudryavtseva.

# Theorem II

A homomorphism $\theta\colon S \to T$ between Boolean inverse semigroups is called *callitic* if it is *proper* (meaning that every element of $T$ lies beneath a finite join of images) and *weakly-meet-preserving*.

A continuous functor $\alpha\colon G \to H$ between étale groupoids is said to be *coherent* if the inverse images of compact-open subsets are compact-open.

**Theorem** [Non-commutative Stone duality II, Lawson & Lenz] *There is a dual equivalence between callitic morphisms and coherent continuous covering functors.*

# Application of Theorem I

**Theorem** [Booleanization]

1. *Let $S$ be an inverse semigroup with zero. Then there is a Boolean inverse semigroup* $\mathsf{B}(S)$ *and an embedding* $\beta \colon S \to \mathsf{B}(S)$ *which is universal for maps to the category of Boolean inverse semigroups.*

2. *The Stone groupoid of* $\mathsf{B}(S)$ *is* Paterson's universal groupoid $\mathsf{G}_u(S)$ *of $S$. The groupoid* $\mathsf{G}_u(S)$ *is constructed from all proper filters of $S$ with the 'patch topology'.*

# III. ABSTRACT CUNTZ-KRIEGER RELATIONS

# 6. Covers

We shall now describe an application of non-commutative Stone duality.

We motivate our definition of covers by the following example. We work in an arbitrary inverse semigroup $S$.

Let $a \in S$ and suppose that $a = \bigvee_{i=1}^{n} a_i$. Thus $A = \{a_1, \ldots, a_n\} \subseteq a^{\downarrow}$, where $a^{\downarrow}$ is the set of all elements below $a$.

Suppose that $0 < x \leq a$. Then it can be proved that $x = \bigvee_{i=1}^{n} a_i \wedge x$.

We deduce that $a_i \wedge x \neq 0$ for some $i$.

# Definition of covers

Let $S$ be an inverse semigroup. Let $a \in S$. A finite subset non-empty $A \subseteq a^{\downarrow}$ is said to be a *cover* of $a$, denoted by $A \to a$, if for every $0 < x \leq a$ there exists $a \in A$ such that $a \wedge x \neq 0$.

It follows that covers of $a$ are 'join-wannabes'.

Let $T$ be a Boolean inverse semigroup. A homomorphism $\theta \colon S \to T$ is called a *cover-to-join map* if for every $a \in S$ and every $A \to a$ we have that $\theta(a) = \bigvee \theta(A)$.

Thus every 'join-wannabe' is converted into an 'honest-to-goodness join'.

# The tight completion

Let $S$ be an inverse semigroup. Let $a \in A$ and let $A \to a$. Then $A$ is a compatible subset. Thus $A$ has a join $\mathbf{a}$ in $\mathsf{B}(S)$. Let $I$ be the ideal in $\mathsf{B}(S)$ generated by the elements $a \setminus \mathbf{a}$; these are the *abstract Cuntz-Krieger relations*. Put $\mathsf{T}(S) = \mathsf{B}(S)/I$ and let $\tau \colon S \to \mathsf{T}(S)$ be the natural map.

**Theorem** *The map $\tau \colon S \to \mathsf{T}(S)$ is a cover-to-join map and is universal for such maps. We call $\mathsf{T}(S)$ the* tight completion of $S$.

We now identify the Stone groupoid of $\mathsf{T}(S)$.

## The tight groupoid

Let $S$ be an inverse semigroup. A proper filter $\mathscr{A}$ is said to be *tight* if $a \in \mathscr{A}$ and $A \rightarrow a$ implies that $A \cap \mathscr{A} \neq \varnothing$.

The *tight groupoid* $\mathsf{G}_t(S)$ of $S$, introduced by Ruy Exel, is the restriction of the universal groupoid $\mathsf{G}_u(S)$ to the subset of tight filters.

**Theorem** *Let $S$ be an inverse semigroup. The Stone groupoid of the tight completion of $S$ is the tight groupoid of $S$.*

## 7. Example: whence Cuntz-Krieger?

Let $A_n = \{a_1, \ldots, a_n\}$ be a finite alphabet with $n \geq 2$ elements. Denote the free *monoid* on $A_n$ by $A_n^*$. Denote the free *semigroup* on $A_n$ by $A_n^+$.

A *morphism* between right ideals of $A_n^*$ is the analogue of a right module morphism.

The *polycyclic inverse monoid* $P_n$ is the inverse monoid of all bijective morphisms between principal right ideals of $A_n^*$ together with the empty partial function.

The non-zero elements of $P_n$ are of the form $xy^{-1}$ meaning: first 'pop $y$' and then 'push $x$'. The identity 1 of $P_n$ is simply $\varepsilon\varepsilon^{-1}$.

# Enter the Cuntz relations . . .

Observe that $a_i a_i^{-1}$ is the identity function on the set $a_i A_n^*$.

Thus the union $a_1 a_1^{-1} \cup \ldots \cup a_n a_n^{-1}$ is the identity on the *free semigroup* $A_n^+$, but it is not the identity on the *free monoid* $A_n^*$.

THIS LOOKS TRIVIAL BUT IS VITAL.

However, we *do* have that $\{a_1 a_1^{-1}, \ldots, a_n a_n^{-1}\} \to 1$.

## Origins

**Theorem** [Lawson] *The tight completion of $P_n$ is the Boolean inverse monoid $C_n$ we call the* Cuntz inverse monoid.

*The group of units of $C_n$ is the Thompson group $G_{n,1}$.*

*The Stone groupoid of $C_n$ is isomorphic to the set of triples $(xw, \mid x \mid - \mid y \mid, yw)$, where $x$ and $y$ are finite strings and $w$ is a right-infinite string, with a groupoid product. This is the usual groupoid associated with the Cuntz $C^*$-algebra.*

FIN: but see following references

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