Boolean inverse monoids

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With the collaboration of many colleagues

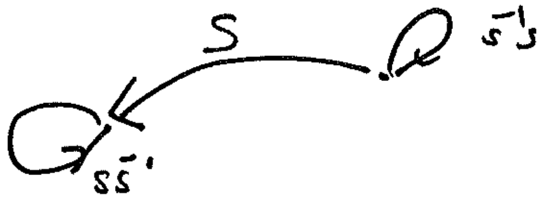
- We deal here only with monoids which corresponds to working with compact spaces. Our results can also be extended to semigroups which corresponds to working with locally compact spaces

- The theory generalizes to distributive inverse semigroups and pseudogroups

Recollections of some terminology

Let S be an inverse semigroup

- If $s \in S$ write $\mathbf{d}(s) = s^{-1}s$ and $\mathbf{r}(s) = ss^{-1}$. Regard s as an arrow



- $E(S)$ is the semilattice of idempotents of S.

- $\leq$ denotes the natural partial order defined by $a \leq b$ iff $a = be$ for some idempotent e.

- $\sim$ denotes the <u>compatibility relation</u> defined by $a \sim b$ iff $a^{-1}b$ and ab^{-1} are both idempotents.

- $\perp$ denotes the <u>orthogonality relation</u> defined by $a \perp b$ iff $a^{-1}b = 0 = ab^{-1}$.

- An inverse monoid is <u>factorizable</u> if every element is beneath a unit (w.r.t. the natural partial order).

- An inverse semigroup is <u>fundamental</u> if the only elements that commute with all idempotents are idempotents.

Key definitions

The inverse monoid S is said to be **distributive** iff:

- $a \sim b \Rightarrow \exists\, a \vee b$.
- If $\exists\, a \vee b$ then
  $c(a \vee b) = ca \vee cb$ and
  $(a \vee b)c = ac \vee bc \quad (\forall c \in S)$

When S is distributive, $E(S)$ is a distributive lattice.

An inverse monoid is said to be <u>Boolean</u> if it is distributive and its semilattice of idempotents is a Boolean algebra.

<u>Morphisms</u> of Boolean inverse monoids are monoid homomorphisms that preserve binary compatible joins and restrict to Boolean algebra maps between the respective Boolean algebras of idempotents.

<u>Notation</u> If $b \leq a$ define

$$a \setminus b = a\overline{\mathbf{d}(b)}$$

Example Let $X \neq \emptyset$ be any set. Then $I(X)$ denotes the symmetric inverse monoid on $I(X)$. It is clear that $I(X)$ is a Boolean inverse monoid. We shall usually denote this monoid by $I_{|X|}$. Thus I_5 is the symmetric inverse monoid on the set $X = \{1, 2, 3, 4, 5\}$.

Direct products of Boolean inverse monoids are Boolean inverse monoids.

Thus $I_{n_1} \times \ldots \times I_{n_r}$

is a Boolean inverse monoid.

There are Boolean inverse monoids which are analogues of AF, Cuntz, and Cuntz-Krieger C^*-algebras

Boolean inverse monoids arise naturally in ring theory (and C^*-algebras).

<u>Theorem</u> Let S be an inverse submonoid (with zero) of the multiplicative monoid of a ring R. Then there is a Boolean inverse submonoid S'' of R s.t.

$$S \subseteq S'' \subseteq R.$$

In fact, Boolean inverse monoids can be constructed from any inverse monoid.

<u>Theorem (Booleanization)</u> Let S be any inverse monoid with zero. Then there is a Boolean inverse monoid $\underline{B}(S)$ together with an embedding

$$\beta : S \to \underline{B}(S)$$

s.t. if $\theta : S \to T$ is any homomorphism to a Boolean inverse semigroup then there is a unique morphism $\gamma : \underline{B}(S) \to T$

s.t. $\theta = \beta\gamma$.

Boolean inverse monoids are more "ring-like" than arbitrary inverse monoids.

Let S be a Boolean inverse monoid.
An additive ideal $I \trianglelefteq S$ is a semigroup ideal I closed under binary compatible joins.

Lemma

(1) The kernels of morphisms between Boolean inverse monoids are additive ideals.

(2) The kernel of a morphism is trivial $\Leftrightarrow$ that morphism is idempotent-separating.

We can use ideals to build quotients (but <u>not</u> Rees quotients).

<u>Lemma</u> Let $I \trianglelefteq S$ be an additive ideal. Define $\equiv_I$ on S by $a \equiv_I b \iff (\exists c \leq a, b);$

$(a \backslash c), (b \backslash c) \in I.$

Then $S/_{\equiv_I} = S/I$ is a Boolean inverse monoid and the natural map $S \to S/I$ is a morphism with kernel I.

A congruence equal to some $\equiv_I$ is said to be *ideal-induced*.

N.B. Not every morphism is ideal-induced. Anja Kudryavtseva proved that ~~a~~ they are precisely the weakly meet preserving ones. That if $\theta : S \twoheadrightarrow T$ then it is weakly meet preserving if

$$t \leq \theta(a), \theta(b) \Rightarrow \exists c \leq a, b \text{ s.t. }$$

$$t \leq \theta(c).$$

<u>0 - simplifying</u> = no, non-trivial additive ideals

<u>Simple</u> = 0-simplifying + fundamental (no additive congruences)

<u>Strongly simple</u> = 0 - simple + fundamental (no congruences of any description)

Program: describe all simple Boolean inverse monoids.

This program may seem overambitious, but there are clues.

An element a of a Boolean inverse monoid is <u>finite</u> if the number of elements below it is finite.

A Boolean inverse monoid is said to be <u>semisimple</u> if every element is finite.

An <u>atom</u> is an element a s.t. $b \leq a \Rightarrow b = a$ or $b = 0$.

<u>The dichotomy theorem</u> Let S be a 0-simplifying Boolean inverse monoid. Then exacty one of the following holds:

(1) S is atomless.

(2) S is semisimple.

The class of countable, simple Boolean inverse monoids looks particulary interesting.

Three theorems

#1

Let $n \geqslant 1$ be a natural number.

Let G^0 be a group with zero adjoined.

$R_n(G^0)$ denotes the set of all $n \times n$ rook matrices over G^0. A rook matrix is a square matrix with at most one non-zero entry in each row and each column.

Theorem 1

(1) The finite Boolean inverse monoids have the form $R_{n_1}(G_1^0) \times \ldots \times R_{n_r}(G_r^0)$.

(2) The finite fundamental Boolean inverse monoids have the form $I_{n_1} \times \ldots \times I_{n_r}$.

(3) The finite simple Boolean inverse monoids have the form I_n.

#2

Let S be a Boolean inverse monoid.

Observe that $\vee$ is partially defined.

Put

- $x \oslash y = \mathbf{r}(x)\,\overline{\mathbf{r}(y)}\,x\,\mathbf{d}(x)\,\overline{\mathbf{d}(y)}$

[$x \oslash y$ is the largest element $\leq x$ and $\perp$ to y]. Called <u>skew difference.</u>

- $x \triangledown y = (x \oslash y) \vee y$.

Called <u>left skew join.</u>

Both $\oslash$ and $\triangledown$ are globally defined.

<u>Theorem 2</u> (Wehrung) Boolean inverse monoids can be axiomatized by means of equations using the signature $(S, 1, 0, {}^{-1}, \cdot, \oslash, \nabla)$. Thus they form a variety. In addition, there is a <u>Mal'cev term</u> for this variety and so they are <u>congruence-permutable</u>.

3

An <u>MV algebra</u> $(A, \boxplus, \neg, 0)$ is a set equipped with a binary operation $\boxplus$, a unary operation $\neg$, and a constant 0 s.t.

(MV1) $\boxplus$ is associative.

(MV2) $\boxplus$ is commutative.

(MV3) 0 is the identity for $\boxplus$.

(MV4) $\neg\neg x = x$.

(MV5) $x \boxplus \neg 0 = \neg 0$. Define

$1 = \neg 0$.

(MV6) $\neg(\neg x \boxplus y) \boxplus y =$

$\neg(\neg y \boxplus x) \boxplus x$.

MV-algebras arise as Lindenbaum algebras of many-valued logics in the same way that Boolean algebras arise as Lindenbaum algebras of classical, two-valued logic.

The type monoid of a Boolean inverse monoid.

Let S be a Boolean inverse monoid.

$E(S)/\mathcal{D}$ is the set of $\mathcal{D}$-classes of idempotents. Denote the $\mathcal{D}$-class containing e by $[e]$.

Define $[e] \oplus [f] = [e' \vee f']$ if $e' \in [e]$, $f' \in [f]$, $e' \perp f'$.

(undefined otherwise).

Put $\underline{0} = [0]$ and $\underline{1} = [1]$.

A commutative monoid M is said to be a <u>refinement monoid</u> if $a_1 + a_2 = b_1 + b_2$ implies there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ s.t. $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$,

$b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

	b_1	b_2
a_1	c_{11}	c_{12}
a_2	c_{21}	c_{22}

<u>Theorem</u> Let S be a Boolean inverse monoid. Then the partial structure $(E(S)/\mathcal{D}, \oplus)$ can be completed in a unique way to a commutative refinement monoid $\mathrm{Typ}(S)$, called the <u>type monoid</u> of S.

Lemma A Boolean inverse monoid is factorizable $\Leftrightarrow (e \mathcal{D} f \Rightarrow \bar{e} \mathcal{D} \bar{f})$.

A Foulis monoid is a factorizable, Boolean inverse monoid in which $S/\mathcal{J}$ is a lattice.

In a Foulis monoid, we may define

$\neg [e] = [\bar{e}]$.

Theorem 3 (Lawson & Scott, Wehrung)

Every MV algebra is isomorphic to the type monoid of some Foulis monoid.

Non-commutative Stone duality

Classical Stone duality

A Boolean space is a compact, Hausdorff space with a basis of clopen subsets.

Stone duality establishes a dual equivalence between the category of Boolean spaces and the category of Boolean algebras.

Let B be a Boolean algebra.

Put

$\underline{X}(B)$ = set of ultrafilters of X.

If $a \in B$, put

U_a = ultrafilters that contain a.

Put $\beta = \{U_a : a \in B\}$.

Then β is the basis of a topology on $\underline{X}(B)$ that makes it a Boolean space.

Let X be a Boolean space. Put $\underline{B}(X)$ = set of clopen subsets of X. Then $\underline{B}(X)$ is a Boolean algebra.

<u>Theorem</u>

(1) If B is a Boolean algebra then $B \cong \underline{B}(\underline{X}(B))$.

(2) If X is a Boolean space then $X \cong \underline{X}(\underline{B}(X))$.

$\underline{X}(B)$ is called the <u>Stone space</u> of B.

<u>Example</u> If X is a finite Boolean space then it is endowed with the discrete topology. Then $\underline{B}(X) = P(X)$, the Boolean algebra of all subsets of X. This is a finite Boolean algebra and every finite Boolean algebra is isomorphic to one of this kind.

<u>Example</u> There is, up to isomorphism, exactly one countable, atomless Boolean algebra. I shall call it the <u>Tarski algebra</u>.

The Stone space of the Tarski algebra is the <u>Cantor space</u>.

Non-commutative Stone duality

We shall replace Boolean algebras by Boolean inverse monoids.

Boolean spaces will be replaced by a suitable class of topological groupoids.

A *groupoid* G is a small category in which every arrow is invertible.

$$\begin{tikzcd} \bullet \arrow[loop left, "gg^{-1}"] & \bullet \arrow[l, "g"'] \arrow[loop right, "g^{-1}g"] \end{tikzcd}$$

$G_0 \subseteq G$ is the set of *identities* of G, meaning elements e s.t. if $\exists ge$ then $ge = g$ and $\exists eg$ then $eg = g$.

A subset $A \subseteq G$ is called a *partial bisection* if $a, b \in A$ and $a^{-1}a = b^{-1}b$ (resp. $aa^{-1} = bb^{-1}$) $\Rightarrow a = b$.

A groupoid G is equipped with maps $d : G \to G$, given by $d(g) = g^{-1}g$, $r : G \to G$, given by $r(g) = g g^{-1}$ and a map $m : G * G \to G$ given by $(g,h) \mapsto gh$ where

$$G * G = \{ (g,h) \in G \times G : d(g) = r(h) \}.$$

and a map $G \to G$ given by $g \mapsto g^{-1}$.

If G is equipped with a topology s.t. these maps are continuous we say

G is a <u>topological groupoid</u>.

A ~~[illegible]~~ topological groupoid G is said to be <u>étale</u> if $\underline{d}$ is a local homeomorphism.

The algebraic significance of étale groupoids is provided by the following result of Pedro Resende.

<u>Theorem</u> Let G be an étale groupoid. Then $\Omega(G)$, the set of all open subsets of G, is a <u>monoid</u> under product of subsets.

An étale topological groupoid is said to be **Boolean** if its space of identities is a Boolean space.

Idea

Boolean algebras $\xrightarrow{\text{replaced by}}$ Boolean inverse monoids

Boolean spaces $\xrightarrow{\text{replaced by}}$ Boolean groupoids.

Let G be a Boolean groupoid. Denote by $\underline{KB}(G)$ the set of all compact-open partial bisections of G.

<u>Proposition</u> Under subset multiplication, $\underline{KB}(G)$ is a Boolean inverse monoid.

Let S be a Boolean inverse monoid. Denote by $\underline{G}(S)$ the set of all ultrafilters on S.

It is not immediately obvious but the set $\underline{G}(S)$ can be endowed with the structure of a groupoid.

If $s \in S$, denote by U_s the set of all ultrafilters in S that contain s. Put $\beta = \{ U_a : a \in S \}$. Then β is the basis for a topology on $\underline{G}(S)$.

<u>Proposition</u> $\underline{G}(S)$ is a Boolean groupoid. We call it the <u>Stone groupoid</u> of S.

Theorem

(1) If S is a Boolean inverse monoid then $S \cong \underline{KB}(\underline{G}(S))$.

(2) If G is a Boolean groupoid then $G \cong \underline{G}(\underline{KB}(G))$.

Refinements

- $\underline{G}(S)$ is Hausdorff $\Leftrightarrow$ S has binary meets.
- $\underline{G}(S)$ is effective $\Leftrightarrow$ S is fundamental.
- $G_1 \sqcup G_2$ corresponds to $\underline{KB}(G_1) \times \underline{KB}(G_2)$.
- $G_1 \times G_2$ corresponds to $\underline{KB}(G_1) \otimes \underline{KB}(G_2)$

<u>Theorem</u> Let S be an inverse monoid with zero. Then the Stone groupoid of $\underline{B}(S)$, the Booleanization of S, is <u>Paterson's universal groupoid</u> of S.

<u>Example</u> If G is a finite Boolean groupoid then it is endowed with the discrete topology. Thus every finite Boolean inverse monoid is isomorphic to the Boolean inverse monoid of all partial bisections of a finite discrete groupoid.

This result can be refined to give the structure theory of finite Boolean inverse monoids described earlier.

<u>Example</u> Second-countable Boolean groupoids whose space of identities is the Cantor space correspond to countably infinite Boolean inverse monoids with a Tarski algebra of idempotents.

These play an important rôle in modern group theory, see work of Matui et al, via their groups of units.

A *Tarski monoid* is a countably infinite, atomless Boolean inverse $\wedge$-monoid.

Theorem (Matui, Boolean inverse monoid version). Let S and T be simple Tarski monoids. Then t.f.a.e.

(1) The monoids S and T are isomorphic.

(2) Their groups of units are isomorphic.