Non-commutative Boolean algebras

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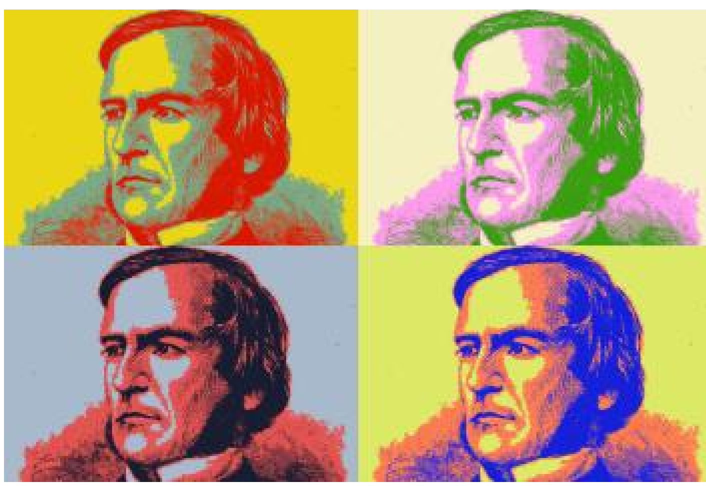

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Boolean algebras

A *Boolean algebra* is a structure $(B, +, \cdot, \bar{\ }, 0, 1)$ where B is a set, $+$ and $\cdot$ are binary operations, $a \mapsto \bar{a}$ is a unary operation, and 0 and 1 are distinguished elements. In addition, the following ten axioms are required to hold.

(B1) $(x + y) + z = x + (y + z)$.

(B2) $x + y = y + x$.

(B3) $x + 0 = x$.

(B4) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(B5) $x \cdot y = y \cdot x$.

(B6) $x \cdot 1 = x$.

(B7) $x \cdot (y + z) = x \cdot y + x \cdot z$.

(B8) $x + (y \cdot z) = (x + y) \cdot (x + z)$.

(B9) $x + \bar{x} = 1$.

(B10) $x \cdot \bar{x} = 0$.

1. The finite Boolean algebras are isomorphic to power set algebras $\mathsf{P}(X)$ where X is a finite set.

2. More generally, Stone's theorem says that the category of Boolean algebras is in duality with the category of compact, Hausdorff 0-dimensional spaces.

3. The Lindenbaum algebra of propositional logic is a Boolean algebra.

4. Boolean algebras used in circuit design.

5. Boolean algebras form the foundations of measure theory.

Non-commutativity

The work of Alain Connes has stimulated interest in non-commutative geometry, closely connected with C^*-algebras.

Out of this work, a theory of non-commutative Boolean algebras has arisen.

In this theory,

- The commutative $\cdot$ is replaced by a non-commutative binary operation.

- The commutative $+$ is replaced by a *partially defined* commutative operation.

Boolean inverse monoids

A *semigroup* is a set with an associative binary operation, a *monoid* is a semigroup with an identity.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

The idempotents in an inverse semigroup commute with each other. We speak of the *semilattice of idempotents* $E(S)$ of the inverse semigroup S.

The set of all partial bijections of a set X forms an inverse monoid $I(X)$ called the *symmetric inverse monoids*. If X has n elements, we denote the symmetric inverse monoid by I_n.

Theorem [Wagner-Preston] *Symmetric inverse monoids are inverse, and every inverse semigroup can be embedded in a symmetric inverse monoid.*

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation $\leq$ is a partial order with respect to which S is a partially ordered semigroup.*

It is called the *natural partial order*.

Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A subset is said to be *compatible* if each pair of distinct elements in the set is compatible.

In the symmetric inverse monoid $I(X)$ the natural partial order is defined by restriction of partial bijections.

The union of two partial bijections is a partial bijection if and only if they are compatible.

- An inverse semigroup is said to have *finite joins* if each finite compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be *Boolean* if it is distributive and its semilattice of idempotents is a Boolean algebra.

Boolean inverse monoids are non-commutative generalizations of Boolean algebras.

The symmetric inverse monoids are Boolean inverse monoids.

To manufacture other examples of Boolean inverse monoids, we use groupoids.

We view categories as 1-sorted structures (over sets): everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as ‘groups with many identities’.

Key definition Let G be a groupoid with set of identities G_o. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

The set of all local bisections of the groupoid G is denoted by $\mathsf{B}(G)$.

Proposition *The set of all local bisections of a groupoid forms a Boolean inverse meet-monoid.*

We can now characterize the *finite* Boolean inverse monoids.

Theorem *Each finite Boolean inverse monoid is isomorphic to a Boolean inverse monoid* $\mathsf{B}(G)$ *where* G *is a finite groupoid.*

Thus in passing from finite Boolean algebras to finite Boolean inverse monoids, we replace finite sets by finite groupoids.

It is possible to define what we mean by a *simple* Boolean inverse monoid.

Theorem *The simple Boolean inverse monoids are precisely the finite symmetric inverse monoids* I_n.

The theory of Boolean inverse monoids has close connections with groups of Thompson-Higman type, via their groups of units, and with étale groupoids under a non-commutative generalization of Stone duality.

But for the remainder of this talk, I will show a (tangential) connection with *multiple-valued (MV) logic.*

We begin by defining *MV-algebras*, another generalization of Boolean algebras.

MV-algebras

An *MV-algebra* $(A, \boxplus, \neg, 0)$ is a set A equipped with a binary operation $\boxplus$, a unary operation $\neg$ and a constant 0 such that the following axioms hold.

(MV1) $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$.

(MV2) $x \boxplus y = y \boxplus x$.

(MV3) $x \boxplus 0 = x$.

(MV4) $\neg\neg x = x$.

(MV5) $x \boxplus \neg 0 = \neg 0$. Define $1 = \neg 0$.

(MV6) $\neg(\neg x \boxplus y) \boxplus y = \neg(\neg y \boxplus x) \boxplus x$.

Examples

1. Every Boolean algebra is an MV-algebra when $\vee$ is interpreted as $\boxplus$ and $^{-}$ as $\neg$.

2. The real closed interval $[0, 1]$ equipped with the operations $x \boxplus y = \min(1, x + y)$ and $\neg x = 1 - x$ is an MV-algebra.

3. For each $n \geq 2$ define
$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$$
equipped with the operations $\boxplus$ and $\neg$ as in (2). These are called *Łukasiewicz chains*.

4. MV-algebras arise as Lindenbaum algebras of many-valued logic in the same way that Boolean algebras arise as Lindenbaum algebras of classical, two-valued logic.

- The idempotents of an MV-algebra form a Boolean algebra. Thus MV-algebras are ‘non-idempotent Boolean algebras’.

- The finite MV-algebras are finite direct products of MV-algebras of the form L_n.

Further reading

Daniele Mundici, Logic of infinite quantum systems, *Int. J. Theor. Phys.* **32** (1993), 1941–1955.

Daniele Mundici, *MV-algebras: A short tutorial*, May 26, 2007.

Boolean algebras as partial algebras

In Boole's original work on Boolean algebras the operation $\boxplus$, that is $\vee$, was a partial operation defined only between orthogonal elements.

Here is an axiomatization of Boolean algebras in these terms due to Foulis and Bennett.

Let $(B, \oplus, 0, 1)$ be a set B equipped with a *partial binary operation* $\oplus$ and two constants 0 and 1 such that the following axioms hold.

(PB1) $p \oplus q$ is defined if and only if $q \oplus p$ is defined, and when both are defined they are equal.

(PB2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined then $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(PB3) For each p there is a unique q such that $p \oplus q = 1$.

(PB4) If $1 \oplus p$ is defined then $p = 0$.

(PB5) If $p \oplus q$ and $p \oplus r$ and $q \oplus r$ are defined then $(p \oplus q) \oplus r$ is defined.

(PB6) Given p and q there exist a, b, c such that $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $p = a \oplus c$ and $q = b \oplus c$.

MV-algebras as partial algebras

Let $(B, \oplus, 0, 1)$ be a set B equipped with a partial binary operation $\oplus$ and two constants 0 and 1. It is called an *effect algebra* if the following axioms hold.

(EA1) $p \oplus q$ is defined if and only if $q \oplus p$ is defined, and when both are defined they are equal.

(EA2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined then $p \oplus q$ is defined and $(p \oplus q) \oplus r$ is defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(EA3) For each p there is a unique p' such that $p \oplus p' = 1$.

(EA4) $1 \oplus p$ is defined if and only if $p = 0$.

Define $p \leq q$ if and only if $p \oplus r = q$ for some r.

The *refinement property* is defined as follows. If $a_1 \oplus a_2 = b_1 \oplus b_2$ then there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} \oplus c_{12}$ and $a_2 = c_{21} \oplus c_{22}$, and $b_1 = c_{11} \oplus c_{21}$ and $b_2 = c_{12} \oplus c_{22}$.

	b_1	b_2
a_1	c_{11}	c_{12}
a_2	c_{21}	c_{22}

Theorem *An effect algebra which is a lattice with respect to $\leq$ and satisfies the refinement property is an MV-algebra when we define*

$$a \boxplus b = a \oplus (a' \wedge b)$$

and every MV-algebra arises in this way.

Further reading

D. J. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.*, **24** (1994), 1331–1352.

M. K. Bennett and D. J. Foulis, Phi-symmetric effect algebras, *Found. Phys.*, **25** (1995), 1699–1722.

D. J. Foulis, MV and Heyting effect algebras, *Found. Phys.*, **30** (2000), 1687–1706.

A question

Boolean inverse monoids are to be viewed as non-commutative generalizations of Boolean algebras.

MV-algebras are to be viewed as non-idempotent generalizations of Boolean algebras

This raises the question of how Boolean inverse monoids are related to MV-algebras.

We now answer this very question.

Let S be a Boolean inverse monoid and let $a \in S$.

We may think of a as an arrow

$$a^{-1}a \xrightarrow{a} aa^{-1}$$

where $a^{-1}a$ is called the *domain idempotent* and aa^{-1} is called the *range idempotent*.

If $e = a^{-1}a$ and $f = aa^{-1}$ we write $e \mathscr{D} f$.

An inverse monoid is *factorizable* if each element is beneath an element of the group of units.

The symmetric inverse monoids, for example, are factorizable if and only if they are finite.

A factorizable Boolean inverse monoid is called a *Foulis monoid*.

Let S be an arbitrary Boolean inverse monoid. Put

$$\mathsf{E}(S) = E(S)/\mathscr{D}.$$

We denote the $\mathscr{D}$-class containing the idempotent e by $[e]$.

Define $[e] \oplus [f]$ as follows. If we can find idempotents $e' \in [e]$ and $f' \in [f]$ such that e' and f' are orthogonal then define $[e] \oplus [f] = [e' \vee f']$, otherwise, the operation $\oplus$ is undefined. Put $\mathbf{0} = [0]$ and $\mathbf{1} = [1]$.

Theorem *Let S be a Boolean inverse monoid. Then $(\mathsf{E}(S), \oplus, \mathbf{0}, \mathbf{1})$ is an effect algebra (satisfying the refinement property) if and only if S is factorizable.*

An inverse monoid S in which the poset of principal ideals is a lattice is said to satisfy the *lattice condition*.

Theorem *Let S be a Foulis monoid satisfying the lattice condition. Then* $\mathsf{E}(S)$ *is an MV-algebra.*

Co-ordinatizations

We say that an MV-algebra A can be *co-ordinatized* if there is a Foulis monoid S satisfying the lattice condition such that E(S) is isomorphic to A.

Theorem 1 [Lawson, Scott, 2014] *Every countable MV-algebra can be co-ordinatized.*

Theorem 2 [Wehrung, 2015] *Every MV-algebra can be co-ordinatized.*

M. V. Lawson, P. Scott, AF inverse monoids and the structure of countable MV-algebras, to appear in *Journal of Pure and Applied Algebra*.

F. Wehrung, Refinement monoids, equidecomposability types, and Boolean inverse semigroups, 205pp, 2015, <hal-01197354>.