Non-commutative Stone dualities

Mark V Lawson
Heriot-Watt University
and the
Maxwell Institute for Mathematical Sciences
March 2016

In collaboration with

Ganna Kudryavtseva (Ljubljana), Daniel Lenz (Jena), Stuart Margolis (Bar Ilan), Pedro Resende (Lisbon), Phil Scott (Ottawa) and Ben Steinberg (CUNY).

1. Inverse semigroups, étale groupoids and C^*-algebras

- J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, **793**, Springer, 1980.

- A. Kumjian, On localizations and simple C^*-algebras, *Pacific J. Math.* **112** (1984), 141–192.

- J. Kellendonk, The local structure of tilings and their integer group of coinvariants, *Comm. Math. Phys* **187** (1997), 115–157.

- A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics, **170**, Birkhäuser, Boston, 1998.

- D. H. Lenz, On an order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.

- P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

Well-known construction of C^*-algebras from étale groupoids.

Goal: to understand the connection between inverse semigroups and étale groupoids.

2. Idea: non-commutative Stone duality

Commutative	**Non-commutative**
Frame	Pseudogroup
Dist. lattice	Dist. inverse semigroup
Boolean algebra	Boolean inverse semigroup
	Boolean inverse meet-semigroup

Algebra	**Topology**
Semigroup	Locally compact
Monoid	Compact
Meet-semigroup	Hausdorff

In this talk, I will concentrate on *Boolean inverse monoids*.

3. Inverse semigroups

> "Symmetry denotes that sort of concordance of several parts by which they integrate into a whole." – Hermann Weyl

Symmetry is more than groups.

As groups are algebraic tools for studying symmetry, so *inverse semigroups* are tools for studying partial symmetry.

Inverse semigroups arose by abstracting pseudogroups of transformations in the same way that groups arose by abstracting groups of transformations.

There were three independent approaches:

1. Charles Ehresmann (1905–1979) in France.

2. Gordon B. Preston (1925–2015) in the UK.

3. Viktor V. Vagner (1908–1981) in the USSR.

They all three converge on the definition of ‘inverse semigroup’.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Example: the symmetric inverse monoid

Let X be a set equipped with the discrete topology. Denote by $I(X)$ the set of all partial bijections of X. This is an example of an inverse semigroup called the *symmetric inverse monoid*. If X is finite with n elements denote $I(X)$ by I_n.

Theorem [Vagner-Preston] *Symmetric inverse monoids are inverse, and every inverse semigroup can be embedded in a symmetric inverse monoid.*

The natural partial order

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation $\leq$ is a partial order with respect to which S is a partially ordered semigroup.*

It is called the *natural partial order*.

Example In symmetric inverse monoids the natural partial order is nothing other than the restriction ordering on partial bijections.

Let S be an inverse semigroup. Elements of the form $a^{-1}a$ and aa^{-1} are idempotents. Denote by $E(S)$ the set of idempotents of S.

Remarks

1. $E(S)$ is a commutative subsemigroup or *semilattice*.

2. $E(S)$ is an order ideal of S.

Observation Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A non-empty subset is said to be *compatible* if each pair of distinct elements in the set are compatible.

Example

The idempotents in $I(X)$ are the identity functions defined on the subsets of X. Denote them by 1_A, where $A \subseteq X$, called *partial identities*. Then

$$1_A \leq 1_B \Longleftrightarrow A \subseteq B$$

and

$$1_A 1_B = 1_{A \cap B}.$$

Thus the semilattice of idempotents on $I(X)$ is isomorphic to $\mathcal{P}(X)$.

Partial bijections f and g are compatible if and only if $f \cup g$ is a partial bijection.

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

- An inverse semigroup is a *meet-semigroup* if has has all binary meets.

Boolean inverse semigroups

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

Symmetric inverse monoids are Boolean.

Theorem [Paterson, Wehrung] *Let S be a subsemigroup of a ring with involution R such that S is an inverse semigroup with respect to the involution. Then there is a Boolean inverse semigroup T such that $S \subseteq T \subseteq R$.*

The above result is significant when viewing inverse semigroups in relation to C^*-algebras.

Theorem *Every inverse semigroup can be embedded in a universal Boolean inverse semigroup.*

Fundamental inverse semigroups

An inverse semigroup is *fundamental* if the only elements that centralize all idempotents are themselves idempotents. Example: symmetric inverse monoids are fundamental.

Theorem [Vagner] *An inverse semigroup is fundamental if and only if it is isomorphic to an inverse semigroup of partial homeomorphisms between the open subsets of a* T_0 *space where the domains of definition of the elements form a basis for the space.*

Fundamental inverse semigroups should therefore be viewed as inverse semigroups of partial homeomorphisms.

Each inverse semigroup is an extension of an inverse semigroup with central idempotents by a fundamental one; inverse semigroups with central idempotents are presheaves of groups.

0-**simplifying Boolean inverse monoids**

A *closed ideal* in a Boolean inverse monoid is an ideal closed under finite compatible joins.

A Boolean inverse monoid is 0-*simplifying* if it contains no non-trivial closed ideals. Example: symmetric inverse monoids are 0-simplifying.

Groupoids

We view categories as 1-sorted structures: everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as 'groups with many identities'.

Let G be a groupoid with set of identities G_o. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Proposition *The set of all local bisections of a groupoid forms a Boolean inverse meet monoid.*

4. Finite Boolean inverse monoids

Theorem

1. The finite 0-simplifying, fundamental Boolean inverse monoids are precisely the finite symmetric inverse monoids.

2. The finite fundamental Boolean inverse monoids are precisely the finite direct products of finite symmetric inverse monoids.

3. The finite Boolean inverse monoids are isomorphic to the inverse monoids of local bisections of finite discrete groupoids.

Remarks

1. Boolean inverse monoids should be viewed as non-commutative unital Boolean algebras.

2. We call finite fundamental Boolean inverse monoids *semisimple*. They have the form $I_{n_1} \times \ldots \times I_{n_r}$. *They are therefore the Boolean inverse monoid analogues of finite dimensional* C^**-algebras.*

3. The groups of units of finite, fundamental Boolean inverse monoids are finite direct products of finite symmetric groups. *This suggests that the groups of units of Boolean inverse monoids are likely to be interesting.*

5. Non-commutative Stone duality

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets.*

Etale groupoids therefore have a strong algebraic character.

A *Boolean space* is a compact Hausdorff space with a basis of clopen subsets.

A *Boolean groupoid* is an étale topological groupoid whose space of identities is a Boolean space.

If G is a Boolean groupoid denote by $\mathsf{KB}(G)$ the set of all compact-open local bisections.

If S is a Boolean inverse monoid denote by $\mathsf{G}(S)$ the set of ultrafilters of S.

Proposition

1. $\mathsf{KB}(G)$ is a Boolean inverse monoid.

2. $\mathsf{G}(S)$ is a Boolean groupoid.

Non-commutative Stone duality

Theorem

1. *Boolean inverse monoids are in duality with Boolean groupoids*

2. *(Countable) Boolean inverse meet-monoids are in duality with (second countable) Hausdorff Boolean groupoids.*

6. Examples

1. There is a family of Boolean inverse meet-monoids C_n, where $n \geq 2$, called *Cuntz inverse monoids* which are congruence-free and whose groups of units are the Thompson groups V_n. Their associated groupoids are the ones derived from Cuntz C^*-algebras.

2. We define a Boolean inverse monoid to be *AF* if it is a direct limit of semisimple inverse monoids. AF inverse monoids are fundamental Boolean inverse meet-monoids and their associated groupoids are the ones derived from AF C^*-algebras.

7. Some sample theorems

The ideas that follow were partly inspired by work of Matui.

- A topological groupoid G is *minimal* if every G-orbit is a dense subset of the space of identities.

- A topological groupoid is *effective* if $\mathrm{Iso}(G)^\circ$ is equal to the space of identities. Here $\mathrm{Iso}(G)$ is the union of the local groups.

We call the countable atomless Boolean algebra the *Tarski algebra*.

Under classical Stone duality the Tarski algebra corresponds to the Cantor space.

A *Tarski inverse monoid* is a countable Boolean inverse meet-monoid whose set of idempotents forms a Tarski algebra.

Theorem *There are bijective correspondences between the following three classes of structures.*

1. *Fundamental (*0*-simplifying) Tarski inverse monoids.*

2. *Second countable Hausdorff étale topological effective (minimal) groupoids with a Cantor space of identities.*

3. Cantor groups*: full countable (minimal) subgroups of the group of homeomorphisms of the Cantor space in which the support of each element is clopen.*

Theorem [*After* Krieger] *There is a bijective correspondence between the following two classes of structures.*

1. *AF Tarski inverse monoids.*

2. Ample groups*: locally finite Cantor groups in which the fixed-point set of each element is clopen.*

In lieu of a definition: MV algebras are to multiple-valued logic as Boolean algebras are to classical two-valued logic.

Denote by $S/\mathscr{J}$ the poset of principal ideals of S. If this is a lattice we say that S satisfies the *lattice condition*. The following is a semigroup version of a theorem of Mundici.

Theorem *Every countable MV algebra is isomorphic to a $S/\mathscr{J}$ where S is AF and satisfies the lattice condition.*

Wehrung (2015) has generalized this result to *arbitrary* MV algebras.

Example The direct limit of $I_1 \to I_2 \to I_4 \to I_8 \to \ldots$ is the *CAR inverse monoid* whose associated MV algebra is that of the dyadic rationals in $[0, 1]$.

8. Concluding remarks

- Inverse semigroup theory provides an abstract setting for connecting results from many different settings (group theory, étale groupoids, C^*-algebras, ...).

- Matui's recent work suggests a programme: classify Tarski inverse monoids.

- The connection with MV-algebras and so with multiple-valued logic raises the question of the *logical content* of the theory of Boolean inverse monoids and the implications of this for applications.