Non-commutative Stone dualities

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This talk is based on work carried out in collaboration with Johannes Kellendonk, Ganna Kudryavtseva, Daniel Lenz, Stuart Margolis, and Ben Steinberg.

Introduction

This talk will consist of four parts and a parting shot:

1. Two important definitions

2. Motivation

3. Statement of main theorems, idea of proofs and simple examples

4. Application to constructing Thompson groups

5. Envoi

1. Two important definitions

- Inverse semigroups

- Topological groupoids

A semigroup S is said to be *inverse* if for each $s \in S$ there exists a unique $s^{-1} \in S$ such that

$$s = ss^{-1}s \text{ and } s^{-1} = s^{-1}ss^{-1}.$$

An inverse semigroup S is equipped with two important relations.

$s \leq t$ is defined if and only if $s = te$ for some idempotent e. Despite appearances ambidextrous. Called the *natural partial order*. Compatible with multiplication.

$s \sim t$ if and only if st^{-1} and $s^{-1}t$ both idempotents. Called the *compatibility relation*. It controls when pairs of elements are *eligible* to have a join.

Examples Symmetric inverse monoids $I(X)$ and pseudogroups of transformations.

Intuitive idea Generalize groups replacing symmetries by partial symmetries.

A *groupoid* G is a (for us, small) category with every arrow invertible. The set of identities (or objects) of G is denoted by G_o. The 'o' stands for 'objects'.

Intuitive idea A groupoid is a 'group with many identities'.

If a groupoid G carries a topology making the multiplication and inversion continuous, it is called a *topological groupoid*.

The most important class of topological groupoids are the *étale groupoids*. We use Resende's characterization to define them.

A topological groupoid G is *étale* if G_o is an open set and the product of any two open sets in G is an open set.

N.B. Hausdorffness is not assumed.

2. Motivation

This work draws on a number of sources

- The idea of a 'non-commutative space'

- The 1980 book by Renault on the relationship between topological groupoids and C^*-algebras

- Paterson's 1999 book on groupoids, inverse semigroups and operator algebras

- Work in the late 90's by Johannes Kellendonk on tiling semigroups

- Birget's semigroup approach to constructing the Thompson groups

- The work of Charles Ehresmann

- Frame theory

I shall talk about some of these informally.

A major theme of contemporary mathematics is non-commutative geometry.

The basis for this is Gel'fand's theorem which describes commutative C^*-algebras as function spaces on locally compact spaces.

The idea is that a C^*-algebra should be regarded as a proxy for a non-commutative space.

But there are many examples of C^*-algebras where this space is actualized: it is a topological groupoid.

So, topological groupoids should be regarded as non-commutative spaces.

This program is particularly associated with the work of Jean Renault and his book

A groupoid approach to C^*-algebras, LNM 793, Springer, 1980.

However, although the ostensible theme is the relationship between topological groupoids and C^*-algebras, inverse semigroups appear throughout the book.

Well-known C^*-algebras, such as the Cuntz algebras, are shown to be closely related to certain inverse semigroups.

Passing from topological groupoids to inverse semigroups is easy.

A *local bisection* A of a groupoid G is a subset such that $A^{-1}A, AA^{-1} \subseteq G_o$. The set of all open local bisections forms an inverse semigroup, with extra properties.

Example Take the groupoid $X \times X$ with the discrete topology. Then the inverse semigroup of open local bisections is the symmetric inverse monoid on X.

This raises the question of the relationships between

inverse semigroups, topological groupoids and C^**-algebras*

with my research focusing on the first two.

What was implicit in Renault's book is explicit in the book by Alan Paterson

Groupoids inverse semgroups and their operator algebras, Birkhäuser, 1999.

Paterson described a, for me, complicated procedure for constructing topological groupoids from inverse semigroups.

Renault's work has been highly influential not least on Johannes Kellendonk:

The Local Structure of Tilings and their Integer Group of Coinvariants, *Commun. Math. Phys.* **187** (1997), 115–157.

Topological equivalence of tilings, *J. Math. Phys.* **38** (1997), 1823–1842.

His goal was to construct a C^*-algebra with every aperiodic tiling, such as a Penrose tiling and then compute invariants for the tiling from the K_0-group of the C^*-algebra.

But this involved constructing an inverse semigroup, the *tiling semigroup*, with any tiling. It can be viewed as the semigroup of *partial translational symmetries*.

He also showed how to construct a topological groupoid from a tiling semigroup.

His work at the algebraic level was extremely interesting.

Kellendonk's work directly influenced Daniel Lenz in his paper, available as a preprint from 2002:

On an order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.

This directly influenced my current research.

To lead you to an overwhelming question . . .

What is the exact nature of the relationship between inverse semigroups and topological groupoic

I shall answer this question providing a framework for all of the above work and, at the same time, providing connections with group theory.

3. Statement of main theorems, idea of proofs and simple examples

Lattices need not have 1's but always have 0's. If they have 1's they will be called *unital*.

Thus: distributive lattices vs. unital distributive lattices; Boolean algebras vs. unital Boolean algebras.

A *distributive inverse semigroup* is one which has joins of compatible pairs of elements and multiplication distributes over such joins.

A *Boolean inverse semigroup* is a distributive inverse semigroup with a Boolean algebra of idempotents.

A *Boolean inverse $\wedge$-semigroup* is a Boolean inverse semigroup with the additional property that all pairs of elements have a meet.

Let P be a poset with zero 0.

A subset $F \subseteq P$ is a *filter* if it is downwardly directed and upwardly closed.

It is *proper* if $0 \notin F$; all filters will be proper.

An *ultrafilter* is a maximal proper filter.

A filter F is *prime* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$.

A topological space is said to be *sober* if each point of the space is uniquely determined by the open sets that contain it (plus a bit more.)

A topological space X is said to be *spectral* if it is sober and has a basis of compact-open sets that is closed under finite non-empty intersections.

We do not assume that X is compact.

An étale groupoid is called *spectral* if its space of identities is a spectral space.

A étale groupoid is called *Boolean* if its space of identities is Boolean.

To avoid piling on definitions, *morphisms* will be kept in the background throughout this talk — they can be defined so that things work.

Classical theorems.

Theorem [Stone duality for distributive lattices] *The category of distributive lattices and their proper homomorphisms is dually equivalent to the category of spectral spaces and their coherent continuous maps.*

A Hausdorff spectral space is called a *Boolean space*.

Theorem [Stone duality for Boolean algebras] *The category of Boolean algebras and their proper homomorphisms is dually equivalent to the category of Boolean spaces and their coherent continuous maps.*

The starting point of our work.

Theorem [Stone duality for distributive inverse semigroups] *The category of distributive inverse semigroups is dually equivalent to the category of spectral groupoids.*

Theorem [Stone duality for Boolean inverse semigroups] *The category of Boolean inverse semigroups is dually equivalent to the category of Boolean groupoids.*

Theorem [Stone duality for Boolean inverse $\wedge$-semigroups] *The category of Boolean inverse $\wedge$-semigroups is dually equivalent to the category of Hausdorff Boolean groupoids.*

Proof sketch

Let G be a spectral groupoid.

The set of all compact-open local bisections of a spectral groupoid is a distributive inverse semigroup.

Let S be a distributive inverse semigroup.

Let P be a prime filter. Define $\mathbf{d}(P) = (P^{-1}P)^{\uparrow}$ and $\mathbf{r}(P) = (PP^{-1})^{\uparrow}$. Define the partial product $P \cdot Q$ to be $(PQ)^{\uparrow}$ iff $\mathbf{d}(P) = \mathbf{r}(P)$. In this way, the set of prime filters becomes a groupoid $\mathsf{G}_P(S)$.

Let $s \in S$. Define X_s to be the set of all prime filters that contain s. These sets form the basis of a topology on $\mathsf{G}_P(S)$.

Examples

Let G be a finite discrete groupoid. The set of all local bisections of G is a finite Boolean inverse $\wedge$-semigroup $I(G)$ and all finite inverse $\wedge$-semigroups are of this form.

Write $G = \bigsqcup_{i=1}^{m} G_i$ where the G_i are the connected components of G. Then

$$I(G) \cong I(G_1) \times \ldots \times I(G_m).$$

Let G be a finite connected discrete combinatorial groupoid and put $G_o = X$. Then $I(G) \cong I(X)$, a finite symmetric inverse monoid.

The fundamental finite Boolean inverse $\wedge$-semigroups are therefore of the form

$$I(X_1) \times \ldots \times I(X_m).$$

Call these *semisimple*.

May construct *AF inverse monoids* from Bratteli diagrams and injective morphisms between semisimple inverse monoids.

Constructing distributive inverse semigroups

Let S be an inverse semigroup. Let $a \in S$ and $b_1, \ldots, b_m \leq a$. We say that the set of elements $\{b_1, \ldots, b_m\}$ is a *(tight) cover* of a if for each $0 \neq x \leq a$ there exists b_i such that $0 \neq z \leq x, b_i$ for some z.

A *tight filter* is a filter A such that if $a \in A$ and $\{b_1, \ldots, b_m\}$ covers a then $b_i \in A$ for some i.

A semigroup homomorphism $\theta: S \to T$ to a distributive inverse semigroup is said to be a *tight map* if for each element $a \in S$ and tight cover $\{a_1, \ldots, a_n\}$ of a we have that $\theta(a) = \bigvee_{i=1}^{n} \theta(a_i)$.

Intuitive idea

The idea is to present distributive inverse semigroups by means of *generators and relations*.

The generating set is in fact an inverse semigroup S.

The relations are given by the tight covers —

if $\{b_1, \ldots, b_m\}$ is a *(tight) cover* of a, then THINK

$$a = \bigvee_{i=1}^{m} b_i.$$

Theorem [Tight completions] *Let S be an inverse semigroup.*

1. *There is a distributive inverse semigroup $\mathsf{D}_t(S)$ and a tight map $\delta\colon S \to \mathsf{D}_t(S)$ which is universal for tight maps from S to distributive inverse semigroups.*

2. *There is an order isomorphism between the poset of tight filters in S and the poset of prime filters in $\mathsf{D}_t(S)$ under which ultrafilters correspond to ultrafilters.*

We call the distributive inverse semigroup $\mathsf{D}_t(S)$ the *tight completion* of S.

If the tight completion of an inverse semigroup is actually Boolean we say that the semigroup is *pre-Boolean*.

It can be proved that every ultrafilter is a tight filters.

Theorem *An inverse semigroup is pre-Boolean if and only if every tight filter is an ultrafilter.*

4. Application to constructing the Thompson groups

The *polycyclic monoid* P_n, where $n \geq 2$, is defined as a monoid with zero generated by the variables $a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}$ subject to the relations

$$a_i^{-1} a_i = 1 \text{ and } a_i^{-1} a_j = 0, i \neq j.$$

Every non-zero element of P_n is of the form yx^{-1} where x and y are elements of the *free monoid* on $\{a_1, \ldots, a_n\}$.

The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable in which case

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some } z \end{cases}$$

The polycyclic monoid P_n is a pre-Boolean inverse monoid.

The set $\{a_1 a_1^{-1}, \ldots, a_n a_n^{-1}\}$ is a tight cover of the identity, and in some sense, determines all other tight covers.

Theorem The Boolean completion of P_n is called (here) the *Cuntz inverse monoid* C_n.

1. This monoid is congruence-free.

2. Its group of units is the Thompson group $V_{n,1}$.

3. Its associated groupoid is the groupoid also associated with the Cuntz C^*-algebra C_n.

- All Thompson-Higman groups $V_{n,r}$ can be constructed in a similar way.

- Self-similar groups actions give rise to generalizations of the polycyclic inverse monoids which are also pre-Boolean.

- Finite directed graphs can be used to construct pre-Boolean inverse semigroups.

- AF inverse monoids are generated by pre-Boolean inverse monoids.

- O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, (1999) is, in fact, a study of tight maps from P_n to $I(X)$.

Our theory can be used to construct interesting groups of the Thompson-Higman variety.

Intuitively, the elements of the group are obtained by *glueing together partial bijections*.

Thus our theory can be used to construct interesting groups from inverse semigroups.

Envoi

A *frame* is a complete infinitely distributive lattice.

The open subsets of a topological space form a frame.

$$\text{points} \longrightarrow \text{topological spaces}$$

$$\text{open sets} \longrightarrow \text{frames}$$

Main idea

Inverse semigroup theory should be viewed as (part of) non-commutative frame theory. This approach provides natural connections with the theories of topoi, quantales and C^*-algebras.

Edinburgh workshop on semigroup representations 2013

When? 10th April to 12th April

Where? ICMS Edinburgh

Who? Ruy Exel, Johannes Kellendonk, Daniel Lenz, Walter Mazorchuk *amongst others*

How? Please register via the ICMS website