3. Non-commutative Stone duality

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I shall assume familiarity with Lecture 2.

Recall that a Boolean inverse monoid is an inverse monoid with all binary compatible joins, multiplication distributes over such joins, and the semilattice of idempotents forms a Boolean algebra with respect to the usual order on the set of idempotents.

1. Commutative (= Classical) Stone duality

This was developed by Marshall Stone in 1936.

He showed that Boolean algebras could be described in topological terms. We shall assume identities but Stone extended the theory to 'generalized Boolean algebras'. We can also do the same for semigroups as opposed to monoids.

The basic concept we shall need is that of an ultrafilter. This will also be used in our generalization.

Let B be a Boolean algebra. A subset F of a Boolean algebra B is called a *filter* if it satisfies the following conditions:

1. $1 \in F$.

2. If $a, b \in F$ then $a \wedge b \in F$.

3. If $a \in F$ and $a \leq b$ then $b \in F$.

The filter F is said to be *proper* if $0 \notin F$.

A proper filter F is said to be *prime* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$.

A maximal proper filter is called an *ultrafilter*.

Lemma *The following are equivalent for a filter F in a Boolean algebra B:*

1. *F is an ultrafilter.*

2. *For each non-zero $a \in B$ either $a \in F$ or $\bar{a} \in F$.*

3. *F is a prime filter.*

The proofs of the following require Zorn's Lemma

Lemma *Every non-zero element of a Boolean algebra is contained in an ultrafilter.*

The following result tells us that we have enough ultrafilters.

Lemma *Let $a \nleq b$ in a Boolean algebra. Then there is an ultrafilter that contains a but omits b.*

A topological space is said to be 0-*dimensional* if it has a base of clopen sets.

A compact Hausdorff space which is 0-dimensional is called a *Boolean space*.

Proposition *The clopen subsets of a Boolean space form a Boolean algebra.*

If X is a Boolean space, we denote the Boolean algebra of clopen subsets of B by $\mathrm{B}(X)$.

Let B be a Boolean algebra. Define $\mathsf{X}(B)$ to be the set of ultrafilters on B.

If $a \in B$ denote by V_a the set of ultrafilters containing a.

Lemma *Let B be a Boolean algebra.*

1. $V_0 = \varnothing$.
2. $V_1 = \mathsf{X}(B)$.
3. $V_a \cap V_b = V_{a \wedge b}$.
4. $V_a \cup V_b = V_{a \vee b}$.
5. $V_{\bar{a}} = \overline{V_a}$.

Define a topology σ on $\mathsf{X}(B)$ whose open sets are unions of the sets of the form V_a.

Proposition *For each Boolean algebra B the space $(\mathsf{X}(B), \sigma)$ is Boolean.*

The topological space $\mathsf{X}(B)$ is called the *Stone space* of the Boolean algebra B.

Theorem

1. *Let* B *be a Boolean algebra. Then* $B \cong \mathsf{BX}(B)$.

2. *Let* S *be a Boolean space. Then* $S \cong \mathsf{XB}(S)$.

Examples

1. Let B be a finite Boolean algebra. Then each ultrafilter is determined by an atom. The Stone space of B is then simply the finite set of atoms equipped with the discrete topology.

2. Tarski proved that any two atomless, countably infinite Boolean algebras are isomorphic. We call any atomless, countably infinite Boolean algebra a *Tarski algebra*. The Stone space of the Tarski algebra is a second-countable, 0-dimensional, compact Hausdorff space with no isolated points; such a space is homeomorphic to the *Cantor space*.

It remains to say a few words about maps. ‘Duality’ means that maps are reversed on the nose.

Geometry = algebra through the looking glass.

Let B be a Boolean algebra. Then there is a bijective map between the ultrafilters in B and the Boolean algebra homomorphisms from B to $\mathbb{B}$, the 2-element Boolean algebra.

This bijection associates with the ultrafilter F its characteristic function χ_F.

Let $\theta\colon B_1 \to B_2$ be a homomorphism between Boolean algebras. Let $F \in \mathsf{X}(B_2)$ be an ultrafilter.

Then $\chi_F\theta$ is the characteristic function of an ultrafilter in B_2.

In this way, we can map homomomomorphisms $B_1 \to B_2$ to continuous functions $\mathsf{X}(B_1) \leftarrow \mathsf{X}(B_2)$ with a consequent reversal of arrows.

In the other direction, let $\phi\colon X_1 \to X_2$ be a continuous function. Then ϕ^{-1} maps clopen sets to clopen sets. In this way, we can map continuous functions $X_1 \to X_2$ to homomorphisms $\mathsf{B}(X_1) \leftarrow \mathsf{B}(X_2)$.

Because maps are reversed, we have the following: *the category of Boolean algebras is dually equivalent to the category of Boolean spaces*.

Our program(me)

We shall generalize the above to a non-commutative setting:

Boolean algebras $\to$ Boolean inverse monoids

topological spaces $\to$ topological groupoids

One can (but I won't here) replace monoids by semigroups (which means that compact is replaced by locally compact) and analogous results can be proved for distributive inverse semigroups.

The correct setting for all of the above is a dual adjunction linking pseudogroups and étale groupoids.

2. Non-commutative Stone duality: Boolean groupoids

A *groupoid* G is a (for us, small) category with every arrow invertible.

The set of identities of G is denoted by G_o. The 'o' stands for 'objects'.

If a groupoid G carries a topology making the multiplication $\mathbf{m}$ and inversion $\mathbf{i}$ continuous, it is called a *topological groupoid*.

The most important class of topological groupoids are the *étale groupoids*.

These are the topological groupoids in which $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms.

Resende's characterization of étale groupoids below explains why they are so important: their topology forms a monoid. They therefore have algebraic alter egos.

Proposition *A topological groupoid G is étale if and only if G_o is an open set and the product of any two open sets in G is an open set.*

Key definition

An étale groupoid G is called *Boolean* if its identity space G_o is a Boolean space.

Passing from Boolean groupoids to Boolean inverse monoids is easy.

Proposition *Let G be a Boolean groupoid. Denote by* $\mathsf{KB}(G)$ *the set of all compact-open local bisections of G. Then* $\mathsf{KB}(G)$ *is a Boolean inverse monoid.*

3. Non-commutative Stone duality: Boolean inverse monoids

Let P be a poset with zero 0.

If $X \subseteq P$, define

$$X^{\downarrow} = \{y \in P \colon y \leq x \text{ if } x \in X\}$$

and

$$X^{\uparrow} = \{y \in P \colon x \leq y \text{ if } x \in X\}.$$

If for any $x, y \in X$ there exists $z \in X$ such that $z \leq x, y$, we say that X is *downwardly directed*. If $X = X^{\uparrow}$ we say that X is *upwardly closed*.

A subset $F \subseteq P$ is a *filter* if it is downwardly directed and upwardly closed.

It is *proper* if $0 \notin F$.

An *ultrafilter* is a maximal proper filter.

A proper filter F is *prime* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$.

The following little result is important.

A filter in an inverse semigroup is said to be *idempotent* if it contains an idempotent.

Lemma *A filter is idempotent if and only if it is an inverse subsemigroup.*

Proposition *In a Boolean inverse monoid prime filters and ultrafilters are the same.*

If A is a prime filter in a Boolean inverse monoid, define

$$\mathbf{d}(A) = (A^{-1}A)^{\uparrow} \text{ and } \mathbf{r}(A) = (AA^{-1})^{\uparrow}$$

both are prime filters and both are idempotent.

Prime filters in Boolean inverse monoids look a little like cosets.

If A is a prime filter and $a \in A$ then

$$A = (a\mathbf{d}(A))^{\uparrow}.$$

Let S be a Boolean inverse monoid. Denote the set of prime filters containing a by V_a.

Lemma *Let S be a Boolean inverse monoid.*

1. $V_0 = \varnothing$.
2. $V_a^{-1} = V_{a^{-1}}$.
3. $V_a \subseteq V_b$ *if and only if* $a \leq b$.
4. $V_a \cap V_b = \bigcup_{c \leq a,b} V_c$.
5. $V_a \cup V_b = V_{a \vee b}$ *if* $a \sim b$.
6. $V_a V_b = V_{ab}$.

Let S be a Boolean inverse monoid. Denote by $\mathsf{G}(S)$ the set of all prime filters on S.

Define a partial binary operation $\cdot$ on $\mathsf{G}(S)$ as follows:

$$A \cdot B = (AB)^{\uparrow}$$

if and only if $\mathbf{d}(A) = \mathbf{r}(B)$.

Lemma $\mathsf{G}(S)$ *is a groupoid whose identities are the idempotent prime filters.*

Let S be a Boolean inverse monoid. Let σ be the topology on $\mathsf{G}(S)$ with basis the set V_a where $a \in S$.

Theorem *Let S be a Boolean inverse monoid. Then $\mathsf{G}(S)$ is a Boolean groupoid.*

We call $\mathsf{G}(S)$ the *Stone groupoid* of S.

Lemma *Let S be a Boolean inverse monoid. Then for each $a \in S$, we have that V_a is a compact-open local bisection of $\mathsf{G}(S)$.*

4. Non-commutative Stone duality: isomorphism theorems

Theorem

1. *Let S be a Boolean inverse monoid. Then $S \cong \mathsf{KB}(\mathsf{G}(S))$.*

2. *Let G be a Boolean groupoid. Then $G \cong \mathsf{G}(\mathsf{KB}(G))$.*

5. Non-commutative Stone duality: maps

A morphism $\theta\colon S \to T$ of Boolean inverse monoids is called *callitic* if it is weakly-meet-preserving and proper (which means that each element in T is a join of elements in the image).

A continuous functor $\alpha\colon G \to H$ of topological groupoids is said to be *coherent* if the inverse images of compact sets are compact.

The following is the full statement of non-commutative Stone duality.

Theorem *The category of Boolean inverse monoids and callitic morphisms is dually equivalent to the category of Boolean groupoids and coherent, continuous, covering functors.*

6. Non-commutative Stone duality: refinements

I do not have time to go into details, so I shall simply summarize the most important results in the following table:

Boolean inverse monoid	**Boolean groupoid**
Meet-monoid	Hausdorff
Fundamental	Effective
Tarski algebra of idempotents	Cantor space of identities
0-simplifying	Minimal
0-simple	Minimal and purely infinite
Group of units	Topological full group
Finite	Discrete
Basic inverse meet-monoids	Hausdorff principal
Countable	Second countable

Papers

1. M. V. Lawson, A noncommutative generalization of Stone duality, *J. Aust. Math. Soc.* **88** (2010), 385–404.

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3. Mark V. Lawson, Daniel H. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170.

4. Mark V. Lawson, Subgroups of the group of homeomorphisms of the Cantor space and a duality between a class of inverse monoids and a class of Hausdorff étale groupoids, **462** (2016), 77–114.

5. Pedro Resende, *Lectures on étale groupoids, inverse semigroups and quantales*, Lecture Notes for the GAMAP IP Meeting, Antwerp, 4–18 September, 2006, 115 pp.

END OF LECTURE 3