1. Inverse semigroups

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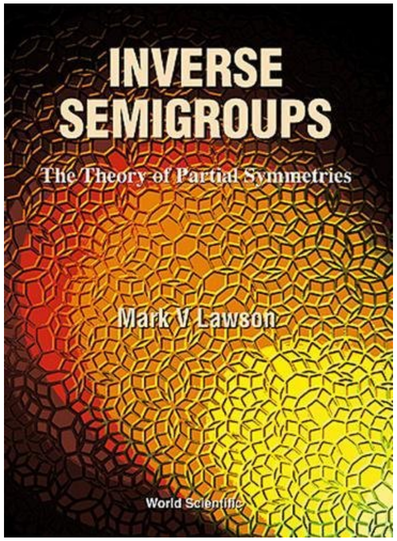

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Please email me with any questions or comments.

With the collaboration of

John Fountain, Peter Hines, Anja Kudryavtseva, Johannes Kellendonk, Daniel Lenz, Stuart Margolis, Pedro Resende, Phil Scott, Aidan Sims, Ben Steinberg, Alina Vdovina and Alistair Wallis.

and developing ideas to be found in (amongst others)

Jean Renault, Alexander Kumjian, Alan Paterson, Ruy Exel, Hiroki Matui, . . .

I shall deliver four linked talks:

1. Inverse semigroups.

2. Boolean inverse monoids.

3. Non-commutative Stone duality.

4. MV-algebras and Boolean inverse monoids (joint work with Phil Scott).

In today's talk, I will introduce inverse semigroups from scratch and also touch on the theory of groupoids (which are related to the theory of inverse semigroups).

If you want to see proofs of some of the unproved assertions I make today, please read my

Primer on inverse semigroups I, arXiv:2006.01628.

On the one hand, inverse semigroups form an important class of semigroups, but on the other hand, they have become important in the theory of C^*-algebras.

Inverse semigroups generalize groups; essentially, bijections are replaced by partial bijections — thus symmetries are replaced by partial symmetries.

Some terminology

A *semigroup* is a set equipped with an associative binary operation.

A *monoid* is a semigroup with an identity. Every monoid S has a group of units $\mathsf{U}(S)$.

We shall also need the idea of a semigroup with zero (for which there is no special term).

An *idempotent* in a semigroup is an element e such that $e^2 = e$.

Distinguish between *functions* and *partial functions*.

If $A \subseteq X$, denote the identity function defined on A by 1_A.

We use $\wedge$ to mean *meets*, and $\vee$ to mean *joins*.

An *ideal* I in a semigroup S is a subset such that $SI, IS \subseteq I$. Ideals in semigroups of the form $SaS \cup \{a\}$ are called *principal ideals*.

1. Symmetric inverse monoids

If you understand these examples then everything that follows will make sense. They are special kinds of pseudogroups of transformations but without the topological luggage.

Let X be a non-empty set. The *symmetric inverse monoid on* X, denoted by $\mathcal{I}(X)$, is the set of all partial bijections of X equipped with the binary operation of composition of partial functions.

If X is finite and has n elements then we usually denote $\mathcal{I}(X)$ by $\mathcal{I}_n$.

To make things concrete, let's consider the case of $\mathcal{I}_5$, the set of all partial bijections of the set $X = \{1, 2, 3, 4, 5\}$.

We can represent elements of $\mathcal{I}_5$ using two-row form. For example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix}$$

is the partial bijection of the set X with domain of definition $\{3, 4\}$, with image $\{1, 5\}$ and which does the following: $3 \mapsto 5$ and $4 \mapsto 1$.

The identity is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

and the zero is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & - & - & - \end{pmatrix}$$

We also have 'inverses'

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & - & - & - & 3 \end{pmatrix}$$

If you compose (from right-to-left) a partial bijection with its inverse you get an idempotent, but not usually the identity

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 3 & 4 & - \end{pmatrix}$$

which is the identity function on the set $\{3, 4\}$, and

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & - & - & - & 5 \end{pmatrix}$$

which is the identity function on the set $\{1, 5\}$.

There is a partial order on partial bijections

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & - & - \end{pmatrix} \subseteq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix} \subseteq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & 2 \end{pmatrix}$$

But we cannot always take unions of partial bijections and get partial bijections.

The union

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix} \cup \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 4 & 1 & - \end{pmatrix}$$

is not a partial bijection because $3 \mapsto \{4,5\}$ and so the union is a binary relation but not a partial bijection.

The union

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & 5 & 1 & - \end{pmatrix} \cup \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & - & 5 & - & - \end{pmatrix}$$

is not a partial bijection because $\{1,4\} \mapsto 1$ and so the union is a binary relation but not a partial bijection.

2. Definition of inverse semigroups

Inverse semigroups arose by abstracting pseudogroups of transformations (see below) — and so, in particular, symmetric inverse monoids — in the same way that groups arose by abstracting groups of transformations.

There were three independent approaches:

1. Charles Ehresmann (1905–1979) in France.

2. Gordon B. Preston (1925–2015) in the UK.

3. Viktor V. Wagner (1908–1981) in the USSR.

All three converge on the definition of ‘inverse semigroup’.

Definition. A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Observe that $a^{-1}a$ and aa^{-1} are idempotents.

Define $\mathbf{d}(a) = a^{-1}a$, which we call the *domain* of a, and $\mathbf{r}(a) = aa^{-1}$, which we call the *range* of a.

Even in the monoid case, there is no reason for these idempotents to equal the identity.

PICTURE

$$\mathbf{r}(a) \xleftarrow{a} \mathbf{d}(a)$$

We also write $\mathbf{d}(a) \mathscr{D} \mathbf{r}(a)$.

Think of the elements of an inverse semigroup as abstract partial bijections.

The following theorem is basic. It was proved by Liber in the USSR, and Munn & Penrose in the UK at pretty much the same time. Penrose is *the* Penrose.

Theorem *In an inverse semigroup, the idempotents commute with each other.*

For those of you with a ring theory background, it is important to stress that the idempotents in an inverse semigroup are not central, in general. When they are, the inverse semigroup can be described as a 'presheaf of groups'.

We denote the set of idempotents of S by $\mathsf{E}(S)$. This set plays an important rôle in the study of inverse semigroups.

Define a partial order on $\mathsf{E}(S)$ by $e \leq f$ if and only if $e = ef (= fe)$. Observe that $ef = e \wedge f$, the greatest lower bound of e and f. In this way, $\mathsf{E}(S)$ becomes a *(meet) semilattice*.

For this reason, the set $\mathsf{E}(S)$ is often called the *semilattice of idempotents* of S.

Inverses behave as you would expect them to behave.

Lemma *Let S be an inverse semigroup.*

1. $(s^{-1})^{-1} = s$.

2. $(st)^{-1} = t^{-1}s^{-1}$.

3. *If e is an idempotent then ses^{-1} is an idempotent.*

Examples

1. Groups are inverse semigroups with a unique idempotent, and all inverse semigroups with a unique idempotent are groups.

2. Meet semilattices are inverse semigroups in which every element is an idempotent, and all inverse semigroups in which every element is an idempotent are meet semilattices.

3. Pseudogroups are inverse semigroups (which is why inverse semigroups were defined in the first place).

As an aside, let me say a little about pseudogroups of transformations. Symmetric inverse monoids are special cases.

Let X be a topological space. A *pseudogroup of transformations on* X is a collection Γ of homeomorphisms between the open subsets of X (called *partial homeomorphisms*) such that the following four properties hold:

1. Γ is closed under composition.

2. Γ is closed under inverses.

3. Γ contains all the identity functions on the open subsets.

4. Γ is closed under arbitrary non-empty unions when those unions are partial bijections.

- Pseudogroups are important in the foundations of geometry; you might have come across them in an introductory course on differential geometry.

- The idempotents in Γ are precisely the identity functions on the open subsets of the topological space. They form a complete, infinitely distributive lattice or *frame*.

- The term *pseudogroup* harks back to the days when partial functions were not clearly demarcated from functions. Pseudogroups were like groups but not actually groups — whence the unfortunate prefix 'pseudo'.

- Johnstone on the origins of frame theory:

  > *It was Ehresmann . . . and his student Bénabou . . . who first took the decisive step in regarding complete Heyting algebras as 'generalized topological spaces'*.

  However, Johnstone does not say *why* Ehresmann was led to his frame-theoretic viewpoint of topological spaces. The reason was pseudogroups.

- Pseudogroups are usually replaced by their *groupoids of germs* but pseudogroups nevertheless persist. The algebraic part of pseudogroup theory became inverse semigroup theory.

If T is an inverse semigroup and $S \subseteq T$ is a subset closed under the binary operation and inverses then we say that S is an *inverse subsemigroup* of T.

Homomorphisms between inverse semigroups are just semigroup homomorphisms.

Monoid homomorphisms between inverse monoids map the identity to the identity.

If both semigroups have a zero, we expect homomorphisms to map the zero to the zero.

Lemma *Let $\theta\colon S \to T$ be a homomorphism between inverse semigroups.*

1. $\theta(s^{-1}) = \theta(s)^{-1}$.

2. *If e is an idempotent then $\theta(e)$ is an idempotent.*

3. *If $\theta(s)$ is an idempotent then there is an idempotent $e \in S$ such that $\theta(s) = \theta(e)$.*

4. *The image of θ is an inverse subsemigroup of T.*

5. *If U is an inverse subsemigroup of T then $\theta^{-1}(U)$ is an inverse subsemigroup of S.*

We know that groups are the abstract version of symmetric groups because Cayley's theorem tells us that every group can be embedded in a symmetric group.

There is an analogue of Cayley's theorem for inverse semigroups. It is called the *Wagner-Preston representation theorem*.

Theorem *Every inverse semigroup can be embedded into a symmetric inverse monoid.*

3. The natural partial order

So far, we have dealt with the algebraic structure on symmetric inverse inverse monoids: the binary operation and inverses.

But there is also a partial order $f \subseteq g$ on the elements of $\mathcal{I}(X)$.

Remarkably, this can be defined *purely algebraically*: $f \subseteq g$ if and only if $f = gf^{-1}f$. This serves as the basis for the following definition.

Let S be an inverse semigroup. Define $s \leq t$ if and only if $s = ts^{-1}s$.

The following results show us that the definition of the above relation is ambidextrous.

Lemma *Let S be an inverse semigroup. Then the following assertions are equivalent:*

1. $s \leq t$.
2. $s = te$ *for some idempotent* e.
3. $s = ft$ *for some idempotent* f.
4. $s = ss^{-1}t$.

Proposition *Let S be an inverse semigroup.*

1. *The relation $\leq$ is a partial order.*

2. *If $s \leq t$ then $s^{-1} \leq t^{-1}$.*

3. *If $a \leq b$ and $c \leq d$ then $ac \leq bd$.*

4. *If e is an idempotent and $a \leq e$ then a is an idempotent.*

5. *The restriction of $\leq$ to the semilattice of idempotents is the usual order there.*

We call $\leq$ the *natural partial order*.

Let me stress that the natural partial order is defined internally using only the algebraic structure and is not imposed from outside.

Every inverse semigroup is partially ordered with respect to the natural partial order.

The natural partial order is usually interesting:

Lemma *The natural partial order on an inverse semigroup S is the equality relation if and only if S is a group.*

An inverse monoid S is said to be *factorizable* if for each $s \in S$ there exists a unit g such that $s \leq g$.

Lemma *The symmetric inverse monoid $\mathcal{I}(X)$ is factorizable if and only if X is finite.*

Don't think that the study of factorizable inverse monoids can be reduced to that of groups; we do not say that the unit above is unique.

4. The compatibility relation

We begin with an observation.

Suppose that $a, b \leq c$; that is, a and b have an upper bound.

Then $a^{-1} \leq c^{-1}$ and $b^{-1} \leq c^{-1}$.

Thus $a^{-1}b \leq c^{-1}c$ and $ab^{-1} \leq cc^{-1}$.

It follows that $a^{-1}b$ and ab^{-1} are idempotents; that is, a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} are idempotents.

There are no preconditions for the existence of meets.

Define $a \sim b$ if and only if $a^{-1}b$ and ab^{-1} are idempotents. We call this the *compatibility relation*. If $a \sim b$ we say that a and b are *compatible*.

A subset of an inverse semigroup is said to be *compatible* if each pair of elements in the subset is compatible.

Lemma *Let S be an inverse semigroup. If $s \sim t$ then $s \wedge t$ exists and $\mathbf{d}(s \wedge t) = \mathbf{d}(s) \wedge \mathbf{d}(t)$ and $\mathbf{r}(s \wedge t) = \mathbf{r}(s) \wedge \mathbf{r}(t)$.*

The compatibility relation is reflexive and symmetric. It is not always transitive. Define an inverse semigroup to be *E-unitary* if $e \leq a$, where e is an idempotent, implies that a is an idempotent.

Proposition *Let S be an inverse semigroup. Then the compatibility relation is transitive if and only if S is E-unitary.*

In lieu of a quiz

The following concepts are important:

- Inverse semigroups.
- The semilattice of idempotents.
- The natural partial order.
- The compatibility relation.

5. Fundamental inverse semigroups

We shall now single out an important class of inverse semigroups. We begin with a construction method.

Let $(P, \leq)$ be a partially ordered set. If $x \in P$ define $x^{\downarrow}$ to be the set of all $y \in P$ such that $y \leq x$. We call $x^{\downarrow}$ a *principal order-ideal* of P.

Let E be a (meet) semilattice. Define T_E to be the set of all order-isomorphisms between the principal order-ideals of E. This is an inverse subsemigroup of $\mathcal{I}(E)$. It is called the *Munn semigroup* of the semilattice E.

Let $\theta\colon S \to T$ be a homomorphism between inverse semigroups. The restriction of θ to $\mathsf{E}(S)$ induces a homomorphism from $\mathsf{E}(S)$ to $\mathsf{E}(T)$. If this restricted homomorphism is injective we say that θ is *idempotent-separating*.

Let S be an inverse subsemigroup of the inverse semigroup T. If $\mathsf{E}(T) = \mathsf{E}(S)$ we say that S is a *wide* inverse subsemigroup. The following is the *Munn representation theorem.*

Theorem *Let S be an inverse semigroup. Then there is an idempotent-separating homomorphism $\delta\colon S \to T_{\mathsf{E(S)}}$ whose image is a wide inverse subsemigroup.*

An inverse semigroup S is said to be *fundamental* if and only if the only elements that commute with all idempotents are themselves idempotents.

Theorem *Let S be an inverse semigroup. Then S is fundamental if and only if it is isomorphic to a wide inverse subsemigroup of the Munn semigroup $T_{\mathsf{E}(S)}$.*

Fundamental inverse semigroups really are 'fundamental'.

6. Groupoids

Groupoids play a starring rôle in our non-commutative Stone duality.

They arise naturally as soon as inverse semigroups are studied.

It is also true that inverse semigroups arise naturally as soon as groupoids are studied. See:

Jean Renault, *A groupoid approach to C^*-algebras*, Springer-Verlag, Lecture Notes in Mathematics 793, 1980.

A category is usually regarded as a *category of structures* of some kind, such as the category of sets or the category of groups.

A (small) category can, however, also be regarded as an algebraic structure; that is, as a set equipped with a partially defined binary operation satisfying certain axioms.

We shall need both perspectives.

We view categories as 1-sorted structures (over sets): everything is an arrow. Objects are identified with identity arrows.

To define the algebraic notion of a category, we begin with a set C equipped with a partial binary operation.

We write $\exists ab$ to mean that the product ab is defined.

An *identity* in such a structure is an element e such that if $\exists ae$ then $ae = a$ and if $\exists ea$ then $ea = a$.

The set of identities of C is denoted by C_o.

A *category* is a set equipped with a partial binary operation satisfying the following axioms:

(C1) $\exists a(bc)$ if and only if $\exists(ab)c$ and when one is defined so is the other and they are equal.

(C2) $\exists abc$ if and only if $\exists ab$ and $\exists bc$.

(C3) For each a there is an identity e, perforce unique, such that $\exists ae$, and there exists an identity f, perforce unique, such that $\exists fa$.

You can check that $\exists ab$ if and only if $\mathbf{d}(a) = \mathbf{r}(b)$.

Think of categories as graphs with a multiplication between edges that abut.

A category with one identity is a monoid; thus, categories are *monoids with many identities*.

Homomorphisms between categories are called *functors*.

An element a of a category is said to be *invertible* if there exists an element b such that ab and ba are identities.

If such an element b exists it is unique and is called the *inverse* of a; we denote the inverse of a when it exists by a^{-1}.

Definition. A category in which every element is invertible is called a *groupoid*. This is the category theory meaning, not the meaning in universal algebra.

Groupoids are useful everywhere. The suffix 'oid' is just as unfortunate as the prefix 'pseudo'. Groups get preferential treatment . . .

We shall need the following notation for the maps involved in defining a groupoid (not entirely standard).

Define $\mathbf{d}(g) = g^{-1}g$ and $\mathbf{r}(g) = gg^{-1}$.

Put

$$G * G = \{(g,h) \in G \times G \colon \mathbf{d}(g) = \mathbf{r}(h)\}.$$

Define $\mathbf{m}\colon G * G \to G$ by $(g,h) \mapsto gh$, the multiplication map, and $\mathbf{i}\colon G \to G$ by $g \mapsto g^{-1}$, the inversion map.

PICTURE

$$\mathbf{r}(g) \xleftarrow{g} \mathbf{d}(g)$$

Examples

1. Groups can be regarded as groupoids. A groupoid with one identity is a group; thus, groupoids are *groups with many identities*.

2. Sets can be regarded as a groupoids. They are the groupoids in which every element is an identity.

3. Equivalence relations can be regarded as groupoids. They correspond to *principal groupoids*; that is, those groupoids in which given any identities e and f there is at most one element g of the groupoid such that $f \xleftarrow{g} e$. A special case of such groupoids are the *pair groupoids*, $X \times X$, which correspond to equivalence relations having exactly one equivalence class.

4. Group actions can be regarded as groupoids. Let $G \times X \to X$ be a left group action of the group G on the set X. We can construct a groupoid $G \ltimes X$, called a *transformation groupoid*. The groupoid represents the pictures you would naturally draw when thinking about group actions.

We now show how to construct all groupoids.

Let G be a groupoid. We say that elements $g, h \in G$ are *connected*, denoted $g \equiv h$, if there is an element $x \in G$ such that $\mathbf{d}(x) = \mathbf{d}(h)$ and $\mathbf{r}(x) = \mathbf{d}(g)$. The $\equiv$-equivalence classes are called the *connected components of the groupoid*. If $\exists gh$ then necessarily $g \equiv h$.

It follows that $G = \coprod_{i \in I} G_i$, the disjoint union, where the G_i are the connected components of G.

It remains to describe the structure of all connected groupoids. Let X be a non-empty set and let H be a group. The set of triples $X \times H \times X$ becomes a groupoid when we define $(x, h, x')(x', h', x'') = (x, hh', x'')$ and $(x, h, y)^{-1} = (y, h^{-1}, x)$. It is easy to check that $X \times H \times X$ is a connected groupoid. All connected groupoids are isomorphic to groupoids of this type.

We now describe a special case. Let X be any non-empty set. Then $X \times X$ is a connected groupoid when we define $\mathbf{d}(x, y) = (y, y)$, $\mathbf{r}(x, y) = (x, x)$, $(x, y)^{-1} = (y, x)$ and $(x, y)(y, z) = (x, z)$. We call this the *pair groupoid*.

We shall need to following definition in Lecture 3.

For each identity e in a groupoid, define the set L_e of all elements g such that $e = \mathbf{d}(g)$.

Let $\alpha\colon G \to H$ be a functor. For each identity e in G, the functor α induces a map from L_e to $L_{\alpha(e)}$. If these restricted maps are bijections for all identities e we say that α is a *covering functor*.

Groupoids and inverse semigroups

Let G be a groupoid. A subset $A \subseteq G$ is called a *partial bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Lemma *Let G be a groupoid. A subset $A \subseteq G$ is a partial bisection if and only if $a, b \in A$ and $\mathbf{d}(a) = \mathbf{d}(b)$ implies that $a = b$ and $\mathbf{r}(a) = \mathbf{r}(b)$ implies that $a = b$.*

Proposition *The set of all partial bisections of a groupoid forms an inverse monoid under subset multiplication.*

A subset $A \subseteq G$ of a groupoid is called a *bisection* if

$$A^{-1}A, AA^{-1} = G_o.$$

Corollary *The set of bisections of a groupoid forms a group which is the group of units of the inverse monoid of all partial bisections of that groupoid.*

The inverse monoid of partial bisections of the pair groupoid $X \times X$ is isomorphic with the symmetric inverse monoid on X, and the group of bisections of the pair groupoid $X \times X$ is isomorphic with the symmetric group on X.

Let S be an inverse semigroup. Define the *restricted product* $a \cdot b$ to be equal to ab when $\mathbf{d}(a) = \mathbf{r}(b)$, and undefined otherwise.

Proposition *An inverse semigroup with respect to its restricted product is a groupoid in which the identities are the idempotents.*

Books

I first learnt about inverse semigroups from Chapter V of

J. M. Howie, *An introduction to semigroup theory*, Academic Press, 1976.

There are two books exclusively dedicated to inverse semigroups:

Mario Petrich, *Inverse semigroups*, John Wiley & Sons, 1984

and

Mark V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific, 1998.

END OF LECTURE 1