# 1. CLASSICAL STONE DUALITY

MARK V. LAWSON

Abstract. In this article, we explain classical Stone duality.

## 1. Boolean algebras

Boolean algebras may not rank highly in the pantheon of algebraic structures but they are, in fact, both mathematically interesting and remarkably useful. Formally, a *Boolean algebra* is a 6-tuple $(B, \vee, \wedge, {}' , 0, 1)$ consisting of a set $B$, two binary operations $\vee$, called *join*, and $\wedge$, called *meet*, one unary operation $a \mapsto a'$, and two constants 0 and 1 satisfying the following ten axioms:

(B1): $(x \vee y) \vee z = x \vee (y \vee z)$.
(B2): $x \vee y = y \vee x$.
(B3): $x \vee 0 = x$.
(B4): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
(B5): $x \wedge y = y \wedge x$.
(B6): $x \wedge 1 = x$.
(B7): $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(B8): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
(B9): $x \vee x' = 1$.
(B10): $x \wedge x' = 0$.

The following lemma summarizes some important properties of Boolean algebras that readily follow from these axioms.

**Lemma 1.1.** *In a Boolean algebra $B$, the following hold for all $x, y \in B$.*

*(1)* $x \vee x = x$. *Idempotence.*
*(2)* $x \wedge x = x$. *Idempotence.*
*(3)* $x \wedge 0 = 0$. *The element* 0 *is the zero for meet.*
*(4)* $1 \vee x = 1$. *The element* 1 *is the zero for join.*
*(5)* $x = x \vee (x \wedge y)$. *Absorption law.*
*(6)* $x \vee y = x \vee (y \wedge x')$. *Difference law.*
*(7)* $x'' = x$. *Double complementation.*
*(8)* $(x \vee y)' = x' \wedge y'$. *De Morgan.*
*(9)* $(x \wedge y)' = x' \vee y'$. *De Morgan.*

The theory of Boolean algebras is described in an elementary fashion in [7] and from a more advanced standpoint in [12] and [17]. The first two chapters of [11] approach the subject of Stone duality from the perspective of frame theory; the whole book can be viewed as a study of Stone's legacy.

**Example 1.2.** The basic example of a Boolean algebra that everyone knows is the *power set Boolean algebra* which consists of the set of all subsets, $\mathsf{P}(X)$, of the set $X$ with the operations $\cup$, $\cap$ and $\overline{A} = X \setminus A$ and the two constants $\varnothing$, $X$. In the finite case, each Boolean algebra is isomorphic to a power set Boolean algebra.

**Example 1.3.** *The 2-element Boolean algebra* $\mathbb{B}$ is defined as follows. Put $\mathbb{B} = \{0, 1\}$. Define operations $'$, $\wedge$, and $\vee$ by means of the following tables:

| $x$ | $x'$ |
|---|---|
| 1 | 0 |
| 0 | 1 |

| $x$ | $y$ | $x \wedge y$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

| $x$ | $y$ | $x \vee y$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

You can check that $\mathbb{B}$ is essentially the same as the power set Boolean algebra on a 1-element set. It is this Boolean algebra that is used in the design of computer circuits [4], [16].

**Example 1.4.** *The Lindenbaum algebra*, named after Adolf Lindenbaum (1834–1923), is constructed from propositional logic by identifying those well-formed formulae which have the same truth table. It is used in more advanced work to prove results about propositional logic in an algebraic way [20, Chapter 6].

**Example 1.5.** Boolean algebras are the basis of measure theory. A *measure space* consists of a set $X$ and a subset $\mathscr{A}$ of $\mathsf{P}(X)$ which contains $\varnothing$ and $X$ and is closed under $\cup$,$\cap$ and complementation and under countable unions. Thus $\mathscr{A}$ is, what is termed, a $\sigma$-complete Boolean algebra.

**Example 1.6.** Recall that a language over an alphabet $A$ is said to be *recognizable* if there is a finite-state automaton that accepts it. By Kleene's theorem [13], the set of recognizable languages over $A$ is equal to the set of regular languages over $A$. Denote the set of regular languages over $A$ by $\mathrm{Reg}(A)$. This set is a Boolean algebra, with extra operations. This Boolean structure can be exploited to provide a sophisticated way of studying families of regular languages [18], [6].

Boolean algebras have their roots in the work of George Boole [2], though the definition of Boolean algebras seems to have been inspired by his work rather than originating there [8]. Until the 1930s, research on Boolean algebras was essentially about axiomatics with the following example being typical.

**Example 1.7.** A non-empty set $S$ is equipped with a binary operation $\vee$ and a unary operation $'$ such that only the following axioms hold:

(1) $a \vee (b \vee c) = (a \vee b) \vee c$.
(2) $a \vee b = b \vee a$.
(3) $(a' \vee b')' \vee (a' \vee b)' = a$.

Define $a \wedge b = (a' \vee b')'$, $1 = a \vee a'$ and $0 = 1'$. Then $(S, \vee, \wedge, ', 0, 1)$ is a Boolean algebra. For a proof, see [9, 10].]

Questions such as this are basic in any new branch of algebra but in the case of Boolean algebras, there was also the problem that there was a plethora of ways of axiomatizing them. With Stone's paper [21], stability in the definition of Boolean algebras emerges because he showed that each Boolean algebra could be regarded as a (unital) ring in which each element was idempotent; rings such as this are called *Boolean rings*. In the language of category theory, his result shows that the category of Boolean algebras is isomorphic to the category of Boolean rings. The following result describes how the correspondence between Boolean algebras and Boolean rings works at the algebraic level.

**Theorem 1.8.**

*(1) Let $B$ be a Boolean algebra. Define $a + b = (a \wedge b') \vee (a' \wedge b)$ and $a \cdot b = a \wedge b$. Then $(B, +, \cdot, 1)$ is a Boolean ring.*

(2) *Let* $(R, +, \cdot, 1)$ *be a Boolean ring. Define* $a \vee b = a + b + a \cdot b$, $a \wedge b = a \cdot b$ *and* $a' = 1 - a$. *Then* $(R, \vee, \wedge, {}', 0, 1)$ *is a Boolean algebra*
(3) *The constructions (1) and (2) above are mutually inverse.*

The above result is satisfying since the definition of Boolean rings could hardly be simpler but also raises the interesting question of why Marshall H. Stone (1903–1989), a functional analyst, should have been interested in Boolean algebras in the first place. the reason is that Stone worked on the spectral theory of symmetric operators and this led to an interest in algebras of commuting projections. Such algebras are naturally Boolean algebras. The following theorem [5] puts this connection in a slightly wider context. If $R$ is a ring (or semigroup) denote its set of idempotents by $\mathsf{E}(R)$.

**Theorem 1.9.** *Let* $R$ *be a commutative ring. Then the set* $\mathsf{E}(R)$ *is a Boolean algebra when we define* $e \vee f = e + f - ef$, $e \wedge f = e \cdot f$ *and* $e' = 1 - e$.

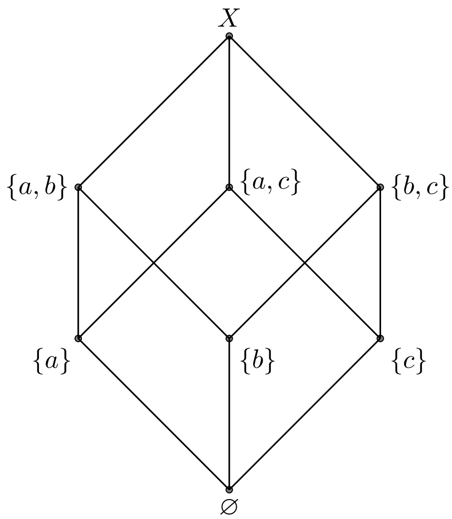
By Theorem 1.8 and Theorem 1.9, it is immediate that each Boolean algebra arises as the Boolean algebra of idempotents of a commutative ring.

So far we have viewed Boolean algebras as purely algebraic objects. But in fact they come equipped with a partial order that underpins this algebraic structure. Let $B$ be a Boolean algebra. For $x, y \in B$, define $x \leq y$ if and only if $x = x \wedge y$. The proofs of the following are routine.

**Lemma 1.10.** *With the above definition, we have the following:*
(1) $\leq$ *is a partial order on* $B$.
(2) $x \leq y$ *if and only if* $y = x \vee y$.
(3) $a \wedge b = \mathrm{glb}\{a, b\}$ *and* $a \vee b = \mathrm{lub}\{a, b\}$.

A non-zero element $x \in B$ of a Boolean algebra is called an *atom* if $y \leq x$ implies that either $x = y$ or $y = 0$.

**Example 1.11.** The following is the Hasse diagram for the finite Boolean algebra $\mathsf{P}(X)$ where $X = \{a, b, c\}$. The atoms are the singleton subsets.




## 2. Duality

In the previous section, we described a great variety of Boolean algebras from a range of mathematical disciplines. What we would like mathematically is some way of describing all Boolean algebras. The partial order defined at the end of the last section plays a crucial role in achieving this goal. We begin with the finite case, where there is an easy description and which provides good motivation for the general case.

2.1. **Finite Boolean algebras.** The description of finite Boolean algebras depends crucially on the properties of atoms. Finiteness tells us that there are atoms but, more strongly than that, each non-zero element of a finite Boolean algebra is either itself an atom or lies above an atome in the partial order. Let $B$ be a finite Boolean algebra. Denote the set of atoms of $B$ by $\mathsf{X}(B)$. For each $a \in B$ denote by $U_a$ the set of all atoms in $B$ below $a$. Observe that $U_0 = \varnothing$, $U_1 = \mathsf{X}(B)$ and $U_a \neq \varnothing$ if $a \neq 0$. The following lemma provides the key properties of the atoms that we shall need.

**Lemma 2.1.** *Let $B$ be a finite Boolean algebra.*

1. *Let $a, b \in B$ such that $a \neq b$. Then there is an atom beneath one of $a$ or $b$ that is not beneath the other.*
2. *Let $a$ be a non-zero element. For each atom $x$ either $x \leq a$ or $x \leq a'$ but not both.*
3. *Let $x$ be an atom. If $x \leq a \vee b$ then $x \leq a$ or $x \leq b$.*
4. *Let $x$ and $y$ be atoms. Then either $x = y$ or $x \wedge y = 0$.*
5. *Let $x \leq x_1 \vee \ldots \vee x_n$ where all elements are atoms. Then $x = x_i$ for some $i$.*

*Proof.* (1) From the fact that $a \neq b$ we deduce that $(a \nleq b) \vee (b \nleq a)$. We assume without loss of generality that $a \nleq b$. Then $a \wedge b' \neq 0$ by basic Boolean algebra. Let $x \leq a \wedge b'$ be an atom. Then $x \leq a$ and $x \leq b'$ from which it follows that $x \nleq b$.

(2) Suppose that $x \nleq a$. Then $x \wedge a' \neq 0$. It follows that $x \leq a'$, as claimed.

(3) We have that $x = (x \wedge a) \vee (x \wedge b)$. Observe that $x \ wedgea = 0$ or $x \leq a$. There are various cases but they all devolve down to $x \leq a$ or $x \leq b$.

(4) Straightforward.

(5) We have that $x = (x \wedge x_1) \vee \ldots \vee (x \wedge x_n)$. The result now follows by part (4) above. □

The above lemma leads to easy proofs of the following.

**Lemma 2.2.** *Let $B$ be a finite Boolean algebra.*

1. $U_a \cap U_b = U_{a \wedge b}$.
2. $U_a \cup U_b = U_{a \vee b}$.
3. $U_{a'} = \overline{U_a}$.

**Theorem 2.3.** *Every finite Boolean algebra is isomorphic to the Boolean algebra of subsets of a finite set.*

*Proof.* Let $B$ be a finite Boolean algebra. Define a function $B \to \mathsf{P}(\mathsf{X}(B))$ by $a \mapsto U_a$. By Lemma 2.2, this is a morphism of Boolean algebras. By part (1) of Lemma 2.9, it is injective. By part (5) of Lemma 2.9, it is surjective. □

In the light of the above result, it is tempting to conjecture that every Boolean algebra is isomorphic to a powerset Boolean algebra. However, this turns out to be false. Define a Boolean algebra to be *atomic* if each non-zero element is above an atom. Define a Boolean algebra to be *atomless* if it has no atoms.

**Example 2.4.** The powerset Boolean algebra $\mathsf{P}(X)$ is always atomic with the atoms being the singleton sets.

**Example 2.5.** Let $X$ be the set $\mathbb{R}$ with the usual ordering with an adjoined maximum element $\infty$ and an adjoined minimum element $-\infty$. The set of all finite unions of semi-open intervals $[a, b)$, where $a, b \in X$, forms a Boolean algebra. This is an example of what is called an *interval algebra* [15]. This Boolean algebra is atomless.

Define a Boolean algebra to be *complete* if it has arbitrary joins with respect to its partial order. The following was proved in [23]

**Theorem 2.6.** *The powerset Boolean algebras are precisely the complete, atomic Boolean algebras.*

2.2. **Arbitrary Boolean algebras.** To describe arbitrary Boolean algebra we have to adopt a different approach, and this was just what Stone did [22]. The approach is symbolized below and can be viewed as a generalization of the finite case described in the previous section.

$$\begin{array}{ccc} \text{atom} & \stackrel{\text{replaced by}}{\longrightarrow} & \text{ultrafilter} \\ \text{powerset} & \stackrel{\text{replaced by}}{\longrightarrow} & \text{topological space} \\ \text{atom } x \leq a & \stackrel{\text{replaced by}}{\longrightarrow} & a \in F \text{ ultrafilter} \end{array}$$

Let $B$ be an arbitrary Boolean algebra. We have seen above that $B$ might not have atoms so we have to find 'atom substitutes' that do always exist. A subset $F$ of a Boolean algebra $B$ is called a *filter* if it satisfies the following conditions:

(1) $1 \in F$.
(2) If $a, b \in F$ then $a \wedge b \in F$.
(3) If $a \in F$ and $a \leq b$ then $b \in F$.

The filter $F$ is said to be *proper* if $0 \notin F$.

**Example 2.7.** Let $a \in B$. Define $a^{\uparrow} = \{b \in B \colon a \leq b\}$. Then $a^{\uparrow}$ is a filter called the *principal filter generated by a* .

Let $X \subseteq B$ be any subset of the Boolean algebra $A$. Define $X^{\wedge}$ to be the set of all finite meets of elements of $X$; we say that $X$ has the *finite intersection property* if $0 \notin X^{\wedge}$. Define $X^{\uparrow}$ to be the set of all elements that are above an element of $X$. Define $\mathsf{F}(X)$ to be the set $(X^{\wedge})^{\uparrow}$ The proof of the following is straightforward.

**Lemma 2.8.** *For each $X \subseteq B$ the set $\mathsf{F}(X)$ is a filter. It is proper if $X$ has the finite intersection property.*

A proper filter $F$ is said to be *prime* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$. A maximal filter is called an *ultrafilter.*

**Lemma 2.9.** *The following are equivalent for a filter $F$ in a Boolean algebra $B$.*

*(1) $F$ is an ultrafilter.*
*(2) For each non-zero $a \in F$ either $a \in F$ or $a' \in F$.*
*(3) $F$ is a prime filter.*

*Proof.* (1)⇒(2). Let $F$ be an ultrafilter. Suppose that $a \notin F$. Then $F \subset \mathsf{F}(F \cup \{a\}) = B$. It follows that $0 \in \mathsf{F}(F \cup \{a\})$. Thus by Lemma 2.8, there is $b \in F$ such that $b \wedge a = 0$. Now $1 = a \vee a'$. Thus $b = (b \wedge a) \vee (b \wedge a')$. So, $b = b \wedge a'$. It follows that $b \leq a'$ giving $a' \in F$, as required.

(2)⇒(3). We prove that $F$ is a prime filter. Suppose that $a \notin F$ and $b \notin F$. Then $a' \in F$ and $b' \in F$ so that $a' \wedge b' \in F$. Thus $(a \vee b)' \in F$.

(3)⇒(1). Suppose that $a \notin F$. Then $a' \in F$, since $F$ is prime and $1 \in F$. It follows that the filter $\mathsf{F}(F \cup \{a\})$ contains 0. □

We can now connect atoms with special kinds of ultrafilters.

**Proposition 2.10.** *Let $B$ be a Boolean algebra. The principal filter $F = a^{\uparrow}$ is an ultrafilter if and only if $a$ is an atom.*

*Proof.* Suppose that $a$ is an atom and that $b \vee b' \in F$. Then $a \leq b \vee b'$. Thus $a = (a \wedge b) \vee (a \wedge b')$; in particular, it cannot happen that both $a \wedge b = 0$ and $a \wedge b' = 0$. Also $a \wedge b \leq a$ and $a \wedge b' \leq a$. But $a$ is an atom. If $a \wedge b = a$ then $a \leq b$ and $b \in F$; if $a \wedge b = 0$ then $a \wedge b' = a$ implying that $a \leq b'$ and so $b' \in F$. It follows by Lemma 2.9 that $F$ is an ultrafilter.

Suppose that $F$ is an ultrafilter. We prove that $a$ is an atom. Suppose not. Then there is $0 < b < a$ where $b$ is an atom. Then $b^{\uparrow}$ is an ultrafilter and $F \subseteq b^{\uparrow}$. But this contradicts the assumption that $F$ is an ultrafilter. It follows that $a$ must be an atom. □

The above lemma is only interesting in the light of the following result. The routine proof uses Zorn's Lemma or see [12, Chapter 1, Proposition 2.16].

**Theorem 2.11** (Boolean Prime Ideal Theorem)**.** *A subset of a Boolean algebra is contained in an ultrafilter if and only if it has the finite intersection property.*

The first corollary is the analogue of the result for finite Boolean algebras that every non-zero element is above an atom.

**Corollary 2.12.** *Every non-zero element of a Boolean algebra is contained in an ultrafilter.*

The second corollary says that there are enough ultrafilters to separate points; this is the analogue of the result that says in a finite Boolean algebra each element is a join of the atoms below it.

**Corollary 2.13.** *Let $a$ and $b$ be distinct non-zero elements of a Boolean algebra. Then there is an ultrafilter that contains one of the elements and omits the other.*

*Proof.* Since $a \neq b$ then either $a \nleq b$ or $b \nleq a$. Suppose that $a \nleq b$. Then it is a simple exercise to check that $a \wedge b' \neq 0$. Thus by Corollary 2.12 there is an ultrafilter $F$ that contains $a \wedge b'$. It follows that $a \in F$ and $b \notin F$. □

Ultrafilters are the first step in generalizing the theory of finite Boolean algebras to arbitrary Boolean algebras. The second is to introduce topological spaces. A topological space is said to be 0-*dimensional* if it has a base of clopen sets. A compact Hausdorff space which is 0-dimensional is called a Boolean space.

**Proposition 2.14.** *The clopen subsets of a Boolean space form a Boolean algebra.*

*Proof.* Let $X$ be a Boolean space and denote by $\mathsf{B}(X)$ the set of all clopen subsets of $X$. Observe that $\varnothing, X \in \mathsf{B}(X)$. If $A, B \in \mathsf{B}(X)$ then $A \cap B, A \cup B \in \mathsf{B}(X)$. Finally, if $A \in \mathsf{B}(X)$ then $\overline{A} \in \mathsf{B}(X)$. □

Let $B$ be a Boolean algebra. Define $\mathsf{X}(B)$ to be the set of ultrafilters on $B$. If $a \in B$ denote by $U_a$ the set of ultrafilters containing $a$. The proof of the following lemma is routine.

**Lemma 2.15.** *Let $B$ be a Boolean algebra.*

(1) $U_0 = \varnothing$.
(2) $U_1 = \mathsf{X}(B)$.
(3) $U_a \cap U_b = U_{a \wedge b}$.
(4) $U_a \cup U_b = U_{a \vee b}$.
(5) $U_{a'} = \overline{U_a}$.

The above lemma tells us that we may define a topology $\sigma$ on $\mathsf{X}(B)$ whose open sets are unions of sets of the form $U_a$. We shall first of all determine the salient properties of the topological space $(\mathsf{X}(B), \sigma)$. We shall refer to the elements of $\mathsf{X}(B)$ as *points*.

**Theorem 2.16.** *For each Boolean algebra $B$ the space $(\mathsf{X}(B), \sigma)$ is Boolean.*

*Proof.* The fact that $(\mathsf{X}(B), \sigma)$ is 0-dimensional follows by part (5) of Lemma 2.15. Let $A$ and $B$ be distinct ultrafilters. Then there exists $a \in A \setminus B$. Observe that $A \in U_a$, $B \in U_{a'}$ and $U_a \cap U_a = \varnothing$. Thus $(\mathsf{X}(B), \sigma)$ is Hausdorff. Finally, we prove that $(\mathsf{X}(B), \sigma)$ is compact. Let $\mathscr{C} = \{U_a : a \in I\}$ be a cover of $\mathsf{X}(B)$. Suppose that no finite subset of $\mathscr{U}$ covers $\mathsf{X}(B)$. Then for any $a_1, \ldots, a_n \in I$ we have that $U_{a_1} \cup \ldots \cup U_{a_n} \neq \mathsf{X}(B)$. It follows that $a_1 \vee \ldots \vee a_n \neq 1$ and so $a_1' \wedge \ldots \wedge a_n' \neq 0$. Thus the set $I' = \{a' : a \in I\}$ has the finite intersection property. By Theorem 2.11, there is an ultrafilter $F$ such that $I' \subseteq F$. By assumption, $F \in U_a$ for some $a \in I$ and so $a, a' \in F$, which is a contradiction. □

The topological space $\mathsf{X}(B)$ is called the *Stone space* of the Boolean algebra $B$.

We can now assemble Proposition 2.14 and Theorem 2.16 into the main result of this article.

**Theorem 2.17.**

(1) *Let $B$ be a Boolean algebra. Then $B \cong \mathsf{BX}(B)$.*

(2) *Let $S$ be a Boolean space. Then $S \cong \mathsf{XB}(S)$.*

*Proof.* (1) Define $\alpha \colon B \to \mathsf{BX}(B)$ by $a \mapsto U_a$. By Lemma 2.15 this is a homomorphism of Boolean algebras. It is injective by Corollary 2.13. It is surjective because a closed subset of a compact space is compact.

(2) Let $x \in X$. Define $O_x$ to be the set of all clopen sets that contain $x$. It is easy to check that this is an ultrafilter in $\mathsf{B}(X)$ and so $O_x \in \mathsf{XB}(X)$. Define $\beta \colon X \to \mathsf{XB}(X)$ by $x \mapsto O_x$ the set of all clopen sets that contain $x$. Since both spaces are compact and Hausdorff, to prove that $\beta$ is a homeomorphism it is enough to prove that it is bijective and continuous. Suppose that $O_x = O_y$. If $x \neq y$ then by the fact that $X$ is Hausdorff we could find disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$. But $X$ is 0-dimensional and so we can assume, without loss of generality, that $U$ and $V$ are clopen from which we would deduce that $O_x \neq O_y$. It follows that $\beta$ is injective. Next, let $F$ be any ultrafilter in $\mathsf{B}(X)$. Then this is an ultrafilter consisting of clopen subsets of a compact space; here we shall only need that these subsets are closed. Since $F$ is a filter, it has the property that the intersection of every finite set of elements is non-empty. By [19, Chapter 4, Theorem D], it follows that there is an element $x$ in the intersection of all the elements of $F$. Thus $F \subseteq O_x$. But $F$ is an ultrafilter and so $F = O_x$. Finally, we prove continuity. Let $U$ be an open subset of $\mathsf{XB}(X)$. Then $U$ is a union of the basic open sets which are clopen. These have the form $U_A$ where $A$ is a clopen subset of $X$. Thus it is enough to calculate $\beta^{-1}(U_A)$. But $O_x \in U_A$ if and only if $x \in A$. Thus $\beta^{-1}(U_A) = A$. □

**Example 2.18.** Let $B$ be a finite Boolean algebra. Then, as we have seen, the ultrafilters of $B$ are in bijective correspondence with the atoms of $B$. We may therefore identify the set of points $\mathsf{X}(B)$ with the set of atoms of $B$. Let $a \in B$. We describe the set $U_a$ in terms of atoms. The ultrafilter $b^{\uparrow} \in U_a$ if and only is $b \leq a$. So, the set $U_a$ is in bijective correspondence with the set of atoms below $a$. It follows that the Boolean space $\mathsf{X}(B)$ is homeomorphic with the the discrete space of atoms of $B$. In this way, the classical theory of finite Boolean algebras can be derived from Stone duality.

**Example 2.19.** Tarski proved that any two atomless, countably infinite Boolean algebras are isomorphic [7, Chapter 16, Theorem 10]. It makes sense, therefore, to define the *Tarski algebra*[1] to be an atomless, countably infinite Boolean algebra.

[1] Not an established term

An element $x$ of a topological space is said to be *isolated* if $\{x\}$ open. Suppose that $a$ is an atom of the Boolean algebra $B$. Then $U_a$ is the set of all ultrafilters that contain $a$. But $a^{\uparrow}$ is an ultrafilter containing $a$ and, evidently, the only one. Thus $U_a$ is an open set containing one point and so the point $a^{\uparrow}$ is isolated. Suppose that $U_a$ contains exactly one point $F$. Then $F$ is the only ultrafilter containing $a$. Suppose that $a$ were not an atom. Then we could find $0 \neq b < a$. Thus $a = b \vee (a \wedge b')$. Let $F_1$ be an ultrafilter containing $b$ and let $F_2$ be an ultrafilter containing $a \wedge b'$. Then $F_1 \neq F_2$ but both contain $a$. This is a contradiction. It follows that $a$ is an atom. Observe that $B$ is an atomic Boolean algebra if and only if the isolated points in its Stone space form a dense subset. We deduce that the Stone space associated with an atomless Boolean algebra has no isolated points. If $B$ is countable then its Stone space is second-countable. The Stone space of the Tarski algebra is therefore a second-countable, 0-dimensional, compact Hausdorff space with no isolated points; such a space is homeomorphic to the Cantor space. It follows that the Stone space of the Tarski algebra is the Cantor space.

**Example 2.20.** We construct the Stone spaces of the powerset Boolean algebras $\mathsf{P}(X)$. The isolated points of the Stone space of $\mathsf{P}(X)$ form a dense subset of the Stone space which is homeomorphic to the discrete space $X$. Thus the Stone space of $\mathsf{P}(X)$ is a compact Hausdorff space that contains a copy of the discrete space $X$. In fact, the Stone-Čech compactification of the discrete space $X$ is precisely the Stone space of $\mathsf{P}(X)$.

It remains to say a few words about maps. Let $B$ be a Boolean algebra. Then there is a bijective map between the ultrafilters in $B$ and the Boolean algebra homomorphisms from $B$ to $\mathbb{B}$, the 2-element Boolean algebra. This bijection associates with the ultrafilter $F$ its characteristic function $\chi_F$. Let $\theta \colon B_1 \to B_2$ be a homomorphism between Boolean algebras. Let $F \in (X)(B_2)$ be an ultrafilter. Then $\chi_F \theta$ is the characteristic function of an ultrafilter in $B_2$. In this way, we can map homomomorphisms $B_1 \to B_2$ to continuous functions $\mathsf{X}(B_1) \leftarrow \mathsf{X}(B_2)$ with a consequent reversal of arrows. In the other direction, let $\phi \colon X_1 \to X_2$ be a continuous function. Then $\phi^{-1}$ maps clopen sets to clopen sets. In this way, we can map continuous functions $X_1 \to X_2$ to homomorphisms $\mathsf{B}(X_1) \leftarrow \mathsf{B}(X_2)$. If Theorem 2.17 is combined with our observations on maps above, we can say the following: *the category of Boolean algebras is dually equivalent to the category of Boolean spaces.*

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Mark V. Lawson, Department of Mathematics and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UNITED KINGDOM

*Email address*: `m.v.lawson@hw.ac.uk`