INTRODUCTION

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Abstract. This is the introduction to a sequence of articles on non-commutative Stone duality. These articles were written to accompany my lectures given at LaBRI, Université de Bordeaux during April 2018. I am grateful to David Janin for the invitation and to the audience for their forbearence.

1. Introduction

It was by accident that I first stumbled across a copy of Renault's monograph [13] in the university library at York when I was a grduate student and it was there that I first learnt that there was a connection between inverse semigroups, groupoids and C^*-algebras. What caught my eye was that amongst the inverse semigroups discussed were ones that I was familiar with — the polycyclic inverse monoids [11]. These inverse monoids are one of the first interesting examples of such monoids with close connections with the theory of push-down automata and context-free languages. What was fascinating to me then was that these selfsame inverse monoids played an important role in constructing an important class of simple C^*-algebras — the Cuntz C^*-algebras [1]. This was the first time that I had seen honest-to-goodness inverse semigroups being used outside of the theory of pseudogroups of transformations. The nature of Renault's use of inverse semigroups in his book was, however, unclear to me. In particular, the connection between inverse semigroups and topological groupoids was not spelt out in the form of a theorem. Nevertheless, I gained the impression that inverse semigroups and C^*-algebras were related and that the nature of this relationship might be interesting. This impression was re-enforced by reading Kumjian's paper [4] in which inverse semigroups took centre stage rather than just being stand-ins for topological groupoids.

Renault's work and what it might mean for inverse semigroup theory remained a nagging memory until a piece of serendipity led me, much later, to the research of Johannes Kellendonk [2, 3]. Kellendonk was interested in aperiodic tilings as models of quasicrystals. Specifically, he wanted to use such tilings to study the physical properties of quasicrystals. To do this, involved constructing a C^*-algebra from a tiling and then computing its K_0-group. However, he realised that there was a combinatorial object, which he termed an 'almost groupoid', that lay behind these computations. In fact, his almost groupoid was an inverse semigroup (once a zero was adjoined) that could be viewed as the inverse semigroup of partial translational symmetries of the tiling. The C^*-algebra that modelled the physics of the tiling could be constructed from a topological groupoid itself constructed from this inverse semigroup. The tiling semigroup and its groupoid are described in my book [5].

An important breakthrough in understanding the connection between inverse semigroups and topological groupoids was the paper of Daniel Lenz [10], motivated by Kellendonk's work, which appeared as preprint in 2002. This paper is the gateway to all subsequent research in this area including my own. One of Lenz's insights in this paper was to provide a purely algebraic description of how to pass from inverse semigroups to topological groupoids. A couple of years earlier, Paterson's book [12] appeared, which updated Renault's, and which developed in much more detail the connection between inverse semigroups and topological groupoids. In particular, this involved the construction of what he termed the universal groupoid of a inverse semigroup. Lenz was able to provide a purely algebraic description of this topological groupoid. Subsequently, Resende [14] clarified the exact connection that exists between inverse semigroups, specifically pseudogroups, and topological groupoids, specifically étale groupoids.

It was subsequently realised [9],[1] that Lenz's approach could be streamlined by using filters (he was essentially working with filter bases). Taking into account Resende's work, it was at this point that it became possible to see an analogy between classical duality theory — which involves filters, prime filters, ultrafilters — and the connection between inverse semigroups and topological groupoids. In fact, it turns out not to be an analogy at all but rather a generalization. This was developed in a sequence of papers [6, 7, 8]. My goal in the subsequent articles is to explain that part of this duality theory that is the most direct generalization of classical Stone duality.

References

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[1]This paper only appeared in 2014 but much of the the groundwork for it was actually carried out in the early 2000's.