Semigroups related to subshifts of graphs

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For Mati Kilp and Ulrich Knauer on the occasion of their 65th birthdays

Abstract. We characterise in terms of Zappa-Szép products a class of semigroups which arise from actions of groupoids on semigroups constructed from subshifts of graphs.

1. Introduction

Before describing the particular questions addressed in this paper, I want to begin by setting the scene. The 'homological classification of monoids' is a phrase that refers to the use of actions of monoids to classify monoids. It goes back to Skornjakov's paper [**16**] and is the subject of [**6**], a book which is the bible for this area of research. There are many homological properties that are of interest, but I want to focus on just one in this introduction, and that property is projectivity. Projectivity can be used to define classes of monoids in the following way. Recall that if I is a right ideal of a monoid S then there is a right monoid action $I \times S \to I$. Accordingly, the following definitions make sense:

- A monoid is said to be *right PP* if all its principal right ideals are projective.
- A monoid is said to be *right semihereditary* if all its finitely generated right ideals are projective.
- A monoid is said to be *right hereditary* if all its right ideals are projective.

The study of monoids defined in this way was initiated by Dorofeeva [**2**] and Kilp [**5**], who obtained some important first results on their structure. In particular, Dorofeeva showed [**2**] that a monoid is right semihereditary iff it is right PP and incomparable principle right ideals are disjoint, and right hereditary iff it is right semihereditary and has the ACC for principal right ideals. The detailed study of classes of right PP monoids was taken up by John Fountain in England. The starting point of Fountain's work is a very neat characterisation of right PP monoids. Define the relation $\mathcal{L}^*$ on a semigroup S by $a\mathcal{L}^*b$ iff for all elements $x, y \in S^1$ we have that $ax = ay \Leftrightarrow bx = by$. The relation $\mathcal{R}^*$ is defined dually. Both relations are equivalence relations. Using this notation, Fountain [**3**] showed that a monoid is right PP if and only if each $\mathcal{L}^*$-class contains an idempotent. A semigroup with

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this property is said to be *right abundant*; *left abundant* semigroups are defined dually, and *abundant* semigroups are those which are both left and right abundant [**4**]. In particular, the right PP monoids are precisely the right abundant monoids. The significance of these characterisations is that they show an analogy between abundant semigroups and regular semigroups, because regular semigroups can be characterised as those semigroups in which each $\mathcal{L}$-class contains an idempotent (equivalently, in which each $\mathcal{R}$-class contains an idempotent).

Clearly, we would like to say something about the structure of classes of abundant semigroups. Fountain's characterisation has provided one fruitful avenue of research by directing us to look for generalisations of results from classes of regular semigroups to classes of abundant semigroups. However, I would like to suggest an alternative approach that will lead naturally to the actual subject matter of this paper. Consider the monoids of each of our three classes which contain a single idempotent: this is a natural special case to focus on given the description of right PP monoids in terms of the $\mathcal{L}^*$-relation. A right PP monoid with a single idempotent is precisely a left cancellative monoid. Thus the theory of right PP monoids is a generalisation of the theory of left cancellative monoids. It is hard to say anything about left cancellative monoids in general so instead I am now going to concentrate on the right hereditary monoids with a single idempotent and then single out a subclass of such monoids in which the ACC is strengthened by the condition that each element is contained in only a finite number of principal right ideals. I call right hereditary monoids of this type *left Rees monoids*.[1] Remarkably, left Rees monoids can be described in more detail: there is a correspondence, originating in the work of Rees and Perrot, between left Rees monoids and self-similar group actions [**12**]. Self-similar group actions are defined in [**14**] and consist of a group acting in a particular way on a free monoid. Such actions are, in fact, of a kind arising in the theory of Zappa-Szép products [**1, 8**]. Thus the correspondence is equivalent to a structural description of left Rees monoids as Zappa-Szép products of free monoids and groups. This success suggests that we try to study the structure of generalisations of right hereditary monoids in a similar way. This paper will not do this exactly, but it is very much in the spirit of this idea.

The actual goal of this paper is to generalise the correspondence between left Rees monoids and self-similar group actions in as direct a way as possible. To carry out this generalisation, we therefore need to do two things: generalise groups and generalise free monoids. The generalisation of groups is easy: we simply replace them by groupoids, where by a groupoid I mean a small category in which every element is an isomorphism. How to generalise free monoids was suggested by Krieger's paper [**7**] and my own on one-dimensional tiling semigroups [**11**]. The semigroups in question are defined in Section 2. We will return to abundant semigroups at the end of the paper.

2. A class of semigroups constructed from free categories

We distinguish between semigroups and monoids, the latter being semigroups with an identity. The group of a units of a monoid is the group of invertible elements in the monoid. We shall say that a monoid is *unit-free* if the only invertible element is the identity. Idempotents are elements equal to their square. The set of

[1] In a left cancellative monoid, the condition that incomparable principal right ideals be disjoint is equivalent to the semigroup being equidivisible. See [**12**].

idempotents of a semigroup S is denoted by $E(S)$. An element a in a semigroup S is said to be *regular* if there is an element b such that $a = aba$ and $b = bab$. The element b is called *an inverse of* a. The set of regular elements in S is denoted by $\mathrm{Reg}(S)$. Clearly, $E(S) \subseteq \mathrm{Reg}(S)$. If $E(S) = \mathrm{Reg}(S)$ we shall say that the set of regular elements is *trivial*. A semigroup in which all elements are regular is said to be *regular*. An *inverse semigroup* is a regular semigroup in which the idempotents commute. If e and f are idempotents we define $e \leq f$ iff $e = ef = fe$. This is a partial order. A semigroup with zero is said to be *primitive* if $e \leq f$ where f is a non-zero idempotent implies that $e = 0$ or $e = f$. Primitive inverse semigroups are isomorphic to groupoids with a zero adjoined; see Theorem 3.3.4 of [**10**] for a proof. In semigroups we can talk about left and right ideals, and ideals, and principal left and right ideals. Given an ideal I in a semigroup we can form its quotient S/I.

An important role will be played in this paper by semigroups with zero which satisfy the following two conditions. A semigroup S has *left and right identities* if for each non-zero $a \in S$ there is a unique idempotent a^* such that $aa^* = a$ and a unique idempotent a^+ such that $a^+a = a$. A semigroup satisfies the *idempotent condition* if whenever e and f are distinct idempotents $ef = 0$. For the purposes of this paper, I shall refer to semigroups satisfying these two conditions as *arrow semigroups*; I shall explain this terminology below. A semigroup with zero is said to be *categorical at zero* if $ab \neq 0$ and $bc \neq 0$ implies that $abc \neq 0$.

Lemma 2.1. *Let S be an arrow semigroup.*

(1) *If $ab \neq 0$ then $a^* = b^+$.*
(2) *S is categorical at zero iff $a^* = b^+ \Leftrightarrow ab \neq 0$.*

Proof. (1) Suppose that $ab \neq 0$ and $a^* \neq b^+$. By the idempotent condition $a^*b^+ = 0$. Thus $ab = a(a^*b^+)b = 0$, a contradiction.

(2) Suppose that S is categorical at zero. Let $a^* = b^+ = e$. Then $ae \neq 0$ and $eb \neq 0$ giving $aeb = ab \neq 0$. The converse is proved in (1) above.

Suppose now that $a^* = b^+$ implies that $ab \neq 0$. Let $ab \neq 0$ and $bc \neq 0$. Then $a^* = b^+$ and $b^* = c^+$ by (1) above. Now ab is non-zero and $(ab)b^* = ab$ and so $(ab)^* = b^*$. But $b^* = c^+$, and so $(ab)^* = c^+$. By assumption, $(ab)c$ is non-zero. $\square$

Categories play a big role in this paper. For us, categories are algebraic structures generalising monoids. To make this precise, we need some definitions. Let C be a set equipped with a partially defined binary operation. We write $\exists ab$ if the product of a and b is defined. An element $e \in C$ is said to be an *identity* if $\exists ea$ implies that $ea = a$ and $\exists be$ implies that $be = b$. A *category*, C, in the sense we use the term in this paper, is a set C equipped with a partially defined binary operation such that for each $a \in C$ there exist unique identities a^* and a^+ such that $\exists aa^*$ and $\exists a^+a$; the partial product ab exists iff $a^* = b^+$, and if it is defined $(ab)^* = b^*$ and $(ab)^+ = a^+$; finally, the product is associative wherever it is defined. A category with exactly one identity is a monoid.

Lemma 2.2. *Let S be an arrow semigroup which is categorical at zero. On the set $S^* = S \setminus \{0\}$ define a partial product by ab is defined iff $a^* = b^+$. Then S^* is a category.*

Proof. Regard a as an arrow starting at a^* and finishing at a^+. If ab is defined then $(ab)^* = b^*$ and $(ab)^+ = a^+$. The fact that we have a category is now clear. $\square$

We can now explain the terminology of *arrow* semigroup. We regard each non-zero element $a \in S$ of an arrow semigroup as being an arrow

$$a^+ \xleftarrow{a} a^*$$

where we regard a^* as the source or domain of a and a^+ as the target or codomain. If $ab \neq 0$ then the domain of a is equal to the codomain of b, but the converse is not true in general. The converse is true, however, by Lemma 2.1, when the semigroup is categorical at zero and by Lemma 2.2 this then gives us a category structure on the set of non-zero elements. We can now see that arrow semigroups are just a way of handling structures a little more general than categories (with the proviso that the only idempotents are identities). In fact, underlying an arrow semigroup is a *precategory* in the sense of Schröder [**15**].[2]

A monoid with exactly one idempotent is an arrow semigroup. In this case, the regular elements are precisely the units.

LEMMA 2.3. *In an arrow semigroup the regular elements form a subsemigroup which is a primitive inverse semigroup.*

PROOF. We begin with an observation. Let a be a non-zero regular element. Then there is an element a' such that $a = aa'a$ and $a' = a'aa'$. The element $a'a$ is an idempotent such that $a(a'a) = a$. The semigroup has left and right idempotents, and so it follows that $a^* = a'a$. Similarly $aa' = a^+$.

Let a and b be regular elements. If $ab = 0$ then we are done. Suppose therefore that $ab \neq 0$. Since $ab = aa^*b^+b$, we clearly have $a^*b^+ \neq 0$ and so $a^* = b^+$ by the idempotent condition. Let a' and b' be such that $a = aa'a$ and $a' = a'aa'$ and $b = bb'b$ and $b' = b'bb'$. We calculate

$$ab(b'a')ab = a(bb')(a'a)b = ab^+a^*b = aa^*b^+b = ab.$$

Similarly $(b'a')ab(b'a') = b'a'$. Thus ab is regular with inverse $b'a'$. It follows that $\mathrm{Reg}(S)$ is a subsemigroup. By assumption, the idempotents commute and so $\mathrm{Reg}(S)$ is inverse. We are given that the idempotents are primitive. It follows that $\mathrm{Reg}(S)$ is a primitive inverse semigroup. $\square$

It follows from the proof of the above lemma that in an arrow category the inverse of a regular element is unique. We denote the unique inverse of g by g^{-1}. Hence if g is regular then $g^* = g^{-1}g$ and $g^+ = gg^{-1}$. Thus the set of regular elements of an arrow semigroup is the analogue of the group of units. We noted above that primitive inverse semigroups are groupoids with an adjoined zero. Thus we essentially replace groups by groupoids.

We have seen that in an arrow semigroup, we cannot in general decide when the product of two elements is zero. However, if one of the terms of the product is regular then we can be more precise. The following lemma will turn out to be crucial in Section 4.

LEMMA 2.4. *Let S be an arrow semigroup. Let $g \in \mathrm{Reg}(S)$ and let $x \in S$. Then $gx \neq 0$ iff $g^{-1}g = x^+$, and $xg \neq 0$ iff $x^* = gg^{-1}$.*

[2]Precategories with a zero adjoined are a little more general than our arrow semigroups, but we shall not need that extra generality here.

Proof. We prove the first result; the proof of the second is similar. One direction is immediate by Lemma 2.1(1). To prove the other direction, suppose that $g^{-1}g = x^{+}$ and $gx = 0$. Then $g^{-1}gx = 0$ and so $g^{-1}g \neq x^{+}$, a contradiction. □

One bugbear of semigroup theory is that because semigroups need not have zeros definitions often come in two flavours: those applied to semigroups, and those applied to semigroups where we acknowledge the zero as a distinguished element. It has been traditional within semigroup theory to designate these two flavours of a definition by distinguishing between a 'definition' and a '0-definition'. I shall follow this tradition here, although Krieger's use of 'definition' versus 'essentially definition' is less ugly. Definitions of left and right cancellative semigroups and cancellative semigroups *tout court* are well-known. Let S be a semigroup with zero. The semigroup S is 0-*left cancellative* if $ab = ac \neq 0$ implies that $b = c$. The definitions of 0-*right cancellative* and 0-*cancellative* are similar. The semigroup S is 0-*equidivisible* if $ac = bd \neq 0$ implies that there is a $u \in S$ such that either $a = bu$ and $uc = d$, or $b = au$ and $c = ud$.

Lemma 2.5. *Let S be a 0-left cancellative semigroup. Then the following are equivalent.*

(1) *If $aS \cap bS \neq \{0\}$ then $aS \subseteq bS$ or $bS \subseteq aS$.*
(2) *S is 0-equidivisible.*

In particular, if either condition holds then $Sa \cap Sb \neq \{0\}$ implies $Sa \subseteq Sb$ or $Sb \subseteq Sa$.

Proof. (1) ⇒ (2). Let $ac = bd \neq 0$. Then $aS \cap bS \neq \{0\}$. Thus by assumption $aS \subseteq bS$ or $bS \subseteq aS$. Suppose that the former holds. Then $a = bu$ for some $u \in S$. Hence $buc = bd \neq 0$. It follows by 0-left cancellation that $uc = d$. Hence S is 0-equidivisible. Suppose that the latter holds. Then $b = au$ for some $u \in S$. Hence $ac = aud \neq 0$. It follows by 0-left cancellation that $c = ud$.

(2) ⇒ (1). Suppose that $aS \cap bS \neq \{0\}$. Then $ac = bd \neq 0$ for some $c, d \in S$. By assumption, it follows that there is a u such that $a = bu$ or $b = au$ so that either $aS \subseteq bS$ or $bS \subseteq aS$.

The proof of the final claim is immediate. □

The following lemma is easy to prove; some of the cases were dealt with in [**11**].

Lemma 2.6. *Let S be a semigroup and let I be an ideal of S.*

(1) *If S is 0-left cancellative so too is S/I.*
(2) *If S is 0-equidivisible so too is S/I.*
(3) *If S has only only trivial regular elements so too does S/I.* □

A free monoid is the set of all strings A^* over an alphabet A equipped with the operation of concatenation. The empty string is denoted ε. There are a number of abstract characterisations of free monoids [**9**], but the one that will be most useful to us is: a monoid is free iff it is cancellative, equidivisible, unit-free and every element is contained in only a finite number of principal right ideals. The maximal proper principal right ideals are of the form aA^* where $a \in A$. Thus the alphabet A is finite iff there are only finitely many maximal proper principal right ideals. However, the principal right ideal xA^* properly contains xaA^*, and the principal left ideal A^*x properly contains A^*ax where $a \in A$ is arbitrary. Thus in the free monoid A^* there are no minimal principal right ideals nor minimal principal left ideals.

Subsets L of A^* are called *languages*. A language is said to be *factorial* if $uv \in L$ implies that $u, v \in L$. Factorial languages arise, for example, in studying 1-dimensional tilings [**11**]. The language L in A^* is factorial iff $A^* \setminus L$ is an ideal in A^*. If $I = A^* \setminus L$ is an ideal then the quotient monoid A^*/I can be identified with the set L with a zero adjoined. We call this the 'obvious' monoid associated with L. By Lemma 2.6, such quotient monoids are 0-cancellative, 0-equidivisible, unit free, and every element is contained in only a finite number of principal right ideals. A language $L \subseteq A^*$ is *prolongable* if for all $x \in L$ there exist non-empty strings u and v such that $uxv \in L$. Prolongable, factorial languages correspond to shift spaces [**13**]. A factorial language L is prolongable iff for all $x \in L$ there exist non-empty strings u and v such that $ux, xv \in L$. In the obvious monoid associated with L this corresponds to the fact that there are no non-zero minimal principal left or principal right ideals.

Free monoids A^* are defined from sets A. Free categories G^* are defined from directed graphs G. The elements of G^* are finite labelled paths in G. The empty paths at each vertex correspond to identities defined by that vertex. If we adjoin a zero to G^* we get a semigroup with zero, which I will denote by $(G^*)^0$. For each non-zero element $a \in (G^*)^0$ there are unique idempotents, being just the identities of the category, which I'll denote by a^* and a^+, such that $a^+a = a = aa^*$. The semigroup $(G^*)^0$ is an arrow semigroup which is 0-cancellative, 0-equidivisible, categorical at zero, and has trivial regular elements.

Following [**7**] we now emulate the theory of factorial languages in free monoids for their analogues in free categories. A subset $L \subseteq G^*$ is said to be *factorial* if $xy \in L$ implies that $x, y \in L$. The complements of factorial subsets are ideals of $(G^*)^0$ and so we can form quotient semigroups. They are arrow semigroups which are 0-cancellative, 0-equidivisible, and have trivial regular elements.

The following results will be proved as Theorem 4.5. We state them now to show how the definitions above are related.

Theorem

(1) *A semigroup with zero is isomorphic to an ideal quotient of a free category with an adjoined zero if and only if it is a* 0*-cancellative,* 0*-equidivisible arrow semigroup having trivial regular elements in which each non-zero element belongs to only a finite number of principal right ideals.*

(2) *A semigroup with zero is isomorphic to a free category with an adjoined zero if and only if it is a* 0*-cancellative,* 0*-equidivisible arrow semigroup which is categorical at zero having trivial regular elements in which each non-zero element belongs to only a finite number of principal right ideals.*

$\square$

3. Left Krieger semigroups

In this section, we shall introduce a class of semigroups generalising those of [**12**]. Unlike the examples we have discussed so far, these semigroups will have non-trivial sets of regular elements in general.

LEMMA 3.1. *Let S be a 0-left cancellative arrow semigroup.*

(1) *If $e = xy \neq 0$ is an idempotent then yx is an idempotent and x is regular with inverse y.*
(2) *Let $a \neq 0$. Then $aS = bS$ iff $a = bg$ where g is a regular element.*
(3) *$aS = eS$ for some non-zero idempotent e iff a is regular.*

PROOF. (1) We have that $x^+e = x^+xy = xy = e$. Thus $x^+ = e$. Similarly, $y^* = e$. Now

$$(yx)^2 = y(xy)x = yex = yx^+x = yx.$$

Thus yx is an idempotent. It is easy to check that $xyx = x$ and $yxy = y$.

(2) Suppose that $aS = bS$. Then $a = bx$ and $b = ay$. Thus $a = ayx$ and so by 0-left cancellation $a^* = yx$. Thus by (i) above, x is regular. Conversely, suppose that $a = bg$ where g is regular with inverse g^{-1}. Then $b^* = gg^{-1}$ and $a^* = g^{-1}g$. But $ag^{-1} = bgg^{-1} = b$, and so $aS = bS$.

(3) Suppose that $aS = eS$. Then by (2), we have that $a = eg$ for some regular element g. Thus a is regular by Lemma 2.3. Conversely, if a is regular then $aS = aa^{-1}aS \subseteq aa^{-1}S \subseteq aS$. Thus $aS = aa^{-1}S$, as required. $\square$

LEMMA 3.2. *Let S be a 0-left cancellative arrow semigroup. Then the maximal principal right ideals are those generated by a non-zero idempotent (equivalently by a non-zero regular element).*

PROOF. Observe that for any non-zero element a we have that $aS \subseteq a^+S$, because $as = a^+(as)$. It follows that if aS is maximal then $aS = a^+S$; observe that this implies that a is regular by Lemma 3.1(3). Conversely, let e be a non-zero idempotent. Suppose that $eS \subseteq aS$. Then $e = ab$ for some b. Thus $a^+e = a^+ab = ab = e$. It follows that $e = a^+$. Thus $eS \subseteq aS \subseteq eS$. Hence $eS = aS$, and so eS is maximal. $\square$

The proof of the following is immediate by the above result.

LEMMA 3.3. *Let S be a 0-left cancellative 0-equidivisible arrow semigroup. Then two maximal principal right ideals either have an intersection in zero or are equal.* $\square$

A non-zero element $a \in S$ is said to be *indecomposable* iff $a = bc$ implies that either a or b is regular. A principal right ideal aS is said to be *submaximal* if $aS \neq a^+S$ and there are no proper principal right ideals between aS and a^+S.

LEMMA 3.4. *Let S be a 0-left cancellative arrow semigroup. The non-regular element a is indecomposable iff aS is submaximal.*

PROOF. Suppose that a is indecomposable, and that $aS \subseteq bS$. Then $a = bc$. By assumption either b or c is regular. If c is regular then $aS = bcS = bS$ by Lemma 3.1(3). If b is regular then bS is a maximal principal right ideal by Lemma 3.2. Thus aS is submaximal.

Conversely, suppose that aS is submaximal. Let $a = bc$. Then $aS = bcS \subseteq bS$. By assumption either $aS = bS$ or b is regular. If the latter we are done; suppose the former. Then $a = bg$ where g is regular by Lemma 3.1(2). By 0-left cancellation $c = g$ and so c is regular. It follows that a is indecomposable. $\square$

LEMMA 3.5. *Let S be a 0-left cancellative arrow semigroup. The set of regular elements is trivial iff for all non-zero idempotents e we have that $e = xy$ implies that either x or y is idempotent.*

PROOF. Suppose that the set of regular elements is trivial. Let e be a non-zero idempotent such that $e = xy$. Then by Lemma 3.1(1), x and y are both regular. By assumption, they must be idempotents.

Conversely, suppose that for all non-zero idempotents e we have that $e = xy$ implies that either x or y is idempotent. If a is a regular element then it has an inverse a'. Thus $e = a'a$ is an idempotent. By assumption, either a' or a is idempotent. But the set of regular elements forms an inverse semigroup by Lemma 2.3 and so a' is idempotent iff a is idempotent. It follows that a is idempotent. Thus the set of regular elements is trivial. □

We now come to our main definition. A *left Krieger semigroup* is a 0-left cancellative 0-equidivisible arrow semigroup in which each non-zero element belongs to only finitely many principal right ideals. A left Krieger *monoid* without zero is called a *left Rees monoid*, such monoids were the subject of [**12**]. A semigroup which is both a left and a right Krieger semigroup is called a *Krieger semigroup*.

LEMMA 3.6. *Let S be a left Krieger semigroup.*

(1) *If S is 0-right cancellative then $Sa = Sb$ iff $ga = b$ for some regular element g.*
(2) *If S is 0-right cancellative then S is a Krieger semigroup.*
(3) *If S has trivial regular elements then S is 0-right cancellative.*

PROOF. (1) This is similar to the proof of Lemma 3.1(2).

(2) By Lemma 2.5, it is enough to prove that each non-zero element of S belongs to only finitely many principal left ideals. Let $a \in S$ be a non-zero element and let $\{Sb_i \colon i \in I\}$ be the set of principal left ideals that a belongs to. Thus for each $i \in I$ there exists $c_i \in S$ such that $a = c_i b_i$. It follows that $a \in c_i S$. By assumption, there are only a finite number of distinct such principal right ideals. Using Lemma 3.1(2), we deduce that there are finitely many elements $c_1, \ldots, c_m$ with the following property: for each c_i, where $i \in I$, there is a c_j, where $1 \leq j \leq m$, and a regular element g_{ij} such that $c_i = c_j g_{ij}$. Let $j = 1$ and consider all the $i \in I$ such that $a = c_1 g_{i1} b_i$. Then by 0-left cancellation and by (1) above, all the b_i's that occur generate the same principal left ideal. We now repeat this argument for each j such that $2 \leq j \leq m$. It follows that the set $\{Sb_i \colon i \in I\}$ is finite, as required.

(3) Let $yx = zx \neq 0$. Without loss of generality we can assume that x is not regular (because then it would be an idempotent and $y = z$ would follow). Then $yS \cap zS \neq \{0\}$. Thus by Lemma 2.5, we have that $yS \subseteq zS$ or $zS \subseteq yS$. Without loss of generality we assume the former. It follows that $y = zw$ for some w. Substituting back into our original equation we get $zwx = zx \neq 0$. Thus by left cancellation $wx = x$. It follows that for all $r \geq 1$ we have that $x = w^r x$. Thus $x \in w^r S$ for all $r \geq 1$. By assumption, the element x is contained in only a finite number of principal right ideals. It quickly follows by Lemma 3.1(2) and the fact that the set of regular elements is trivial that not all powers of w can be distinct. It follows from this and 0-left cancellation that there is $s \geq 1$ such that $w^s = w^*$. Now $y = zw$ and so $yw^{s-1} = zw^s = zw^* = z$. Hence $yS = zS$. Thus from Lemma 3.1(2) and

the fact that the regular elements are trivial, we deduce that $y = z$, as required. □

By the above lemma, it follows that Krieger semigroups are the left Krieger semigroups which are also 0-right cancellative.

4. The structure of left Krieger semigroups

In this section, we shall show how left Krieger semigroups can be described in terms of groupoids and left Krieger semigroups having trivial regular elements.

THEOREM 4.1. *Let S be a left Krieger semigroup. Let X be a transversal of generators of the submaximal principal right ideals, and denote by $X^\star$ the subsemigroup generated by X and $E(S)$. Then*

(1) *The semigroup $X^\star$ is a left Krieger semigroup in which the set of regular elements is trivial.*
(2) *Each non-zero element of S can be written uniquely as a product xg where $x \in X^\star$ and $g \in \mathrm{Reg}(S)$.*

PROOF. (1) We prove that the only regular elements of $X^\star$ are the idempotents. Suppose that $x_1 \dots x_m$ is regular where each $x_i \in X$. Then $x_1 \dots x_m S$ is a maximal principal right ideal. But $x_1 \dots x_m S \subseteq x_1 S$ and so $x_1 \dots x_m S = x_1 S$. Thus x_1 is regular by Lemma 3.1, which is a contradiction.

To prove that $X^\star$ is 0-equidivisible, it is enough to prove that if $x = yz$ where $x, y \in X^\star$ then $z \in X^\star$. Let $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$ where $x_i, y_j \in X$. Now $x_1 \dots x_m S = y_1 \dots y_n z S$ and $x_1 \dots x_m S \subseteq x_1 S$ and $y_1 \dots y_n z S \subseteq y_1 S$. Thus $x_1 S \cap y_1 S \neq \{0\}$, and so either $x_1 S \subseteq y_1 S$ or vice-versa. But both are submaximal and neither is regular and so $x_1 S = y_1 S$, which implies $x_1 = y_1$. By 0-left cancellation we have that $x_2 \dots x_m = y_2 \dots y_n z$. Suppose that $m < n$. Then repeated 0-left cancellation yields $x_m^* = y_{m+1} \dots y_n z$. It follows that $y_{m+1} \dots y_n z S$ is maximal and so $y_{m+1} \dots y_n z S = y_{m+1} S$ giving that y_{m+1} is regular, which is a contradiction. Thus $m \geq n$, from which it follows immediately by 0-left cancellation that $z \in X^\star$.

(2) We show first that each non-zero element can be written in the stated way. Let $s \in S \setminus \mathrm{Reg}(S)$. Consider the set of all non-maximal principal right ideals that contain s: such exist because s is non-regular, and they are finite in number by assumption. This set contains a maximal ideal $x_1 S$, which is necessarily submaximal, and where $x_1 \in X$. Thus $s = x_1 s_1$. If s_1 is regular or an element of X we are done. Otherwise, repeat this process with s_1 to get $s_1 = x_2 s_2$ and so on. Thus we can write $s = x_1 \dots x_i s_i$. To show that this process terminates, observe that

$$sS \subset x_1 \dots x_i S \subset \dots \subset x_1 S.$$

Thus termination follows from our assumption that each non-zero element belongs to only finitely many principal right ideals. It follows that we can write $s = x_1 \dots x_n g$ where g is regular.

We now verify the uniqueness claim. Let $s = xu = yv \neq 0$ where $x, y \in X^\star$ and $u, v \in \mathrm{Reg}(S)$, and let $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$. Observe that $xuS = yvS$ but $x_1 \dots x_m u S \subseteq x_1 S$ and $y_1 \dots y_n v S \subseteq y_1 S$. Thus $x_1 S \cap y_1 S \neq \{0\}$. But both are submaximal and neither is regular and so $x_1 S = y_1 S$. Hence $x_1 = y_1$. By 0-left cancellation, we have that $x_2 \dots x_m u = y_2 \dots y_n v$. Assume that $m < n$. Repeating the above argument, we get that $u = y_{m+1} \dots y_n v$. Let v' be an inverse of v. Then

$uv' = y_{m+1} \dots y_n$. Thus $y_{m+1} \dots y_n$ is regular and so $y_{m+1} \dots y_n S$ is maximal. But $y_{m+1} \dots y_n S \subseteq y_{m+1} S$. Hence $y_{m+1} \dots y_n S = y_{m+1} S$, and so y_{m+1} is regular. This is a contradiction. We also get a contradiction if $m > n$. Thus in fact, we must have that $m = n$, and so $x_i = y_i$ for all i. By 0-left cancellation again we get $u = v$. □

Theorem 4.1 suggests that left Krieger semigroups can be constructed from groupoids and those left Krieger semigroups having trivial regular elements. We shall now show how this can be done.

Let G be a groupoid with set of identities G_0 and let S be a 0-left cancellative arrow semigroup. We shall suppose that there is a bijection between G_0 and the set of non-zero idempotents of S, although to simplify notation we shall assume that this is actually an equality. The element g of G can be regarded as an arrow starting at $g^{-1}g$ and ending at gg^{-1}; the element x of $S \setminus \{0\}$ can also be regarded as an arrow starting at x^* and ending at x^+, although it is important to remember that in general the partial multiplication on $S \setminus \{0\}$ does not form a category. Denote by $G * S$ the set of pairs (g, x) such that $g^{-1}g = x^+$. We suppose that there is a function $G * S \to S$, denoted by $(g, x) \mapsto g \cdot x$, and a function $G * S \to G$, denoted by $(g, x) \mapsto g|_x$, such that

(C1): $(g \cdot x)^+ = gg^{-1}$.
(C2): $(g \cdot x)^* = g|_x (g|_x)^{-1}$.
(C3): $x^* = (g|_x)^{-1} g|_x$.

This information is summarised by the following diagram

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "g\cdot x"'] \\ \cdot \arrow[u, "g"] & \cdot \arrow[l, "x"] \arrow[u, "g|_x"'] \end{tikzcd}$$

We also require that the following axioms are satisfied:

(SS1): $x^+ \cdot x = x$.
(SS2): If gh is defined in the groupoid G then $(gh) \cdot x = g \cdot (h \cdot x)$.
(SS3): $gg^{-1} = g \cdot g^{-1}g$.
(SS4): $x^+|_x = x^*$.
(SS5): $g|_{g^{-1}g} = g$.
(SS6): If $xy \neq 0$ and $gg^{-1} = x^+$ then $g|_{xy} = (g|_x)|_y$.
(SS7): If gh is defined in the groupoid and $h^{-1}h = x^+$ then $(gh)|_x = g|_{h \cdot x} h|_x$.
(SS8): If $xy \neq 0$ and $gg^{-1} = x^+$ then $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$.

We shall say that if there are maps $g \cdot x$ and $g|_x$ satisfying (C1)–(C3) and (SS1)–(SS8) then there is a *ZS action of G on S*. Let

$$S \bowtie G = \{(x, g) \in S \setminus \{0\} \times G \colon \; x^* = gg^{-1}\} \cup \{0\}$$

where we define

$$(x, g)(y, h) = (x(g \cdot y), g|_y h)$$

if $x(g \cdot y) \neq 0$ and $g|_y h$ is defined, and zero otherwise. The zero acts as a zero. A necessary condition for $(x, g)(y, h) \neq 0$ is that $g^{-1}g = y^+$. If this condition holds then the product is non-zero precisely when $x(g \cdot y) \neq 0$. If we represent (x, g) by the diagram

$$\begin{tikzcd} {} & {} \arrow[l, "x"'] \\ & {} \arrow[u, "g"'] \end{tikzcd}$$

then this necessary condition is represented by the following diagram

$$\begin{tikzcd} {} & {} \arrow[l, "x"'] & \\ & {} \arrow[u, "g"'] & {} \arrow[l, "y"'] \\ & & {} \arrow[u, "h"'] \end{tikzcd}$$

THEOREM 4.2. *Let G be a groupoid having a ZS action on the semigroup S which is* 0*-left cancellative arrow semigroup.*

(1) *$S \bowtie G$ is a* 0*-left cancellative arrow semigroup.*
(2) *$S \bowtie G$ contains copies S' and G' of $S \setminus \{0\}$ and G respectively such that each non-zero element of $S \bowtie G$ can be written as a product of a unique element from S' followed by a unique element from G'.*
(3) *If S has trivial regular elements then the set of non-zero regular elements of $S \bowtie G$ is G'.*
(4) *If S is* 0*-equidivisible so too is $S \bowtie G$.*

PROOF. (1) We begin by proving associativity. Suppose first that

$$[(x, g)(y, h)](z, k)$$

is non-zero. The product $(x, g)(y, h)$ is non-zero and so we have the following diagram

$$\begin{tikzcd} {} & {} \arrow[l, "x"'] & {} \arrow[l, "g \cdot y"'] \\ & {} \arrow[u, "g"'] & {} \arrow[l, "y"'] \arrow[u, "g|_y"'] \\ & & {} \arrow[u, "h"'] \end{tikzcd}$$

similarly $[(x,g)(y,h)](z,k)$ is non-zero and so we have the following diagram

$$\begin{tikzcd}
{} & {} \arrow[l, "x(g\cdot y)"'] & {} \arrow[l, "(g|_y h)\cdot z"'] \\
 & {} \arrow[u, "g|_y h"] & {} \arrow[l, "z"'] \arrow[u, "(g|_y h)|_z"'] \\
 & & {} \arrow[u, "k"']
\end{tikzcd}$$

resulting in the product

$$(x(g\cdot y)[(g|_y h)\cdot z], (g|_y h)|_z k).$$

By assumption, $x(g\cdot y)[(g|_y h)\cdot z]$ is non-zero and so $(g\cdot y)[(g|_y h)\cdot z]$ is non-zero. We now use (SS8) and (SS4) and (SS7), to get that $y(h\cdot z)$ is non-zero, and we use (SS7) and (SS6) to show that

$$(g|_y h)|_z k = g|_{y(h\cdot z)} h|_z k.$$

By (SS2),

$$x(g\cdot y)[(g|_y h)\cdot z] = (x(g\cdot y)((g|_y \cdot (h\cdot z)).$$

It now follows that

$$(y,h)(z,k) = (y(h\cdot z), h|_z k)$$

is non-zero. It also follows that $(x,g)[(y,h)(z,k)]$ is non-zero and equal to

$$[(x,g)(y,h)](z,k).$$

Next suppose that

$$(x,g)[(y,h)(z,k)]$$

is non-zero. This multiplies out to give $(x[g\cdot(y(h\cdot z))], g|_{y(h\cdot z)} h|_z k)$. By (SS6) and (SS7) we get that

$$g|_{y(h\cdot z)} h|_z k = (g|_y h)|_z k,$$

and by (SS8) and (SS2) we get that $x[g\cdot(y(h\cdot z))] = x(g\cdot y)[(g|_y h)\cdot z]$. This completes the proof that $S \bowtie G$ is a semigroup with zero.

It is easy to check that it is 0-left cancellative.

We now locate the non-zero idempotents. We have that $(x,g)^2 = (x,g)$ iff both x and g are idempotents. Thus idempotents have the form (e,e) where e is an idempotent. It is now easy to check that distinct idempotents multiply to zero. Define $(x,g)^* = (g^{-1}g, g^{-1}g)$ and $(x,g)^+ = (x^+, x^+)$. Then $S \bowtie G$ is an arrow semigroup.

(2) Define $\iota_S\colon S\setminus\{0\} \to S \bowtie G$ by $\iota_S(x) = (x, x^*)$. Let $xy \neq 0$. Then in particular $x^* = y^+$. It is easy to check using (SS4) and (SS1) that $\iota_S(x)\iota_S(y) = \iota_S(xy)$. In fact, $\iota_S(x)\iota_S(y) \neq 0$ iff $xy \neq 0$. Thus the partial semigroups $S\setminus\{0\}$ and S' are isomorphic. Now define $\iota_G\colon G \to S \bowtie G$ by $\iota_G(g) = (gg^{-1}, g)$. Then once again the partial semigroups G and G' are isomorphic. Finally, if we now pick an arbitrary non-zero element (x,g), then we can write it as $(x,g) = (x,x^*)(gg^{-1}, g)$ using the fact that $gg^{-1}|_{gg^{-1}} = gg^{-1}$ and $x^*\cdot gg^{-1} = gg^{-1}$.

(3) Suppose that S has trivial regular elements. It is not hard to check that (x,g) is regular precisely when x is a non-zero idempotent.

(iv) Suppose now that S is 0-equidivisible. We prove that $S \bowtie G$ is 0-equidivisible. Suppose that

$$(x, g)(y, h) = (u, k)(v, l) \neq 0.$$

From the definition of the product it follows that $x(g \cdot y) = u(k \cdot v)$ and $g|_y h = k|_v l$. From the first equation we know that xS and uS are comparable. Suppose that $x = uw$ (the other case, where $u = xw$, follows similarly). Then by 0-left cancellation $w(g \cdot y) = k \cdot v$. Observe that $k^{-1} \cdot (k \cdot v)$ is defined and so $k^{-1} \cdot (w(g \cdot y))$ is defined by (SS2). Thus by (SS8), $k^{-1} \cdot w$ is defined. It is now easy to check that

$$(x, g) = (u, k)(k^{-1} \cdot w, (k|_{k^{-1} \cdot w})^{-1} g).$$

□

We call $S \bowtie G$ the *Zappa-Szép product* of the semigroup S by the groupoid G [**1, 8**].

For later reference, it is worth observing that the axioms (C1)–(C3) and (SS1)–(SS8) can be modified in the obvious way to define the notion of the Zappa-Szép product of a category by a groupoid.

Lemma 4.3. *Let $S = MG$ where M is a 0-left cancellative arrow semigroup and G is a groupoid, and each non-zero element of S can be written uniquely as a product of an element of M followed by an element of G. Then the posets of principal right ideals of S and M are isomorphic.*

Proof. Let $xg \in S$ where $x \in M$ and $g \in G$. Then $xgS = xS$. We prove that $xS \subseteq yS$ iff $xM \subseteq yM$. Suppose that $xS \subseteq yS$. Then $x = y(wh)$ where $w \in M$ and $h \in G$. By uniqueness, h must be an idempotent. Thus $x = yw$ and so $xM \subseteq yM$. The converse is clear. □

Theorem 4.4. *A semigroup S is a left Krieger semigroup iff it is isomorphic to a Zappa-Szép product of a left Krieger semigroup with trivial regular elements by a groupoid.*

Proof. Let S be a left Krieger semigroup. By Theorem 4.1, each non-zero element can be written uniquely as product of an element of $X^\star$ followed by an element of $\mathrm{Reg}(S)$. Put $G = \mathrm{Reg}(S) \setminus \{0\}$, regarded as a groupoid (see the first paragraph of Section 2). If $g \in G$ and $x \in X^\star$ and $gx \neq 0$ then we know that $gx = x'g'$ where $x' \in X^\star$ and $g' \in G$. Define $g \cdot x = g'$ and $g|_x = x'$. Taking into account Lemma 2.4, it is routine to check that (C1)–(C3) and (SS1)–(SS8) all hold. Thus there is a ZS action of G on $X^\star$. According to Theorem 4.2, we can form the Zappa-Szép product $X^\star \bowtie G$. We show that S is isomorphic to $X^\star \bowtie G$. Clearly, the nonzero element xg is mapped to (x, g). We therefore have a bijection, and it is a homomorphism by construction.

Conversely, a Zappa-Szép product of a left Krieger semigroup with trivial regular elements by a groupoid is a left Krieger semigroup by Theorem 4.2 and Lemma 4.3. □

The results of this section show that the structure of left Krieger semigroups is determined by ZS actions of groupoids on left Krieger semigroups with trivial regular elements. We now look at some special cases.

We first determine the structure of left Krieger semigroups having trivial regular elements generalising slightly Theorem 2.2 of [**7**].

THEOREM 4.5.

(1) *A semigroup S is a left Krieger semigroup having trivial regular elements iff it is isomorphic to an ideal quotient of a free category with an adjoined zero.*
(2) *A semigroup S is a left Krieger semigroup having trivial regular elements which is categorical at zero iff it is isomorphic to a free category with an adjoined zero.*

PROOF. (1) The proof in one direction is straightforward and was essentially covered in Section 2.

Let S be a left Krieger semigroup having trivial regular elements. By Lemma 3.1, and the fact that the set of regular elements is trivial, the set X in Theorem 4.1 is just the set of generators of the submaximal principal right ideals. The semigroup $X^\star$ is equal to S.

Define a graph G as follows. The vertices are labelled by the non-zero idempotents of S. For each $x \in X$ there is an edge $\overline{x}$ that starts at the vertex labelled x^* and ends at the vertex labelled x^+. Let G^* be the free category on G where the identity arising from the vertex labelled e is denoted by $\overline{e}$. An element of G^* is a finite sequence of composable edges $\overline{x_1} \ldots \overline{x_r}$ which we also denote by $\overline{x_1 \ldots x_r}$. Adjoin a zero to get the semigroup $(G^*)^0$. Let I be the set of elements x (finite composable sequences) of $(G^*)^0$ such that $x = 0$ in S. Then I is an ideal of $(G^*)^0$. We may therefore form the semigroup $S' = (G^*)^0/I$. The elements of S' are in bijective correspondence with the elements of S by means of the obvious map. By construction, this map is a homomorphism and so S and S' are isomorphic, as required.

(2) By (1) and Lemma 2.2. □

By Lemma 2.2, a left Krieger semigroups which is categorical at zero is a category with a zero adjoined, so by omitting the zero such a semigroup can be described in purely categorical terms: it is a left cancellative category which is equidivisible, and in which each element is contained in only a finite number of principal right ideals. In view of the semigroups introduced in Section 1, it is natural to call such categories *left Rees categories*. A *Rees category* is a category which is both a left Rees category and a right Rees category. By Lemma 3.6, a left Rees category is a Rees category precisely when it is cancellative. The following can be deduced from Theorem 4.4 and Theorem 4.5(2) together with a suitable definition of the Zappa-Szép product of two categories which follows readily from axioms (C1)–(C3) and (SS1)–(SS8).

THEOREM 4.6. *A left Rees category is isomorphic to the Zappa-Szép product of a free category by a groupoid, and conversely.* □

We shall refer to a Rees category with a zero adjoined as a *Rees semigroup*. Such semigroups are in fact of a type considered in Section 1. Recall that a semigroup is said to be *right abundant* if each element is $\mathcal{L}^*$-related to an idempotent. *Left abundant* is defined dually, and a semigroup which is left and right abundant is said to be *abundant*.

LEMMA 4.7. *Let S be 0-left cancellative having left and right identities which is categorical at zero. Then S is right abundant.*

PROOF. Suppose that $ab = ac$. Assume first that $ab = ac \neq 0$. Then $a = aa^*$ and so $a(a^*b) = a(a^*c) \neq 0$ giving $a^*b = a^*c$ by 0-left cancellation. Conversely, suppose that $a^*b = a^*c \neq 0$. Then $aa^* \neq 0$ and $a^*b \neq 0$ implies that $aa^*b = ab \neq 0$. Thus $ab = ac \neq 0$. Now suppose that $ab = ac = 0$. I claim that $a^*b = a^*c = 0$, but this is immediate from the fact that $aa^* = a$ and the fact that the semigroup is categorical at zero. Conversely, if $a^*b = a^*c = 0$ then $ab = ac = 0$. It follows that $a \,\mathcal{L}^*\, a^*$, and so S is right abundant. □

The above lemma and its dual yield the following.

COROLLARY 4.8. *Let S be a 0-cancellative semigroup having left and right identities which is categorical at zero. Then S is abundant.* □

The following now follows by the above result and Proposition 5.5 of [**4**].

PROPOSITION 4.9. *Let S be a 0-left cancellative arrow semigroup. Then S is also 0-right cancellative and categorical at zero if and only if S is a primitive abundant semigroup whose idempotents commute.* □

Thus the 0-cancellative arrow semigroups which are categorical at zero are precisely the primitive adequate semigroups of Fountain [**4**]. It follows that Rees semigroups are examples of primitive adequate semigroups.

We can construct simple examples of Rees semigroups as follows. Let $\mathcal{M} = M^0(S^0; I, I; P)$ be a Rees matrix semigroup with zero over the Rees monoid S with zero adjoined. Here I is a non-empty set and each diagonal entry of the $I \times I$ sandwich matrix P is the identity and all off-diagonal entries are zero. Such semigroups are primitive adequate semigroups and they satisfy the conditions on the principal right ideals that ensure they are Rees semigroups. The subsemigroup, $\mathcal{G}$, of regular elements of $\mathcal{M}$ is just $M^0(G^0; I, I; P)$, where G is the group of units of S. Thus it is a 0-bisimple primitive inverse semigroup. Let X^* be a free monoid such that $S = X^*G$ uniquely. Put $\mathcal{X} = \{(i, x, i)\colon\ x \in X, i \in I\} \cup \{0\}$. Then $\mathcal{X}$ is isomorphic to the disjoint union of $|I|$ copies of the free monoid X^* — a free category — with a zero adjoined. The fact that $S = X^*G$ uniquely implies that $\mathcal{M} = \mathcal{X}\mathcal{G}$, with each non-zero element being uniquely represented.

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